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CURVATURES OF TYPICAL CONVEX BODIES— THE COMPLETE PICTURE

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Abstract. It is known that a typical n-dimensional convex body, in the Baire category sense, has the property that its set of umbilics of zero curvature has full measure in the boundary of the body. We show that a typical convex body has in addition the following properties. The spherical image of the set of umbilics of zero curvature has measure zero. The set of umbilics of infinite curvature is dense in the boundary and uncountable and its spherical image has full measure in the unit sphere.

1. Introduction and main result

The boundary of a convex body with interior points in n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$) is a closed convex hypersurface. If it is regular and of class C^2, then its curvature behavior is unspectacular; it is described by its principal curvatures and can be visualized by the Dupin indicatrix together with Euler's theorem. If the boundary hypersurface $\mathrm{bd}\, K$ is not regular of class C^2, then the principal curvatures still exist, and Euler's theorem holds, almost everywhere in $\mathrm{bd}\, K$, due to a theorem of A. D. Alexandroff. Here 'almost everywhere' means that the set of exceptional points has $(n-1)$-dimensional Hausdorff measure zero. The set of these exceptional points may still carry important geometric information. For example, if K is a polytope, then the spherical image of the set of exceptional points has full measure in the unit sphere. This can also happen if $\mathrm{bd}\, K$ is regular and of class C^1. A more bizarre curvature behavior can take place at these exceptional points. This has become clear when the curvature behavior of typical convex bodies, in the Baire category sense, was investigated. In the present paper we supplement the known results of this type by taking the spherical image into account, which allows us to answer a longstanding question in a stronger sense and which altogether gives, we think, a fairly complete picture of the curvature behavior of typical convex bodies.

The space $\mathcal{K}^n$ of convex bodies (nonempty compact convex sets) in $\mathbb{R}^n$ with the Hausdorff metric is a complete metric space and thus a Baire space. By definition, a topological space is a *Baire space* if the intersection of any countable family of dense open subsets of the space is dense. A subset of a Baire space is called *meager* or of the *first category* if it is a countable union of nowhere dense sets, and *comeager* if its complement is meager (we avoid the synonymous term 'residual',

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since it is misleading, for etymological reasons). We shall make use of the fact that a comeager subset Y of a Baire space X is again a Baire space, with the induced topology, and that a comeager subset of Y is comeager in X (see, e.g., [3, chap. IX, §5.3, Prop. 5] or [17, Thm. 3.1]). Comeager subsets of a Baire space can be considered as 'large' since the intersection of countably many such sets is still dense. If P is a property that a convex body can have, we say that the typical convex body in $\mathcal{K}^n$ has property P if the set of convex bodies with this property is a comeager subset of $\mathcal{K}^n$. Alternatively, this is expressed by saying that most convex bodies in $\mathcal{K}^n$ have property P.

Let $\mathcal{K}^n_*$ denote the set of smooth and strictly convex bodies in $\mathbb{R}^n$. Here a convex body is called smooth if each boundary point lies in a unique supporting hyperplane. The boundary $\operatorname{bd} K$ of a convex body $K \in \mathcal{K}^n_*$ is a regular hypersurface of class C^1. We recall (e.g., from Busemann [4, p. 14]) a general notion of curvature. Let $K \in \mathcal{K}^n_*$ and $x \in \operatorname{bd} K$. Let u be the outer unit normal vector of K at x and let $t \in \mathbb{R}^n$ with $o \neq t \perp u$ be a tangent vector of K at x. We denote by L the line through x spanned by u and by E the half-plane $L + \mathbb{R}_{>0}t$. For $z \in E \cap \operatorname{bd} K$ there is a unique half-circle in E with centre on L that passes through x and z. Let $r(z)$ denote its radius. Then

$$\kappa_i(x,t) := \liminf_{z\to x} r(z)^{-1} \in [0,\infty], \qquad \kappa_s(x,t) := \limsup_{z\to x} r(z)^{-1} \in [0,\infty]$$

are called, respectively, the *lower curvature* and the *upper curvature* of K at x in direction t. If

$$\kappa_s(x,t) = 0 \qquad \text{for all } t \perp u,\ t \neq o,$$

then x is called an *umbilic of zero curvature*, and if

$$\kappa_i(x,t) = \infty \qquad \text{for all } t \perp u,\ t \neq o,$$

then x is called an *umbilic of infinite curvature*. To the other extreme, if

$$\kappa_i(x,t) = 0 \quad \text{and} \quad \kappa_s(x,t) = \infty \qquad \text{for all } t \perp u,\ t \neq o,$$

then we call x a *point of oscillating curvature*. We denote by $U_0(K)$ the set of umbilics of zero curvature and by $U_\infty(K)$ the set of umbilics of infinite curvature of the convex body K.

