

## POSITIVE GORENSTEIN IDEALS

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ABSTRACT. We introduce positive Gorenstein ideals. These are Gorenstein ideals in the graded ring  $\mathbb{R}[x]$  with socle in degree  $2d$  which when viewed as a linear functional on  $\mathbb{R}[x]_{2d}$  is nonnegative on squares. Equivalently, positive Gorenstein ideals are apolar ideals of forms whose differential operator is nonnegative on squares. Positive Gorenstein ideals arise naturally in the context of nonnegative polynomials and sums of squares, and they provide a powerful framework for studying concrete aspects of sums of squares representations. We present applications of positive Gorenstein ideals in real algebraic geometry, analysis and optimization. In particular, we present a simple proof of Hilbert's nearly forgotten result on representations of ternary nonnegative forms as sums of squares of rational functions. Drawing on our previous work (2012), our main tools are Cayley-Bacharach duality and elementary convex geometry.

### 1. INTRODUCTION

A real polynomial in  $n$  variables is called nonnegative if it is greater than or equal to 0 on all points in  $\mathbb{R}^n$ . The relationship between nonnegative polynomials and sums of squares (of polynomials or more general objects) is a fundamental question in real algebraic geometry. Algorithmic approaches to this question have been quite useful in polynomial optimization [12], [14], [15], [17].

Any nonnegative polynomial can be made homogeneous by adding an extra variable and it will remain nonnegative. The same holds for sums of squares. We will therefore work with homogeneous polynomials (forms).

Let  $\mathbb{R}[x]_{n,d}$  be the vector space of real forms in  $n$  variables of degree  $d$ . Nonnegative forms and sums of squares both form full dimensional closed convex cones in  $\mathbb{R}[x]_{n,2d}$ , which we call  $P_{n,2d}$  and  $\Sigma_{n,2d}$  respectively:

$$P_{n,2d} = \{p \in \mathbb{R}[x]_{n,2d} \mid p(x) \geq 0 \text{ for all } x \in \mathbb{R}^n\},$$

and

$$\Sigma_{n,2d} = \left\{ p \in \mathbb{R}[x]_{n,2d} \mid p(x) = \sum q_i^2 \text{ for some } q_i \in \mathbb{R}[x]_{n,d} \right\}.$$

In 1888 Hilbert showed that nonnegative polynomials are the same as sums of squares only in the following three cases:  $n = 2$ ,  $2d = 2$  and  $n = 3$ ,  $2d = 4$ . In all other cases  $P_{n,2d}$  is strictly larger than  $\Sigma_{n,2d}$  [9].

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The defining linear inequalities of a cone are described by its dual cone. Let  $P_{n,2d}^*$  and  $\Sigma_{n,2d}^*$  be the dual cones of  $P_{n,2d}$  and  $\Sigma_{n,2d}$ :

$$P_{n,2d}^* = \{ \ell \in \mathbb{R}[x]_{n,2d}^* \mid \ell(p) \geq 0 \text{ for all } p \in P_{n,2d} \},$$

and

$$\Sigma_{n,2d}^* = \{ \ell \in \mathbb{R}[x]_{n,2d}^* \mid \ell(p) \geq 0 \text{ for all } p \in \Sigma_{n,2d} \}.$$

The dual cone  $P_{n,2d}^*$  is easy to describe: its extreme rays are point evaluations, i.e. the linear functionals  $\ell_v \in \mathbb{R}[x]_{n,2d}^*$  given by

$$\ell_v(f) = f(v) \quad \text{for } f \in \mathbb{R}[x]_{n,2d} \quad \text{and } v \in \mathbb{R}^n.$$

Linear functionals  $\ell_v$  are also extreme rays of  $\Sigma_{n,2d}^*$ , but when nonnegative polynomials are not the same as sums of squares, there are linear functionals that are nonnegative on squares and do not come from point evaluations. In order to understand properties of such linear functionals we look at the Gorenstein ideals that they generate. To a linear functional  $\ell : \mathbb{R}[x]_{n,2d} \rightarrow \mathbb{R}$  we can associate the Gorenstein ideal  $I(\ell)$ :

$$I(\ell) = \{ p \in \mathbb{R}[x] \mid \deg(p) > 2d \text{ or } \ell(pq) = 0 \text{ for all } q \in \mathbb{R}[x]_{2d-\deg(p)} \}.$$

By a slight abuse of terminology we will call  $\ell$  the socle of  $I(\ell)$ . We call the ideal  $I(\ell)$  a *positive Gorenstein ideal* if the socle  $\ell$  belongs to  $\Sigma^*$ , i.e. if  $\ell$  is nonnegative on squares.

We can identify  $\mathbb{R}[x]_{n,2d}$  with its dual space  $\mathbb{R}[x]_{n,2d}^*$  by sending  $x_i$  to the differential operator  $\frac{\partial}{\partial x_i}$  and replacing multiplication with composition of differential operators. For  $f \in \mathbb{R}[x]_{n,2d}$  let  $\partial f \in \mathbb{R}[x]_{n,2d}^*$  denote the corresponding differential operator. Then the Gorenstein ideal  $I(\partial f)$  corresponds precisely to the *apolar ideal* of  $f$ . We remark that with identification of  $\mathbb{R}[x]_{n,2d}$  and  $\mathbb{R}[x]_{n,2d}^*$  the dual cone  $P_{n,2d}^*$  is the conic hull of the real Veronese variety of degree  $2d$ , and thus the dual cone of  $P_{n,2d}$  is the conical hull of the Veronese Orbitope [19], [21]. In the following we prefer to keep the language of linear functionals and Gorenstein ideals.

We call a positive Gorenstein ideal  $I$  with socle  $\ell$  of degree  $2d$  *maximal*, if  $\ell$  is an extreme ray of  $\Sigma_{n,2d}^*$ . We showed in [1] that  $I(\ell)$  is a maximal positive Gorenstein ideal if and only if  $I_d(\ell)$  is maximal (by inclusion) over all Gorenstein ideals with socle of degree  $2d$ . We provided a classification of extreme rays of  $\Sigma_{n,2d}^*$  for the two smallest cases where there exist nonnegative polynomials that are not sums of squares:  $n = 3, 2d = 6$  and  $n = 4, 2d = 4$ . We summarize some of the results of [1] in the language of positive Gorenstein ideals:

**Theorem 1.1.** *Let  $I$  be a maximal positive Gorenstein ideal with socle  $\ell \in \mathbb{R}[x]_{3,6}^*$  (resp.  $\mathbb{R}[x]_{4,4}^*$ ). Then  $I$  is a complete intersection of three cubics (resp. four quadrics).*

*Let  $S$  be the linear span of the three cubics (resp. four quadrics) in  $I$ . Then any two cubics (resp. three quadrics) in  $S$  intersecting transversely contain at most one conjugate pair of complex zeroes, with the rest of the zeroes being real.*

This characterization was later used in [2] to study the *algebraic boundaries* of the cones  $\Sigma_{3,6}$  and  $\Sigma_{4,4}$  via a surprising connection with K3 surfaces.

We now undertake the study of positive Gorenstein ideals and present applications in real algebraic geometry, analysis and optimization. Our main tool is Theorem 3.1 which uses Cayley-Bacharach duality of [7] and allows us to give a unified presentation.

## 2. RESULTS

Let  $I$  be a maximal positive Gorenstein ideal with socle of  $\ell$  of degree  $2d$ . We first investigate the possible dimensions of  $I_d$ . As can be seen from Theorem 1.1, we expect maximal positive Gorenstein ideals to possess significant structure, and the choice of possible dimensions for  $I_d$  should be limited.

**Theorem 2.1.** *Let  $I$  be a maximal positive Gorenstein ideal in  $\mathbb{R}[x]$  with socle  $\ell$  of degree  $2d$ , which is not a point evaluation. Then the forms in  $I_d$  have no common zeroes, real or complex, and  $I_d$  generates  $I_{2d}$ . Additionally we have,*

$$\text{codim } I_d \geq 3d - 2 \quad \text{if } d \geq 3 \quad \text{or} \quad \text{codim } I_d \geq 6 \quad \text{if } d = 2.$$

*The bounds are tight for a complete intersection of a cubic and two forms of degree  $d$  in  $\mathbb{R}[x_1, x_2, x_3]$  with  $d \geq 3$ , and a complete intersection of four quadrics in  $\mathbb{R}[x_1, x_2, x_3, x_4]$  for  $d = 2$ .*

We remark that the case  $d = 2$  in the above theorem and subsequent corollaries will always get special treatment. This is due to the fact that tight examples for  $d \geq 3$  occur when the number of variables  $n$  is 3. However, for ternary quartics, nonnegative polynomials are equal to sums of squares by Hilbert's theorem. Therefore the first examples of extreme rays of  $\Sigma_{n,4}^*$  that are not point evaluations occur when  $n = 4$  and this case leads to tight examples.