If $K \in \mathcal{K}^n$ and $A \subset \operatorname{bd} K$, the *spherical image* of the set A is the set of all outer unit normal vectors to K at points of A. It is a subset of the unit sphere, $\mathbb{S}^{n-1}$, of $\mathbb{R}^n$. If $K \in \mathcal{K}^n_*$, then the *spherical image map* or *Gauss map* u_K is defined by associating with each point $x \in \operatorname{bd} K$ the unique outer unit normal vector $u_K(x)$ of K at x. The spherical image map u_K of $K \in \mathcal{K}^n_*$ is a homeomorphism from $\operatorname{bd} K$ to $\mathbb{S}^{n-1}$.

The subsequent theorem presents information about how large a set of points with specific curvature properties is for a typical convex body. Here 'large' can have one of the following interpretations: full measure, comeager, dense, uncountable. When we say that a subset $A \subset \operatorname{bd} K$ has *full measure*, then this means that $\mathcal{H}^{n-1}(\operatorname{bd} K \setminus A) = 0$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Similarly, A is said to be of measure zero if $\mathcal{H}^{n-1}(A) = 0$. That A is dense means that A is dense in $\operatorname{bd} K$. Corresponding ways of speaking are used for subsets of the unit sphere $\mathbb{S}^{n-1}$.

The following theorem collects old and new results.

Theorem. *A typical convex body $K \in \mathcal{K}^n$ has the following properties:*

(a) *K is smooth and strictly convex.*

(b) *The set of umbilics of zero curvature has full measure. Its spherical image has measure zero, but is dense and uncountable.*

(c) *The set of umbilics of infinite curvature has measure zero, but is dense and uncountable. Its spherical image has full measure.*

(d) *The set of points of oscillating curvature is comeager in* $\operatorname{bd} K$.

We now list which of the preceding results are known and which new information we add (surveys on Baire category type results in convexity were given by Gruber [6, 7] and Zamfirescu [16, 17]). Assertion (a), that most convex bodies in $\mathcal{K}^n$ are smooth and strictly convex, was first proved (in an infinite dimensional version) by Klee [8]. It was rediscovered by Gruber [5], who also showed that most convex bodies in $\mathcal{K}^n$ have boundaries that are not of class C^2. The latter observation was strengthened by Zamfirescu [12], who proved that for most convex bodies $K \in \mathcal{K}^n$, for each boundary point x of K and for each tangent direction t at x it holds that $\kappa_i(x,t) = 0$ or $\kappa_s(x,t) = \infty$. By a theorem of Alexandroff [2], almost all boundary points of $K \in \mathcal{K}^n_*$, in the sense of measure, are normal points (we repeat the definition in Section 2). If x is a normal point, then $\kappa_i(x,t) = \kappa_s(x,t) < \infty$ for any tangent direction t at x. Hence, the results of Zamfirescu and Alexandroff together imply that, for most convex bodies $K \in \mathcal{K}^n_*$, the set $U_0(K)$ of umbilics of zero curvature has full measure. This is the first part of assertion (b) above. In particular, $U_0(K)$ must be dense and uncountable, and since the spherical image map of $K \in \mathcal{K}^n_*$ is a homeomorphism, the spherical image of $U_0(K)$ is also dense and uncountable. We shall explain in Section 2 why it is of measure zero.

In 1985, Zamfirescu [14, Problem 1] asked (in other words) the following question: Do most convex bodies possess an umbilic of infinite curvature? In the plane, Zamfirescu [15] even showed the following. For each smooth, strictly convex curve C for which the curvature is zero almost everywhere, there is an uncountable dense set of points on C where the curvature is infinite. The proof, however, does not extend to higher dimensions. In fact, Zamfirescu's original question had to wait over 25 years for an answer. Finally, Adiprasito [1] proved that a typical convex body in $\mathcal{K}^n_*$ has at least one umbilic of infinite curvature. Here this result is strengthened considerably by part of assertion (c), saying that typically the set $U_\infty(K)$ of umbilics of infinite curvature is dense and uncountable. This follows from the fact that for a typical convex body the spherical image of $U_\infty(K)$ has full measure, which will be proved in Section 2, together with the fact that the spherical image map of $K \in \mathcal{K}^n_*$ is a homeomorphism. That $U_\infty(K)$ itself has measure zero follows, of course, from (b).

The results of Zamfirescu mentioned so far left open the possibility that a typical convex body might have a nonempty set of boundary points x at which $\kappa_i(x,t) = 0$ *and* $\kappa_s(x,t) = \infty$ for each tangent direction t at x. That typically there is a dense set of such points was proved by Schneider [9], and Zamfirescu [13] improved this by showing that the set in question is even comeager. Of this fact, Zamfirescu gave a simpler proof in [14], which is reproduced in [10, Sec. 2.6]. We have included this result as assertion (d) above, to complete the picture.

We recall that, up to some remarks, only assertion (c) is new. It will be proved in the rest of this note.

2. Umbilics of zero or infinite curvature

First we recall the notion of normal points. For our purpose, it suffices to consider a convex body $K \in \mathcal{K}^n_*$. Let $x \in \operatorname{bd} K$, let u be the unique outer unit normal vector of K at x and $H(K,u)$ the supporting hyperplane of K at x. The subspace $T_xK := H(K,u) - x$ is the tangent space of K at x. For sufficiently small $h > 0$ we consider the sets

$$\mathcal{D}_x(K,h) := (2h)^{-1/2}\left\{[K \cap (H(K,u) - hu)] + hu - x\right\}.$$

If the topological (Painlevé–Kuratowski) limit

$$\mathcal{D}_x(K) = \lim_{h\to 0+} \mathcal{D}_x(K,h)$$

exists, it is called the *generalized Dupin indicatrix*, or briefly the *indicatrix*, of K at x. If K is of class C^2, then $\mathcal{D}_x(K)$ does exist and can be represented in the form

$$\mathcal{D}_x(K) = \left\{y \in T_xK : k_1\langle y, e_1\rangle^2 + \cdots + k_{n-1}\langle y, e_{n-1}\rangle^2 \le 1\right\}, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^n$, $(e_1,\ldots,e_{n-1})$ is an orthonormal basis of T_xK and the real numbers $k_1,\ldots,k_{n-1} \ge 0$ are the principal curvatures of K at x. In general, the point x is called a *normal point* of K if the indicatrix $\mathcal{D}_x(K)$ exists and can be represented by (1), with a suitable orthonormal basis $(e_1,\ldots,e_{n-1})$ and real numbers $k_1,\ldots,k_{n-1} \ge 0$. These numbers are then again called the principal curvatures of K at x. According to Alexandroff [2], the set of normal points of K has full measure in $\operatorname{bd} K$.

The set $\mathcal{D}_x(K,h)$ considered above contains o in its relative interior, for all sufficiently small $h > 0$ (and we consider only these). Let $\rho(\mathcal{D}_x(K,h),\cdot) > 0$ denote the radial function of $\mathcal{D}_x(K,h)$, so that $\rho(\mathcal{D}_x(K,h),t)t$ is in the relative boundary of $\mathcal{D}_x(K,h)$, for $t \in T_xK \setminus \{o\}$. For $\|t\| = 1$ we then have

$$\frac{1}{\kappa_i(x,t)} = \limsup_{h\to 0+} \rho^2(\mathcal{D}_x(K,h),t), \qquad \frac{1}{\kappa_s(x,t)} = \liminf_{h\to 0+} \rho^2(\mathcal{D}_x(K,h),t). \tag{2}$$

Therefore, x is an umbilic of zero curvature if and only if $\mathcal{D}_x(K) = T_xK$. This is easily seen from (2) if one uses a suitable criterion for the topological convergence (see, e.g., [11, Thm. 12.2.2]), together with the fact that all considered sets $\mathcal{D}_x(K,h)$ are convex bodies containing o. In particular, each umbilic of zero curvature is a normal point. Similarly, x is an umbilic of infinite curvature if and only if $\mathcal{D}_x(K) = \{o\}$. Note that an umbilic of infinite curvature is not a normal point.

To deal with umbilics of infinite curvature, we use polarity. Let $K \in \mathcal{K}^n_*$ and assume that $o \in \operatorname{int} K$ (the interior of K). The *polar body* of K is defined by

$$K^\circ := \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } y \in K\}.$$

It is again smooth and strictly convex, contains o in the interior, and $(K^\circ)^\circ = K$. Since K and K° are both in $\mathcal{K}^n_*$, each boundary point x of K determines uniquely a boundary point y of K° such that x is an outer normal vector of K° at y, and then y is an outer normal vector of K at x (see, e.g., [10, p. 44]). We say that y *corresponds* to x under polarity, and we define the mapping $p_K : \operatorname{bd} K \to \operatorname{bd} K^\circ$ by $p_K(x) := y$.