To a linear functional  $\ell : \mathbb{R}[x]_{n,2d} \rightarrow \mathbb{R}$  we associate a quadratic form  $Q_\ell$  on  $\mathbb{R}[x]_{n,d}$  by setting  $Q_\ell(p) = \ell(p^2)$  for  $p \in \mathbb{R}[x]_{n,d}$ . It is easy to see that  $I(\ell)_d$  is equal to the kernel of  $Q_\ell$ . Restated in convex geometry terms, Theorem 2.1 immediately leads to the following corollary.

**Corollary 2.2.** *Let  $\ell \in \mathbb{R}[x]_{n,2d}^*$  span an extreme ray of  $\Sigma_{n,2d}^*$ . If  $\text{rank } Q_\ell \leq 3d - 3$  for  $d \geq 3$  or  $\text{rank } Q_\ell \leq 5$  for  $d = 2$ , then  $\ell$  is a point evaluation and  $\text{rank } Q_\ell = 1$ . Furthermore these bounds are tight and there exist extreme rays of  $\Sigma_{n,2d}$  of rank  $3d - 2$  for  $d \geq 3$ ,  $n \geq 3$  and 6 for  $d = 2$ ,  $n \geq 4$  that do not come from point evaluations.*

Below we present some direct consequences of Theorem 2.1 in analysis, algebraic geometry and optimization. We hope that the different interpretations highlight the interdisciplinary nature of convex algebraic geometry.

The *truncated moment problem* in real analysis asks for a classification of linear functionals  $\ell \in \mathbb{R}[x]_{n,2d}^*$  that come from integration with respect to a Borel measure:  $\ell(f) = \int_{\mathbb{R}^n} f \, d\mu$  [15]. For  $\ell \in \mathbb{R}[x]_{n,2d}^*$  the matrix  $M_\ell$  of the associated quadratic form  $Q_\ell$  with respect to the monomial basis is known as the *moment matrix* of  $\ell$ . If  $\ell$  comes from integration with respect to a measure, then  $M_\ell$  must be positive semidefinite. We show that if rank of  $M_\ell$  is sufficiently small, then  $\ell$  indeed comes from a measure, and furthermore  $\ell$  can be written as a sum of precisely rank  $M_\ell$  point evaluations.

**Theorem 2.3.** *Let  $\ell : \mathbb{R}[x]_{n,2d} \rightarrow \mathbb{R}$  be a linear functional and suppose that the moment matrix  $M_\ell$  is positive semidefinite and  $\text{rank } M_\ell \leq 3d - 3$  with  $d \geq 3$  or  $\text{rank } M_\ell \leq 5$  for  $d = 2$ . Then  $\ell$  comes from integration with respect to a measure and it can be written as a conical combination of rank  $M_\ell$  point evaluations. Furthermore this bound is tight, and there exist linear functionals  $\ell : \mathbb{R}[x]_{n,2d} \rightarrow \mathbb{R}$  with  $\text{rank } M_\ell = 3d - 2$  for  $d \geq 3$ ,  $n \geq 3$  and  $\text{rank } M_\ell = 6$  for  $d = 2$ ,  $n \geq 4$ , such that*

$M_\ell$  is positive semidefinite, but  $\ell$  does not come from integration with respect to a measure.

The *symmetric tensor decomposition problem* (also known as the *Waring problem*) for a given  $f \in \mathbb{R}[x]_{n,d}$  (or  $f \in \mathbb{C}[x]_{n,d}$ ) asks for the minimum number of linear forms  $\ell_i$  such that  $f = \sum c_i \ell_i^d$ . The minimal number of linear forms is known as the *Waring rank* of  $f$  [13]. By using the identification of  $f \in \mathbb{R}[x]_{n,2d}$  with  $\partial f \in \mathbb{R}[x]_{n,2d}^*$  we can associate to  $f$  a quadratic form  $Q_f : \mathbb{R}[x]_{n,d} \rightarrow \mathbb{R}$  by setting  $Q_f(p) = \partial f(p^2)$ . The matrix of the quadratic form  $Q_f$  is known as the *middle catalecticant matrix* of  $f$ . It is well known that the Waring rank of  $f$  is at least the rank of  $Q_f$  [11]. Theorem 2.1 implies that if  $Q_f$  is positive semidefinite and its rank is sufficiently small, then it can be decomposed as linear combination of  $2d$ -th powers of real linear forms, with strictly positive coefficients, and the Waring rank of  $f$  is precisely  $\text{rank } Q_f$ . We remark that the *real Waring problem* where we require that all the forms used in the decomposition are real tends to be more complicated than the Waring problem over  $\mathbb{C}$  [4], [5].

**Theorem 2.4.** *Let  $f \in \mathbb{R}[x]_{n,2d}$  be a form such that the middle catalecticant matrix  $Q_f$  of  $f$  is positive semidefinite and  $\text{rank } Q_f \leq 3d - 3$  if  $d \geq 3$  or  $\text{rank } Q_f \leq 5$  if  $d = 2$ . Then  $f$  can be decomposed as a sum of  $2d$ -th powers of linear forms with positive coefficients, and the Waring rank of  $f$  is equal to  $\text{rank } Q_f$ :*

$$f = \sum_{i=1}^{\text{rank } Q_f} c_i \ell_i^{2d} \quad \text{with} \quad \ell_i \in \mathbb{R}[x]_{n,1} \quad \text{and} \quad c_i \geq 0.$$

Furthermore this bound is tight, and there exist forms  $f \in \mathbb{R}[x]_{n,2d}$  with positive semidefinite form  $Q_f$  of rank  $3d - 2$  whose real Waring rank is strictly greater than  $\text{rank } Q_f$ .

We note that if the positive semidefiniteness assumption of  $Q_f$  is dropped, then it is possible, using for instance lower bounds on the Waring rank from [13], to construct forms  $f$  with small rank of  $Q_f$ , such that the Waring rank is strictly greater than the rank of  $Q_f$ , even if we allow  $2d$ -th powers of complex linear forms in the decomposition.

Theorem 2.1 also leads to an interesting stopping criterion for sum of squares relaxations. Sum of squares methods lead to a hierarchy of relaxations indexed by degree. Under a variety of conditions the hierarchy is guaranteed to converge to the optimal solution [14], [15], [17]. One of the important questions in this area is as follows: when can we guarantee that we obtained the actual optimum, and thus stop computing relaxations of higher degree? Primal-dual methods for semidefinite programming, when applied to sums of squares relaxations, lead to a solution, along with a certifying dual linear functional on  $\mathbb{R}[x]_{n,2d}$ . If we can conclude that the certifying functional comes from a measure, then the sum of squares relaxation is exact and we obtained the optimal solution. From Theorem 2.3 we see the following stopping criterion:

**Stopping criterion for SOS relaxations.** Suppose that a sum of squares relaxation truncated in degree  $2d$  with  $d \geq 3$  returns an optimal linear functional with moment matrix of rank at most  $3d - 3$ . Then the relaxation is exact.

The codimension bound in Theorem 2.1 comes from the following theorem, which we hope is interesting in itself.

**Theorem 2.5.** *Let  $S$  be a subspace of  $\mathbb{C}[x]_{n,d}$  with  $n, d \geq 3$  such that  $\mathcal{V}(S) = \emptyset$ . Let  $I = I(S)$  be the ideal generated by  $S$  and suppose that  $\text{codim } I_{2d} \geq 1$ , so that  $S$  does not generate all forms of degree  $2d$ . Then*

$$\text{codim } I_d \geq 3d - 2.$$

*The bound is tight when  $S$  is the degree  $d$  part of the complete intersection of a form of degree 3 and two forms of degree  $d$  in  $\mathbb{C}\mathbb{P}^2$ .*

*Let  $S$  be a subspace of  $\mathbb{C}[x]_{n,2}$  with  $n \geq 4$  such that  $\mathcal{V}(S) = \emptyset$ . Let  $I = I(S)$  be the ideal generated by  $S$  and suppose that  $\text{codim } I_4 \geq 1$ . Then*

$$\text{codim } I_2 \geq 6.$$

*The bound is tight when  $S$  is the complete intersection of 4 quadrics in  $\mathbb{C}\mathbb{P}^3$ .*

We remark that the tight examples in Theorem 2.5 come from three or four variable constructions, i.e.  $I_1$  has codimension 3 for  $d \geq 3$  or 4 for  $d = 2$ . If we want to instead bound the size of  $I_d$  as the function of the codimension of  $I_1$ , then better bounds are possible. For instance it was shown in [16] that for the case of socle of degree 4 we have

$$\lim_{(\text{codim } I_1) \rightarrow \infty} \min \frac{\text{codim } I_2}{(\text{codim } I_1)^{2/3}} = 6^{2/3}$$

for all Gorenstein ideals with socle of degree 4. We are not aware of any results for socles of higher degree, but these would be very interesting.