Lemma 1. *If $K \in \mathcal{K}^n_*$ and $o \in \operatorname{int} K$, then $p_K(U_0(K)) = U_\infty(K^\circ)$ and $p_K(U_\infty(K)) = U_0(K^\circ)$.*

Proof. First, we remark that the notion of umbilic point of zero or infinite curvature is invariant under affine transformations. This is clear for translations. Let $K \in \mathcal{K}^n_*$ and $x \in \operatorname{bd} K$. If $\Lambda \in \mathrm{GL}(n)$ is a nonsingular linear transformation of $\mathbb{R}^n$, then for all sufficiently small $h > 0$ we have

$$\mathcal{D}_{\Lambda x}(\Lambda K, h) = \Lambda_1 \mathcal{D}_x(K, \lambda h)$$

with a nonsingular linear transformation Λ_1 of $T_x K$ and a positive number λ, both depending only on Λ and $T_x K$. This immediately gives the assertion.

Let $K \in \mathcal{K}^n_*$ with $o \in \operatorname{int} K$. We assume, first, that $x \in U_0(K)$. Let u be the outer unit normal vector of K at x. Without loss of generality we can assume that $x = \mu u$ with $\mu > 0$, since this can be achieved by applying a suitable nonsingular linear transformation. Let $y := p_K(x)$, then u is also the outer unit normal vector to K° at y. For $a > 0$ let E_a be the ellipsoid of revolution with centre o, axis of revolution parallel to u, semi-axis length orthogonal to u equal to a, and containing x in the boundary.

Let $a > 0$ be given. The indicatrix $\mathcal{D}_x(E_a)$ is an $(n-1)$-dimensional ball with centre o in $T_x K$. Since

$$\lim_{h \to 0+} \mathcal{D}_x(K, h) = T_x K, \qquad \lim_{h \to 0+} \mathcal{D}_x(E_a, h) = \mathcal{D}_x(E_a),$$

there is a number $h_0 > 0$ such that

$$\mathcal{D}_x(E_a, h) \subset 2\mathcal{D}_x(E_a) \subset \mathcal{D}_x(K, h) \qquad \text{for } 0 < h < h_0.$$

Hence, for $0 < h < h_0$ we have

$$E_a \cap (H(K, u) - hu) \subset K \cap (H(K, u) - hu).$$

Thus, there is a neighbourhood N of x with $E_a \cap N \subset K \cap N$. From this it follows (e.g., using [10, Theorem 1.7.6]) that there is a neighbourhood N' of y with

$$K^\circ \cap N' \subset E_a^\circ \cap N'.$$

Therefore, there is a number $h_1 > 0$ such that

$$\mathcal{D}_y(K^\circ, h) \subset \mathcal{D}_y(E_a^\circ, h) \qquad \text{for } 0 < h < h_1.$$

From this, we conclude that

$$\mathcal{D}_y(K^\circ, h) \subset 2\mathcal{D}_y(E_a^\circ)$$

for all sufficiently small $h > 0$. Since $\lim_{a \to \infty} \mathcal{D}_y(E_a^\circ) = \{o\}$, it follows that $\mathcal{D}_y(K^\circ) = \{o\}$ and hence that $y \in U_\infty(K^\circ)$. As $x \in U_0(K)$ was arbitrary, we have shown that

$$p_K(U_0(K)) \subset U_\infty(K^\circ). \tag{3}$$

If we assume, second, that $x \in U_\infty(K)$, then a completely analogous argument gives $y \in U_0(K)$ and thus

$$p_K(U_\infty(K)) \subset U_0(K^\circ). \tag{4}$$

Interchanging the roles of K and K° in (4) and applying (3), we get

$$(p_K \circ p_{K^\circ})(U_\infty(K^\circ)) \subset p_K(U_0(K)) \subset U_\infty(K^\circ).$$

Since $p_K \circ p_{K^\circ}$ is the identity mapping, we conclude that $p_K(U_0(K)) = U_\infty(K^\circ)$. Similarly we get $p_K(U_\infty(K)) = U_0(K^\circ)$. □

The following simple fact will be needed. Let $K \in \mathcal{K}^n$ and $o \in \operatorname{int} K$. Then the map $\varphi_K : \operatorname{bd} K \to \mathbb{S}^{n-1}$ defined by $\varphi_K(x) := x/\|x\|$ is Lipschitz. In fact, there are numbers $0 < r < R$ with $r \leq \|x\| \leq R$ for all $x \in \operatorname{bd} K$, and easy estimates give

$$\|\varphi_K(x) - \varphi_K(y)\| \leq \frac{2R}{r^2}\|x - y\| \quad \text{for } x, y \in \operatorname{bd} K.$$