**2.1. Positive forms and positive Gorenstein ideals.** In 1893 Hilbert showed the following [10]:

**Theorem 2.6** (Hilbert). *Let  $p$  be a nonnegative form of degree  $2d$  in 3 variables. Then there exists a nonnegative form  $q$  of degree  $2d - 4$  such that  $pq$  is a sum of squares.*

By reducing degrees of multipliers  $q$  Hilbert concluded that for every  $p \in P_{3,2d}$  there exists a sum of squares multiplier  $q$  such that  $pq$  is a sum of squares, where the degree of  $q$  is at most  $d(d/2 - 1)$  if  $d$  is even, and at most  $(d - 1)^2/2$  if  $d$  is odd. This allowed Hilbert to conclude that any  $p \in P_{3,2d}$  is a sum of squares of rational functions. Later, Hilbert posed his 17th problem, which asked whether for any number of variables, a nonnegative polynomial is a sum of squares of rational functions. This was answered in the affirmative by Artin and Schreier [3]. The development of general theory, and the difficulty of Hilbert's proof led to his result on trivariate forms being nearly forgotten.

However, the bounds for the degrees of sums of squares multipliers are very poorly understood in the general case. In fact, the general approach leads to significantly worse bounds than Hilbert's bounds in the case of trivariate forms. We use Theorem 2.7, stated below, to reprove Hilbert's result. We note that in fact Hilbert proved more: the form  $pq$  was not just a sum of squares, but a sum of 3 squares, which allowed Hilbert to conclude that any  $p \in P_{3,2d}$  is a sum of squares of at most 4 rational functions. We do not provide a bound on the number of squares, however, we hope that a simple proof of this result will be useful in understanding the bounds on degrees of sum of squares multipliers.

Our main theorem on the structure of trivariate positive Gorenstein ideals states that such ideals cannot contain positive forms of low degree. This leads to a new, significantly simpler proof of Theorem 2.6.

**Theorem 2.7.** *Let  $I$  be a positive Gorenstein ideal in  $\mathbb{R}[x_1, x_2, x_3]$  with socle  $\ell$  of degree  $4d$ . Then  $I$  does not contain a strictly positive form of degree  $2d + 2$  or less.*

We also consider the question of optimality of the degree of multipliers in Hilbert's theorem. We note that since  $P_{3,2} = \Sigma_{3,2}$ , Theorem 2.6 implies that for  $p \in P_{3,6}$  there exists  $q \in \Sigma_{3,2}$  such that  $pq$  is a sum of squares. This is an optimal bound on the degree of  $q$  since  $P_{3,6} \neq \Sigma_{3,6}$ . Similarly, since  $P_{3,4} = \Sigma_{3,4}$  we see that for  $p \in P_{3,8}$  there exists  $q \in \Sigma_{3,4}$  such that  $pq$  is a sum of squares. However it is not known whether quadratic multipliers would suffice for degree 8 ternary forms, and more generally, whether bounds of Theorem 2.6 are optimal.

Construction (or proof of existence) of nonnegative forms that are not sums of squares is already nontrivial [20]. Our task is to construct forms that are not sums of squares even after multiplication by forms of certain degree. We use an extension of Hilbert's ideas from [9] together with tools from convex geometry to prove the following:

**Corollary 2.8.** *For all  $k \in \mathbb{N}$  there exist forms  $p \in P_{3,4k+6}$  such that  $pq$  is not a sum of squares for all  $q \in \Sigma_{3,2k}$ . There exist forms in  $P_{4,6}$  and  $P_{7,4}$  such that  $pq$  is not a sum of squares for all  $q \in \Sigma_{4,2}$  (resp.  $\Sigma_{7,2}$ ).*

We note that this implies, in particular, that there exist  $p \in \Sigma_{3,10}$  for which quadratic multipliers do not suffice, and  $p \in \Sigma_{3,14}$  for which quartic multipliers do not suffice. While Theorem 2.6 optimally settles the case of  $P_{3,6}$  we still leave open the case of  $P_{4,4}$ , the other smallest case where nonnegative polynomials are not the same as sums of squares.

**Question.** Let  $p \in P_{4,4}$ . Does there exist  $q \in P_{4,2} = \Sigma_{4,2}$  such that  $pq$  is a sum of squares?

We note that Corollary 2.8 implies that if we increase the degree from  $P_{4,4}$ , then quadratic multipliers will not be sufficient.

Theorem 2.7 also has interesting consequences in polynomial optimization. It directly leads to the following corollary:

**Corollary 2.9.** *Let  $p$  be a strictly positive form in  $\mathbb{R}[x]_{3,2d}$ . Then  $p^2$  lies strictly in the interior of the cone of sums of squares  $\Sigma_{3,4d}$ .*

We observe that a form  $p \in \mathbb{R}[x]_{n,2d}$  is in the interior of  $\Sigma_{n,2d}$  if and only if there exists a positive definite Gram matrix for  $p$  (cf. Lemma 6.2 and [18]). We also note that a feasible semidefinite program will return Gram matrix of maximal rank. Therefore we see that  $p \in P_{3,2d}$  is strictly positive if and only if  $p^2$  has a Gram matrix of full rank. This provides us with a numerical way to test strict positivity of  $p \in \mathbb{R}[x]_{3,2d}$ .

This idea can be extended to testing feasibility of systems of equations over  $\mathbb{R}$ . Consider a system of equations  $f_1(x) = \dots = f_k(x) = 0$  with  $f_i \in \mathbb{R}[x]_{3,d}$ . The system is infeasible if and only if  $q = f_1^2 + \dots + f_k^2$  is strictly positive. We can decide strict positivity of  $q$  for ternary forms, by checking whether  $q^2$  lies on the boundary of  $\Sigma_{3,4d}$ .

This observation leads to the following question:

**Question.** Let  $p \in \mathbb{R}[x]_{n,2d}$  be a strictly positive form. When can  $p^2$  lie on the boundary of  $\Sigma_{n,4d}$ ?

We show that this is not possible for ternary forms, and results of [1] also show that this is not possible when  $p \in \mathbb{R}[x]_{4,2}$ , but the general case is open, and has potentially interesting consequences for deciding complexity of semidefinite programming.

### 3. MASTER THEOREM

We begin by proving our main tool from Cayley-Bacharach duality. Let  $\Gamma$  be a finite collection of points in  $\mathbb{C}\mathbb{P}^{n-1}$ , let  $I = I(\Gamma)$  be the ideal of  $\Gamma$  and let  $A(\Gamma)$  be the coordinate ring of  $\Gamma$ :  $A(\Gamma) = \mathbb{C}[x]/I$ . Evaluations on points of  $\Gamma$  are linear functionals on  $\mathbb{C}[x]$ . Let  $\text{Ind}_d(\Gamma)$  be the number of linearly independent evaluations on points of  $\Gamma$  for forms of degree  $d$ . It follows that  $\text{Ind}_d(\Gamma) = \dim A(\Gamma)_d$ . Let  $\text{Dep}_d(\Gamma)$  be the number of linearly dependent relations between the evaluation functionals. Then  $\text{Dep}_d(\Gamma) = |\Gamma| - \text{Ind}_d \Gamma$ .

**Theorem 3.1.** *Let  $p$  and  $q$  be two curves in  $\mathbb{C}\mathbb{P}^2$ , with  $\deg p = k$ ,  $\deg q = s$  and  $k \geq s \geq 3$ , intersecting transversely in a 0-dimensional variety  $\Gamma$ . Then for any subvariety  $\Gamma' \subseteq \Gamma$  such that  $\text{Dep}_s(\Gamma') \geq 1$  we have:*

$$\text{Ind}_s(\Gamma') - \text{Dep}_s(\Gamma') \geq \text{Ind}_s(\Gamma) - \text{Dep}_s(\Gamma).$$

*Proof.* We note that by application of Bertini's Theorem (on the level of generality of [8, Theorem 17.16]) to the degree  $k$  part of the ideal generated by  $p$  and  $q$ , we may choose  $p$  to be smooth. Let  $\Gamma''$  be the complement of  $\Gamma'$  in  $\Gamma$ . Let  $J$ ,  $J'$  and  $J''$  be the ideals of  $\Gamma$ ,  $\Gamma'$  and  $\Gamma''$  respectively.

By applying Theorem CB7 of [7] we see that  $\text{Dep}_s(\Gamma')$  is equal to the dimension of the vector space of forms of degree  $k-3$  vanishing on  $\Gamma''$  modulo vanishing on  $\Gamma$ . Therefore  $\text{Dep}_s(\Gamma') = \dim J''_{k-3} - \dim J_{k-3}$ .

We know that  $\text{Dep}_s(\Gamma') \geq 1$ , and therefore there exists a form  $r$  of degree  $k-3$  vanishing on  $\Gamma''$  but not on  $\Gamma$ . Let  $G$  be the intersection of  $r$  with  $p$ . Since  $p$  is a smooth curve, we see that  $G$  is a zero dimensional scheme and  $G$  contains  $\Gamma''$ . Let  $G'$  be the subscheme residual to  $\Gamma''$  in  $G$ . Let  $I'$  be the ideal of curves containing  $G'$ . We can now apply Theorem CB7 to  $\Gamma''$  as a subscheme of  $G$ . It follows that  $\text{Dep}_{k-3}(\Gamma'')$  is equal to the dimension of the vector space of curves of degree  $k-3$  containing  $G'$  modulo containing all of  $G$ . Since  $G$  is a 0-dimensional complete intersection of curves of degree  $k-3$  and  $k$ , it follows that  $r$  is the unique form of degree  $k-3$  vanishing on all of  $G$ . Therefore  $\text{Dep}_{k-3} \Gamma'' = \dim I'_{k-3} - 1$  and also  $J''_{k-3} \cap I'_{k-3}$  is a 1-dimensional vector space spanned by  $r$ . Therefore we obtain the following inequality:

$$\dim \mathbb{C}[x]_{3,k-3} \geq \dim J''_{k-3} + \text{Dep}_{k-3}(\Gamma'').$$

By another application of Theorem CB7 we see that  $\dim J'_s - \dim J_s = \text{Dep}_{k-3}(\Gamma'')$  and  $\dim J''_{k-3} - \dim J_{k-3} = \text{Dep}_s(\Gamma')$ . It follows that

$$-\text{Dep}_s(\Gamma') \geq \dim J'_s - \mathbb{C}[x]_{3,k-3} - \dim J_s + \dim J_{k-3}.$$

We now observe that  $\text{Ind}_{k-3}(\Gamma) + \dim J_{k-3} = \dim \mathbb{C}[x]_{3,k-3}$ , which gives us

$$(3.1) \quad -\text{Dep}_s(\Gamma') \geq \dim J'_s - \dim J_s - \text{Ind}_{k-3}(\Gamma).$$

Similarly,  $\text{Ind}_s(\Gamma') + \dim J'_s = \text{Ind}_s(\Gamma) + \dim J_s = \dim \mathbb{C}[x]_{3,s}$  and therefore

$$\dim J'_s - \dim J_s = \text{Ind}_s(\Gamma) - \text{Ind}_s(\Gamma').$$

Substituting this into (3.1) we obtain

$$\text{Ind}_s(\Gamma') - \text{Dep}_s(\Gamma') \geq \text{Ind}_s(\Gamma) - \text{Ind}_{k-3}(\Gamma).$$

We now apply Theorem CB7 for the final time to note that  $\text{Ind}_{k-3}(\Gamma) = \text{Dep}_s(\Gamma)$  and the theorem follows.  $\square$

#### 4. PROOFS OF DIMENSION RESULTS

We begin by deriving a codimension bound from Theorem 3.1.

**Theorem 4.1.** *Let  $S$  be a subspace of  $\mathbb{C}[x]_{3,d}$  with  $d \geq 3$ , such that  $\mathcal{V}(S) = \emptyset$ . Let  $I = I(S)$  be the ideal generated by  $S$  and suppose that  $\text{codim } I_{2d} \geq 1$ . Then*

$$\text{codim } I_d \geq 3d - 2.$$

*The bound is tight when  $S$  is the degree  $d$  part of the complete intersection of a cubic and two forms of degree  $d$ .*

*Proof.* Since  $\text{codim } I_{2d} \geq 1$  there exists a linear functional  $\ell : \mathbb{C}[x]_{3,2d} \rightarrow \mathbb{C}$  vanishing on  $I_{2d}$ . By applying Bertini's Theorem we know that there exist two forms  $p, q \in S$  such that they intersect transversely in  $d^2$  points  $\Gamma = \{\gamma_1, \dots, \gamma_{d^2}\} \subset \mathbb{C}\mathbb{P}^2$ . Let  $z_1, \dots, z_{d^2}$  be affine representatives for the points  $\gamma_i$ . Since the ideal generated by  $p$  and  $q$  is radical, we know that we can write  $\ell$  as a linear combination of points evaluations on the points  $z_i$ :

$$\ell(f) = \sum_{i=1}^{d^2} \mu_i f(z_i) \quad \text{for all } f \in \mathbb{C}[x]_{3,2d}.$$

Now define a symmetric bilinear form  $B_\ell$  on  $\mathbb{C}[x]_{3,d}$  given by  $B_\ell(f, g) = \ell(fg) = \sum_{i=1}^{d^2} \mu_i f(z_i)g(z_i)$ . It follows that  $S$  is contained in the kernel of  $B_\ell$ . Let  $\Gamma'$  be the subvariety of  $\Gamma$  consisting of points  $\gamma_i$  for which the coefficient  $\mu_i$  is not zero. Evaluation at any nonzero point of  $\mathbb{C}^3$  leads to a rank 1 symmetric bilinear form. Let  $X$  be the set of  $\text{Ind}_d(\Gamma')$  many points in  $\Gamma'$  such that these points impose independent conditions on forms of degree  $d$ . Consider the quadratic form  $B_X$  given by  $B_X = \sum_{z_i \in X} \mu_i f(z_i)g(z_i)$ . By linear independence of evaluations on  $X$  it follows that  $\text{rank } B_X = \text{Ind}_d(\Gamma')$ . Now as we add evaluations at dependent points to  $B_X$  to obtain  $B_\ell$  the rank can drop by at most  $\text{Dep}_d(\Gamma')$ . Therefore it follows that

$$\text{rank } B_\ell \geq \text{Ind}_d(\Gamma') - \text{Dep}_d(\Gamma').$$

We observe that if  $\text{Dep}_d(\Gamma') = 0$ , then the forms in the kernel of  $B_\ell$  must vanish on all of  $\Gamma'$ , and therefore forms in  $S$  will have a common projective zero. Thus  $\text{Dep}_d(\Gamma') \geq 1$  and we can apply Theorem 3.1. We see that

$$\text{Ind}_d(\Gamma') - \text{Dep}_d(\Gamma') \geq \dim \mathbb{C}[x]_{3,d} - 2 - (d^2 - (\dim \mathbb{C}[x]_{3,d} - 2)) = 3d - 2.$$

Since  $S \subseteq \ker B_\ell$ , we see that  $\text{codim } S \geq \text{rank } B_\ell$  and the inequality of the theorem follows.

The complete (empty) intersection of a form of degree 3 and two forms of degree  $d$  generates a Gorenstein ideal with socle of degree  $2d$  by Theorem CB8 in [7]. Therefore the conditions of the theorem are satisfied. The degree  $d$  part of the ideal has dimension  $\binom{d-1}{2} + 2$  and thus codimension  $3d - 2$  in  $\mathbb{C}[x]_{3,d}$ .  $\square$

We remark that the idea of representing linear function  $\ell$  in terms of point evaluations is well known, and contained for instance in the Apolarity Lemma of [6], where necessary and sufficient conditions are provided. We can straightforwardly generalize Theorem 4.1 to more than 3 variables, while also adding the case of degree 2 to obtain Theorem 2.5.

*Proof of Theorem 2.5.* We prove the first statement by induction on  $n$ . The base case  $n = 3$  is Theorem 4.1. Let  $S$  be a subspace of  $\mathbb{C}[x]_{n+1,d}$  such that  $\mathcal{V}(S) = \emptyset$  and  $\text{codim } I_{2d}(S) \geq 1$ . Then there exists a nontrivial linear functional  $\ell$  on  $\mathbb{C}[x]_{n+1,2d}$  such that  $\ell(I_{2d}(S)) = 0$ .