Lemma 2. *Let $K \in \mathcal{K}^n_*$ be a convex body with $o \in \operatorname{int} K$ and such that $U_0(K^\circ)$ has full measure in $\operatorname{bd} K^\circ$. Then the spherical image of $U_0(K)$ has measure zero and the spherical image of $U_\infty(K)$ has full measure.*

Proof. We have $u_K = \varphi_{K^\circ} \circ p_K$; hence

$$u_K(U_0(K)) = \varphi_{K^\circ}(p_K(U_0(K))) = \varphi_{K^\circ}(U_\infty(K^\circ))$$

by Lemma 1. Since $U_0(K^\circ)$ has full measure by assumption, the set $U_\infty(K^\circ)$ has measure zero. Since φ_{K° is a Lipschitz map, the set $u_K(U_0(K))$ has measure zero.

Similarly as above, $u_K(U_\infty(K)) = \varphi_{K^\circ}(U_0(K^\circ))$ and hence $\mathbb{S}^{n-1} \setminus u_K(U_\infty(K)) = \varphi_{K^\circ}(\operatorname{bd} K^\circ \setminus U_0(K^\circ))$. Since $U_0(K^\circ)$ has full measure by assumption, the set $\operatorname{bd} K^\circ \setminus U_0(K^\circ)$ has measure zero. Since φ_{K° is a Lipschitz map, the set $\mathbb{S}^{n-1} \setminus u_K(U_\infty(K))$ has measure zero; hence $u_K(U_\infty(K))$ has full measure. □

3. Finishing the proof

To finish the proof of the theorem, we make use of the fact that all the properties of convex bodies appearing in the theorem are invariant under translations. Let $f : \mathcal{K}^n_* \to \mathbb{R}^n$ be either the centroid map c or the Santaló point map s. We choose these maps since $f(K) \in \operatorname{int} K$ and $c(K) = o \Leftrightarrow s(K^\circ) = o$ (see, e.g., [10, p. 420]). We consider the subspace $\mathcal{K}^n_f := \{K \in \mathcal{K}^n_* : f(K) = o\}$. It contains precisely one element from each translation class of convex bodies in $\mathcal{K}^n_*$. Since f is continuous, the following holds. If a translation invariant subset $\mathcal{M} \subset \mathcal{K}^n_*$ is open, closed, or dense in $\mathcal{K}^n_*$, then the set $\mathcal{M}_f := \{K - f(K) : K \in \mathcal{M}\}$ is, respectively, open, closed, or dense in $\mathcal{K}^n_f$. And if $\mathcal{N}$ is open, closed, or dense in $\mathcal{K}^n_f$, then the set $\tau(\mathcal{N}) := \{K + t : K \in \mathcal{N},\, t \in \mathbb{R}^n\}$ is, respectively, open, closed, or dense in $\mathcal{K}^n_*$. It follows that a translation invariant set $\mathcal{M} \subset \mathcal{K}^n_*$ is comeager in $\mathcal{K}^n_*$ if and only if the set $\mathcal{M}_f$ is comeager in $\mathcal{K}^n_f$.

Let $\mathcal{M}$ be the set of convex bodies $K \in \mathcal{K}^n_*$ with the properties that $U_0(K)$ has full measure in $\operatorname{bd} K$ and that the set of points of K of oscillating curvature is comeager in $\operatorname{bd} K$. By the known results mentioned in the introduction, the translation invariant set $\mathcal{M}$ is comeager in $\mathcal{K}^n_*$ and hence $\mathcal{M}_c$ is comeager in $\mathcal{K}^n_c$. The set $\mathcal{N} := \{K^\circ : K \in \mathcal{M}_c\}$ is comeager in $\mathcal{K}^n_s$, since the polarity mapping is a homeomorphism from $\mathcal{K}^n_c$ to $\mathcal{K}^n_s$. Therefore, the set $\tau(\mathcal{N})$ is comeager in $\mathcal{K}^n_*$. For each $K \in \tau(\mathcal{N})$ the set $U_0(K^\circ)$ has full measure in $\operatorname{bd} K^\circ$. The set $\mathcal{L} := \mathcal{M} \cap \tau(\mathcal{N})$ is comeager in $\mathcal{K}^n_*$. For $K \in \mathcal{L}$, the set $U_0(K)$ has full measure in $\operatorname{bd} K$ and $U_0(K^\circ)$ has full measure in $\operatorname{bd} K^\circ$. Now it follows from Lemma 2 that for $K \in \mathcal{L}$ the spherical image of $U_0(K)$ has measure zero and the spherical image of $U_\infty(K)$ has full measure. This completes the proof. □

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