Suppose that  $\text{codim } S < 3d - 2$ . Let  $W$  be the intersection of  $S$  with a copy of  $\mathbb{C}[x]_{n,d}$  obtained by mapping  $x_{n+1}$  to 0. Then  $W$  is a subspace of  $\mathbb{C}[x]_{n,d}$  and  $\text{codim } W < 3d - 2$ .

We claim that  $\mathcal{V}(W) = \emptyset$ . Suppose not, and  $p(z) = 0$  for some nonzero  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$  and all  $p \in W$ . Then  $f(z_1, \dots, z_n, 0) = 0$  for all  $f \in S$ , which is a contradiction. Let  $\langle W \rangle$  be the ideal generated by  $W$ . By the induction assumption we must have  $\langle W \rangle_{2d} = \mathbb{C}[x]_{n,2d}$  and therefore  $\ell$  is identically 0 on  $\mathbb{C}[x]_{n,2d}$ .

Let  $m_1, \dots, m_n$  be any collection of  $n$  linearly independent forms in  $\mathbb{C}[x]_{n+1,1}$ . We can apply the above argument to the copy of the polynomial ring  $\mathbb{C}[x]_n$  generated by  $m_i$ , not just with the choice  $m_i = x_i$  above. Therefore it follows that  $\ell$  is identically zero on  $\mathbb{C}[m_1, \dots, m_n]_{2d}$  for any choice of  $m_i$ . However, even just the forms  $m^{2d}$  with  $m$  a linear form in  $\mathbb{C}[x]_{n+1,1}$  span  $\mathbb{C}[x]_{n+1,2d}$ . Therefore  $\ell$  is identically zero on  $\mathbb{C}[x]_{n+1,2d}$ , which is a contradiction.  $\square$

We show the following characterization of maximal positive Gorenstein ideals.

**Proposition 4.2.** *Let  $I$  be a positive Gorenstein ideal with socle  $\ell$  of degree  $2d$ . Then  $I$  is maximal if and only if  $I_d$  generates  $I_{2d}$ .*

*Proof.* We recall that by Lemma 2.2 of [1], a positive Gorenstein ideal  $I$  with socle of degree  $2d$  is maximal if and only if  $I_d$  is maximal over all Gorenstein ideals with socle of degree  $2d$ .

Suppose that  $I$  is a positive Gorenstein ideal with socle  $\ell \in \mathbb{R}[x]_{n,2d}^*$ , and  $I_d$  generates  $I_{2d}$ . Then  $I_d$  is maximal, since  $I_d$  uniquely determines  $\ell$ , and  $I_d$  already includes all forms  $p \in \mathbb{R}[x]_{n,d}$  such that  $\ell(pq) = 0$  for all  $q \in \mathbb{R}[x]_{n,d}$ . Therefore  $I$  is a maximal positive Gorenstein ideal.

Now suppose that  $I$  is a maximal positive Gorenstein ideal, but  $I_d$  does not generate  $I_{2d}$ . Let  $J$  be the ideal generated by  $I_d$ . Then there exist two linearly independent functionals  $\ell_1$  and  $\ell_2$  vanishing on  $J_{2d}$ . When we take Gorenstein ideals with socles  $\ell_1$  and  $\ell_2$  we see that they both include  $I_d$  and therefore  $I_d$  is not maximal, which is a contradiction.  $\square$

We note that the condition that  $I_d$  generates  $I_{2d}$  does not imply that  $I$  is generated in degree  $d$ ; it is possible for  $I$  to have additional generators of degree greater than  $d$ , while  $I_d$  generates  $I_{2d}$ . We are now ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* The fact that the forms in  $I_d$  have no common zeroes, real or complex, follows from [1], Corollary 2.3. Now we can apply Theorem 2.5 to  $I_d$  and the dimensional conclusion follows. The statement about maximal positive Gorenstein ideals was proved in Proposition 4.2.

To show that the bounds are tight, let  $\Gamma = \{\gamma_1, \dots, \gamma_{3d}\}$  be a fully real transverse intersection in  $\mathbb{R}\mathbb{P}^2$  of a cubic  $h \in \mathbb{R}[x]_{3,3}$  and a form  $g \in \mathbb{R}[x]_{3,d}$  of degree  $d$ .

Let  $v_1, \dots, v_{3d}$  be affine representatives of the points  $\gamma_i$ . By Theorem CB7 of [7] there exists a unique linear relation for forms in  $\mathbb{R}[x]_{n,d}$  evaluated on the points  $v_i$ :

$$u_1 f(v_1) + \dots + u_{3d} f(v_{3d}) = 0 \quad \text{for all } f \in \mathbb{R}[x]_{n,d},$$

and furthermore all coefficients  $u_i \in \mathbb{R}$  are nonzero. The case of linear functionals based on point evaluations in the presence of unique linear relation was examined in detail in [1]. It follows from Theorem 6.1 of [1] that any linear functional  $\ell \in \mathbb{R}[x]_{3,2d}^*$  given by

$$\ell(f) = \sum_{i=1}^{3d} a_i f(v_i) \quad \text{with all } a_i > 0 \text{ except one and } \sum_{i=1}^{3d} \frac{u_i^2}{a_i} = 0$$

defines an extreme ray of  $\Sigma_{3,2d}^*$ , and  $\ell \notin P_{3,2d}^*$ . As explained in [1] the specific choice of coefficients  $a_i$  adds a form  $\bar{g}$  of degree  $d$  to the kernel of  $B_\ell$ , such that  $\bar{g}$  is not zero at any of the  $v_i$ . It follows by Theorem CB8 of [7] that  $\ell$  is a socle of a maximal positive Gorenstein ideal  $I(\ell)$ , which is the complete intersection of a cubic with two forms of degree  $d$ . We can extend  $\ell$  to be a linear functional on  $\mathbb{R}[x]_{n,2d}$  by adding 0 coordinates to points  $v_i$  and the corresponding  $\ell$  will still define a maximal positive Gorenstein ideal.

For the case  $d = 2$ , we take a fully real transverse intersection of 3 quadrics in  $\mathbb{R}\mathbb{P}^3$  and follow the same construction. This case was explicitly discussed in [1] and the same construction as above will yield an extreme ray of  $\Sigma_{4,4}^*$  that is not a point evaluation. The linear functional  $\ell$  can again be extended to  $\mathbb{R}[x]_{n,4}$ .  $\square$

Corollary 2.2 immediately follows from Theorem 2.1, since  $\text{rank } Q_\ell = \text{codim } I_d(\ell)$ .

*Proof of Theorem 2.3.* We know that any  $\ell \in \Sigma_{n,2d}^*$  can be written as sum of the extreme rays of  $\Sigma_{n,2d}^*$ . The kernel of the moment matrix  $M_\ell$  is equal to  $I_d(\ell)$ . By Corollary 2.2 the only extreme rays of  $\Sigma_{n,2d}^*$  that have moment matrices of rank at most  $3d - 3$  are point evaluations. Therefore, if  $\text{rank } M_\ell \leq 3d - 3$ , then  $M_\ell$  is a sum of point evaluations. Now suppose that  $\ell = \sum \ell_{v_i}$  and  $M_\ell = \sum M_{v_i}$ , where  $\ell_{v_i}$  is point evaluation on  $v_i \in \mathbb{R}^n$  and  $M_{v_i}$  is its moment matrix. It follows that  $\ker M_\ell \subset \ker M_{v_i}$  for all  $i$ . Therefore we can find  $\lambda_1 > 0$  such that  $M_1 = M_\ell - \lambda_1 M_{v_1}$  is positive semidefinite and  $\text{rank } M_1 = \text{rank } M_\ell - 1$ . Then  $M_1$  is again a sum of point evaluations and we can keep reducing the rank of  $M_i$  until we obtain a decomposition of  $M_\ell$  as a sum of exactly  $\text{rank } M_\ell$  point evaluations.  $\square$

*Proof of Theorem 2.4.* Theorem 2.4 is essentially a restatement of Theorem 2.3. The only claim that is left to be shown is that the bounds are tight: there exist forms  $f \in \mathbb{R}[x]_{n,2d}$  with positive semidefinite quadratic form  $Q_f$  of rank  $3d - 2$  for  $d \geq 3$ ,  $n \geq 3$  and rank 6 for  $d = 2$ ,  $n \geq 4$ , such that the real Waring rank of  $f$  is strictly greater than  $\text{rank } Q_f$ .

We take  $f \in \mathbb{R}[x]_{n,2d}$  such that  $\partial f$  is an extreme ray of  $\Sigma_{n,2d}^*$  with  $\text{rank } Q_f = 3d - 2$  for  $d \geq 3$  or rank 6 for  $d = 2$ , which we know exist by Corollary 2.2. Since  $Q_f$  is positive semidefinite, we know that any decomposition of  $\partial f$  as a sum point evaluation must include at least  $\text{rank } Q_f$  point evaluations with positive signs. But  $\partial f$  is not a linear combination of point evaluations with positive coefficients, since  $\partial f$  is an extreme ray of  $\Sigma_{n,2d}^*$  and therefore any decomposition of  $f$  as  $2d$ -th powers of linear forms must include negative signs and we have more than  $\text{rank } Q_f$  powers.  $\square$

## 5. POSITIVE FORMS CONTAINED IN POSITIVE GORENSTEIN IDEALS

We now begin the investigation of positive forms contained in positive Gorenstein ideals. Let  $\ell \in \mathbb{R}[x]_{n,2d}^*$  be the socle of a positive Gorenstein ideal  $I(\ell)$ . Suppose that  $I(\ell)$  contains a strictly positive form  $p$  of degree  $2k$  with  $k \leq d$ . Let  $C_p$  be the cone of linear functionals  $m$  in  $\Sigma_{n,2d}^*$  such that  $p$  is contained in  $I(m)$ :

$$C_p = \{m \in \Sigma_{n,2d}^* \mid p \in I(m)\}.$$

The condition  $p \in I(m)$  is equivalent to  $m(pq) = 0$  for all  $q \in \mathbb{R}[x]_{n,2d-2k}$ . Therefore the cone  $C_p$  is the section of  $\Sigma_{n,2d}^*$  with the subspace  $L_p$  of  $\mathbb{R}[x]_{n,2d}^*$  consisting of linear functionals vanishing on  $p \cdot \mathbb{R}[x]_{n,2d-2k}$ :

$$C_p = \Sigma_{n,2d}^* \cap L_p.$$

Thus  $C_p$  is a closed convex cone and it contains a nontrivial linear functional  $\ell$ . We consider extreme rays of  $C_p$ .

Recall that to a linear functional  $\ell \in \mathbb{R}[x]_{n,2d}^*$  we can associate a quadratic form  $Q_\ell$  on  $\mathbb{R}[x]_{n,2d}$  given by

$$Q_\ell(g) = \ell(g^2) \quad \text{for } g \in \mathbb{R}[x]_{n,d}.$$

It follows that a linear functional  $\ell$  is in  $\Sigma_{n,2d}^*$  if and only if the form  $Q_\ell$  is positive semidefinite.

**Lemma 5.1.** *Let  $\ell$  be an extreme ray of  $C_p$  and let  $W \subset \mathbb{R}[x]_{n,d}$  be the kernel of  $Q_\ell$ . Then  $p$  and  $W$  have no common zeroes (real or complex), i.e.  $\mathcal{V}_{\mathbb{C}}(p, W) = \emptyset$ .*

*Proof.* Let  $\ell$  and  $W \subset \mathbb{R}[x]_{n,d}$  be as above. Since  $p$  is strictly positive it follows that  $p$  and forms in  $W$  have no common real zeroes. Suppose that there is a common complex zero  $z$ . We recall that  $C_p$  is a section of  $\Sigma_{n,2d}^*$  with the subspace  $L_p$ . It follows that the kernel  $W$  of  $Q_\ell$  is strictly maximal among all the linear functionals in  $L_p$ , ([1], Lemma 2.2). In other words, if  $s \in L_p$  and  $W \subseteq \ker Q_s$ , then  $\ell = \lambda s$  for some  $\lambda \in \mathbb{R}$ .

Now let  $s \in \mathbb{R}[x]_{n,2d}^*$  be given by  $s(f) = \operatorname{Re} f(z)$ , the real part of  $f(z)$ . Then  $s \in L_p$  and  $W \subseteq \ker Q_s$ . Therefore  $\ell(f) = \lambda \operatorname{Re} f(z)$  for some  $\lambda \in \mathbb{R}$ . Thus  $Q_\ell(g) = \lambda \operatorname{Re} g^2(z)$  for  $g \in \mathbb{R}[x]_{n,d}$ . It follows that  $Q_\ell$  is not positive semidefinite, and we arrive at a contradiction.  $\square$

The following is a generalization of Lemma 2.8 of [1].

**Lemma 5.2.** *Suppose that  $p \in \mathbb{R}[x]_{n,k}$  and  $q_1, \dots, q_{n-1} \in \mathbb{R}[x]_{n,d}$  together form a sequence of parameters and  $n \geq 3$ . Then there exist  $f_1, \dots, f_{n-2}$  in the real linear span of  $q_i$  such that the forms  $p, f_1, \dots, f_{n-2}$  intersect transversely in  $kd^{n-2}$  (possibly complex) points.*

*Proof.* Let  $W$  be the linear span of  $q_1, \dots, q_{n-1}$  with complex coefficients. We begin by showing that there exist linear combinations  $f_1, \dots, f_{n-2} \in W$  such that  $p, f_1, \dots, f_{n-2}$  intersect transversely in  $\mathbb{C}\mathbb{P}^{n-1}$ .

Let  $\mathcal{V}_0$  be the projective variety defined by  $p$ . Then  $W$  defines a linear system of divisors on  $\mathcal{V}_0$ . By Bertini's Theorem a general element of  $W$  intersects  $\mathcal{V}_0$  in a smooth variety of dimension  $n-3$ . Let  $f_1$  be such a form in  $W$ , let  $\mathcal{V}_1$  be the smooth variety defined by  $p$  and  $f_1$ , and let  $W_1$  be a subspace of  $W$  complementary to  $f_1$ . Then  $W_1$  defines a linear system of divisors on  $\mathcal{V}_1$ , and by Bertini's Theorem the intersection of  $\mathcal{V}_1$  with a general element of  $W_1$  is a smooth variety of dimension

$n - 4$ . Let  $f_2$  be such an element of  $W_1$ . Now we can let  $\mathcal{V}_2$  be the smooth variety defined by  $p$ ,  $f_1$  and  $f_2$ , let  $W_2$  be the complementary subspace to  $f_1$  and  $f_2$  and repeatedly apply Bertini's Theorem until we get a 0-dimensional smooth intersection. Hence the forms  $p, f_1, \dots, f_{n-2}$  we constructed intersect transversely.

Now we argue that there exist *real* linear combinations  $f_1, \dots, f_{n-2}$  which intersect transversely with  $p$ . Suppose not and let  $f_i = \sum_{j=1}^n \alpha_{ij} q_j$ . Then for all  $\alpha_{ij} \in \mathbb{R}$  the forms  $p$  and  $f_i$  do not intersect transversely. This is an algebraic condition on the coefficients  $\alpha_{ij}$ , given by vanishing of some polynomials in the variables  $\alpha_{ij}$ . However, if a polynomial vanishes on all real points, then it must be identically zero. Therefore, no complex linear combinations of  $q_i$  intersect transversely, which is a contradiction.  $\square$

We recall the Lemma 4.2 from [1], which allows us to limit the number of complex point evaluations that define the socle of a positive Gorenstein ideal. Let  $S$  be a finite set of points in  $\mathbb{C}^n$  that is invariant under conjugation:  $\bar{S} = S$ . Let  $S$  be given by  $S = \{r_1, \dots, r_k, z_1, \dots, z_m, \bar{z}_1, \dots, \bar{z}_m\}$  with  $r_i \in \mathbb{R}^n$  and  $z_i, \bar{z}_i \in \mathbb{C}^n$ ,  $z_i \neq \bar{z}_i$ . Let  $\ell : \mathbb{R}[x]_{n,2d} \rightarrow \mathbb{R}$  be a linear functional given as a combination of evaluations on  $S$ :

$$\ell(p) = \sum_{i=1}^k \lambda_i p(r_i) + \sum_{i=1}^m (\mu_i p(z_i) + \bar{\mu}_i p(\bar{z}_i)), \quad p \in \mathbb{R}[x]_{n,2d}$$

with  $\lambda_i \in \mathbb{R}$ ,  $\mu_i \in \mathbb{C}$  and  $\lambda_i, \mu_i \neq 0$ .

**Lemma 5.3** ([1, Lemma 4.2]). *Suppose that  $Q_\ell$  is positive semidefinite. Then the number of complex conjugate pairs in  $S$  is at most equal to  $\text{Dep}_d(S)$ .*

## 6. PROOFS OF POSITIVE FORMS THEOREMS

With the preparatory work in the previous section we are in position to prove Theorem 2.7.

*Proof of Theorem 2.7.* The case  $d = 1$  is straightforward since  $4d = 2d + 2 = 4$  and  $\Sigma_{3,4} = P_{3,4}$ , and therefore any linear functional  $\ell \in \Sigma_{3,4}^*$  cannot vanish on a strictly positive form of degree 4. We now consider  $d \geq 2$ .

Suppose not and let  $p$  be a strictly positive polynomial of degree  $2d + 2$  contained in  $I$ . Then by Lemma 5.1 we may assume that  $p$  and  $I_{2d}$  have no common zeroes. Therefore, by Lemma 5.2 we can find a form  $q \in I_{2d}$  such that  $p$  and  $q$  intersect transversely in  $(2d + 2)2d$  points. Since  $p$  is strictly positive on  $\mathbb{R}P^2$  we know that the intersection of  $p$  and  $q$  consists entirely of complex points.

Let  $k = d(2d + 2)$  and let  $\Gamma = \{\gamma_1, \dots, \gamma_{2k}\} \subset \mathbb{C}P^2$  be the intersection of  $p$  and  $q$ . Let  $z_i$  be affine representatives of  $\gamma_i$ . Since points  $\gamma_i$  are strictly complex we can choose  $z_i$  in conjugate pairs. As before we can express the linear functional  $\ell$  as a linear combination of point evaluations on  $z_i$ :

$$\ell(f) = \sum_{i=1}^k (\mu_i f(z_i) + \bar{\mu}_i f(\bar{z}_i)) \quad \text{for all } f \in \mathbb{R}[x]_{3,4d}.$$

The coefficients  $\mu_i$  can be chosen in conjugate pairs since  $\ell$  is a real functional. Let  $\Gamma'$  be the subset of  $\Gamma$  corresponding to nonzero coefficients  $\mu_i$ . We know by Lemma 5.3 that the  $\text{Dep}_{2d}(\Gamma')$  is at least  $\frac{1}{2}|\Gamma'|$ . It follows that  $\text{Dep}_{2d}(\Gamma') \geq 1$  and we can apply Theorem 3.1 to  $\Gamma'$ . Since  $\Gamma$  is a transverse intersection of a form of degree  $2d$  and a form of degree  $2d + 2$  it follows that there is a unique form of

degree  $2d$  vanishing on  $\Gamma$ , which leads to  $\text{Ind}_{2d}(\Gamma) = \binom{2d+2}{2} - 1$  and  $\text{Dep}_{2d}(\Gamma) = |\Gamma| - \text{Ind}_{2d}(\Gamma) = (2d)(2d+2) - \binom{2d+2}{2} + 1$ . Thus we see that

$$\text{Ind}_{2d}(\Gamma') - \text{Dep}_{2d}(\Gamma') \geq \text{Ind}_{2d}(\Gamma) - \text{Dep}_{2d}(\Gamma) = 2d.$$

Since  $\text{Ind}_{2d}(\Gamma') + \text{Dep}_{2d}(\Gamma') = |\Gamma'|$  and  $\text{Dep}_{2d}(\Gamma') \geq \frac{1}{2}|\Gamma'|$  we reach a contradiction.  $\square$

We can quickly deduce Theorem 2.6 from Theorem 2.7.

*Proof of Theorem 2.6.* Suppose not and for some  $r \in P_{3,2d}$  we have  $rq \notin \Sigma_{3,4d-4}$  for all  $q \in P_{3,2d-4}$ . Let  $T$  be the set of nonnegative forms  $f$  of degree  $2d$  such that  $fg$  is a sum of squares for some nonnegative form  $g$  of degree  $2d-4$ :

$$T = \{f \in P_{3,2d} \mid fg \in \Sigma_{3,4d-4} \text{ for some } g \in P_{3,2d-4}\}.$$

It is easy to see that the set  $T$  is closed. Since  $T \subset P_{3,2d}$  there must exist a strictly positive form  $p \notin T$ .

We have  $pq \notin \Sigma_{3,4d-4}$  for all  $q \in P_{3,2d-4}$  and therefore  $pq \notin \Sigma_{3,4d-4}$  for all nonzero  $q \in \mathbb{R}[x]_{3,2d-4}$ . In other words, the linear subspace  $p \cdot \mathbb{R}[x]_{3,2d-4}$  of  $\mathbb{R}[x]_{3,4d-4}$  intersects the convex cone  $\Sigma_{3,4d-4}$  only at the origin. It follows that there exists a linear functional  $\ell \in \Sigma_{3,4d-4}^*$  such that  $\ell$  is zero on  $p \cdot \mathbb{R}[x]_{3,2d-4}$ . In other words there exists a positive Gorenstein ideal with socle  $\ell$  of degree  $4d-4$  which contains a strictly positive form  $p$  of degree  $2d$ . This is a contradiction by Theorem 2.7.  $\square$

**6.1. Positive forms on the boundary of  $\Sigma_{n,2d}$ .** Corollary 2.9 follows immediately from Theorem 2.7: if  $p$  is a strictly positive form on the boundary of  $\Sigma_{3,4d}$ , then  $p$  cannot be a square of a form in  $\mathbb{R}[x]_{3,d}$ . In fact we can do better.

**Corollary 6.1.** *Let  $p$  be a strictly positive form on the boundary of  $\Sigma_{3,2d}$ . Then  $p$  is a sum of at most  $\binom{d+2}{2} - 3d + 2$  squares and  $p$  cannot be written as a sum of fewer than 3 squares.*

*Proof.* Let  $p$  be a strictly positive form on the boundary of  $\Sigma_{3,2d}$  with  $d \geq 3$ . Then there exists an extreme ray  $\ell$  of the dual cone  $\Sigma_{3,2d}^*$ , such that  $\ell(p) = 0$ , defining a maximal positive Gorenstein ideal  $I(\ell)$ . Now suppose that  $p = \sum f_i^2$  for some  $f_i \in \mathbb{R}[x]_{3,d}$ . It follows that  $Q_\ell(f_i) = 0$  for all  $i$ , and since  $Q_\ell$  is a positive semidefinite quadratic form we see that all  $f_i$  lie in the kernel  $I_d(\ell)$  of  $Q_\ell$ . By Theorem 2.1 we know that  $\dim I_d(\ell) \leq \binom{d+2}{2} - 3d + 2$  and the upper bound follows.

Now suppose that  $p$  is a sum of 2 squares,  $p = f_1^2 + f_2^2$ . Since  $p$  is strictly positive we know that the forms  $f_i$  have no common real zeroes. Therefore we found two forms  $f_i \in I_d(\ell)$  that have no common real zeroes. This is not possible when  $d$  is odd and so we reach a contradiction.

For even  $d$  from the proof of Lemma 5.2 we know that 2 generic forms in  $I_d(\ell)$  intersect transversely, and hence we can perturb the forms  $f_i$  and find forms  $f'_i \in I_d(\ell)$  in a neighborhood of the respective  $f_i$  such that  $f'_1$  and  $f'_2$  intersect transversely in a 0-dimensional variety  $\Gamma$  consisting of  $d^2$  complex points. Then the linear functional  $\ell$  can be defined as a linear combination of evaluations on points of  $\Gamma$ . Let  $\Gamma'$  be the subvariety of  $\Gamma$  consisting of points whose coefficients in  $\ell$  are nonzero. Since  $\ell$  is a real functional, by adding the conjugate  $\bar{\ell}$  if necessary, we may assume that  $\Gamma'$  is conjugation-invariant, and since  $\ell \in \Sigma_{3,2d}^*$  we must have  $\text{Dep}_d(\Gamma') \geq 1$ .

Then by Theorem 3.1 we have  $\text{Ind}_d(\Gamma') > \text{Dep}_d(\Gamma')$  and therefore  $\text{Dep}_d(\Gamma') < \frac{1}{2}|\Gamma'|$ . This is a contradiction by Lemma 5.3.  $\square$

We observe that for  $d = 3$  the upper bound is equal to the lower bound and therefore the bounds are tight. This was discussed in [1] and used in [2] to study the algebraic boundary of  $\Sigma_{3,6}$ . The lower bound is always tight; i.e. we can always find a positive form on the boundary of  $\Sigma_{3,2d}$  which is a sum of 3 squares. In fact this happens for any maximal positive Gorenstein ideal  $I$  whose socle is not a point evaluation. We know that the forms in  $I_d$  have no common zeroes by Theorem 2.1. Therefore we can find three forms  $f_1, f_2, f_3 \in I_d$  with no common zeroes and  $f_1^2 + f_2^2 + f_3^2$  will be strictly positive and on the boundary of  $\Sigma_{3,2d}$ .

We remark that the upper bound of  $\binom{d+2}{2} - 3d + 2$  is also sharp for  $d = 4$ , where it is equal to 5. This can be seen by considering the maximal positive Gorenstein ideals constructed in the proof of Theorem 2.1, which come from a complete intersection of a cubic and two quartics. However we expect that the bound is not optimal for higher  $d$ .

**Connection to semidefinite programming.** Let  $s = \dim \mathbb{R}[x]_{n,d}$ , let  $\mathcal{B} = \{f_1, \dots, f_s\}$  be a basis of  $\mathbb{R}[x]_{n,2d}$  and let  $b = (f_1, \dots, f_s)$ . For  $p \in \mathbb{R}[x]$  let  $c(p) = (c_1, \dots, c_s)$  be the vector of coefficients expressing  $p$  as a linear combination of  $f_i$ :  $p = \sum_{i=1}^s c_i f_i$ . The following is a well known connection between sums of squares and positive semidefinite matrices:

**Lemma 6.2.** *Let  $p \in \mathbb{R}[x]_{n,2d}$ . The form  $p$  is a sum of squares if and only if there exists a positive semidefinite matrix  $M$  such that  $p = b^T M b$ . Furthermore  $p$  is in the interior of  $\Sigma_{n,2d}$  if and only if there exists a positive definite  $M$  with the above property.*

*Proof.* The key observation is that if  $M$  is a rank one psd matrix, then  $M = c(p)c(p)^T$  for some vectors  $c(p)$  and therefore  $b^T M b = p^2$ . Thus if  $p = \sum q_i^2$  then we can take  $M = \sum c(q_i)c(q_i)^T$ . On the other hand, any psd matrix can be written as a sum of rank 1 psd matrices, and therefore if  $M$  is psd, then  $p = b^T M b$  is a sum of squares. This shows that  $\Sigma_{n,2d}$  is a linear projection of the cone of  $s \times s$  psd matrices. We now observe the interior of the cone of  $s \times s$  psd matrices consists of positive definite matrices. A linear projection cannot send an interior point to the boundary of the image and therefore the second implication follows.  $\square$

## 7. MULTIPLIER DISCUSSION

The following is a generalization of the ideas of Hilbert's original method of showing that there exist nonnegative forms that are not sums of squares. We show that values on complete intersections can also be used to certify that a nonnegative form has no sum of squares multipliers  $q$  of some degree, such that  $pq$  is a sum of squares.

Let  $S = \{v_1, \dots, v_m\}$  be a finite subset of  $\mathbb{R}^n$ . Let  $E_S$  be the evaluation map that sends  $p \in \mathbb{R}[x]$  to its values on the points  $v_i$ :

$$E_S : \mathbb{R}[x] \longrightarrow \mathbb{R}^m, \quad E_S(p) = (p(v_1), \dots, p(v_m)).$$

Let  $P_{2d}(S)$  and  $\Sigma_{2d}(S)$  be the images of  $P_{n,2d}$  and  $\Sigma_{n,2d}$  under  $E_S$  respectively, and let  $H_d(S)$  be the image of  $\mathbb{R}[x]_{n,d}$ . We first show that the cone  $\Sigma_{2d}(S)$  is always closed.

**Lemma 7.1.**  $\Sigma_{2d}(S)$  is a closed convex cone for all  $d$  and all finite subsets  $S$  of  $\mathbb{R}^n$ .

*Proof.* Consider  $H_d(S)$ , the image of  $\mathbb{R}[x]_{n,d}$  under the evaluation projection  $E(S)$ . Let  $\mathbb{S}_d$  be the unit sphere in  $H_d(S)$  with respect to the standard inner product on  $\mathbb{R}^m$ . Let  $C$  be the points in  $\mathbb{R}^m$  that are pointwise squares of points in  $\mathbb{S}_d$ :

$$C := \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid (x_1, \dots, x_m) = (w_1^2, \dots, w_m^2) \text{ for some } (w_1, \dots, w_m) \in \mathbb{S}_d\}.$$

We see that  $C$  is a compact subset of  $\mathbb{R}^m$ . Let  $K$  be the convex hull of  $C$ . It follows that  $K$  is a compact convex set and  $0 \notin K$ . We now observe that  $\Sigma_{2d}(S)$  is the conical hull of  $K$ . Therefore  $\Sigma_{2d}(S)$  is a closed convex cone.  $\square$

We now restrict ourselves to finite subsets of  $\mathbb{R}^n$  coming as affine representatives of a complete intersection. Let  $\Gamma = \{\gamma_1, \dots, \gamma_m\}$  be a fully real transverse intersection of  $n - 1$  forms from  $\mathbb{R}[x]_{n,d}$  in  $m = d^{n-1}$  points in  $\mathbb{R}P^{n-1}$ . Let  $S = \{v_1, \dots, v_m\} \subset \mathbb{R}^n$  be a set of affine representatives for  $\gamma_i$ . The following theorem is a special case of Theorem 3.1 in [1].

**Theorem 7.2.** Let  $\mathbb{R}_{++}^m$  be the positive orthant of  $\mathbb{R}^m$ . The intersection of  $H_{2d}(S)$  with the positive orthant is contained in  $P_{2d}(S)$ :

$$H_{2d}(S) \cap \mathbb{R}_{++}^m \subset P_{2d}(S).$$

We now show how to use values of forms on a complete intersection to show existence of nonnegative forms that have no sum of squares multipliers of certain degree.

**Theorem 7.3.** Let  $\Gamma$  be a generic completely real transverse intersection of  $n - 1$  forms from  $\mathbb{R}[x]_{n,d}$  in  $d^{n-1}$  points in  $\mathbb{R}P^{n-1}$ . The genericity assumption on  $\Gamma$  is that for all  $t \leq 2d$ , any collection of at most  $\text{Ind}_t(\Gamma)$  points of  $\Gamma$  imposes linearly independent conditions on forms in  $\mathbb{R}[x]_{n,t}$ . Suppose that for some  $k < d$  we have

$$\text{Ind}_{2d}(\Gamma) \geq \text{Ind}_{d+k}(\Gamma) + \text{Ind}_k(\Gamma).$$

Then there exists a nonnegative form  $p \in P_{n,2d}$  such that  $pq$  is not a sum of squares for any sum of squares  $q \in \Sigma_{n,2k}$ .

*Proof.* Suppose not and for every nonnegative form  $p \in P_{n,2d}$  there exists  $q \in \Sigma_{n,2k}$  such that  $pq$  is a sum of squares. Let  $S \subset \mathbb{R}^n$  be the set of affine representatives of  $\Gamma$ . From Lemma 7.1 we know that  $\Sigma_{2k}(S)$  and  $\Sigma_{2d+2k}(S)$  are closed convex cones.

For  $x, y \in \mathbb{R}^m$  let  $x \cdot y$  denote pointwise multiplication of  $x$  and  $y$ :  $x \cdot y = (x_1y_1, \dots, x_my_m)$ . Let  $\mathbb{R}_+^m$  be the nonnegative orthant of  $\mathbb{R}^m$ . We claim that for any  $x \in H_{2d}(S) \cap \mathbb{R}_+^m$  there exists  $y \in \Sigma_{2k}(S)$  such that  $x \cdot y \in \Sigma_{2d+2k}(S)$ . By our assumption, and Theorem 7.2 we know that this holds for all  $x \in H_{2d}(S) \cap \mathbb{R}_{++}^m$ .

Let  $x \in H_{2d}(S) \cap \mathbb{R}_+^m$  and let  $x_1, x_2, \dots$  be a sequence of points in  $H_{2d}(S) \cap \mathbb{R}_{++}^m$  approaching  $x$ . Let  $y_1, y_2, \dots$  be a sequence of multipliers in  $\Sigma_{2k}(S)$  such that  $x_i \cdot y_i \in \Sigma_{2d+2k}(S)$ . Since the condition of belonging to  $\Sigma_{2d+2k}(S)$  is conical, it follows that we may choose  $y_i$  lying on the unit sphere in  $\mathbb{R}^m$ . Let  $y$  be an accumulation point of  $y_i$ . Since the cone  $\Sigma_{2k}(S)$  is closed, it follows that  $y \in \Sigma_{2k}(S)$ , and since  $\Sigma_{2d+2k}(S)$  is closed we see that  $x \cdot y \in \Sigma_{2d+2k}(S)$ .

We now examine  $T = H_{2d}(S) \cap \mathbb{R}_+^m$ .  $T$  is a polyhedral cone and we have  $\dim H_{2d}(S) = \text{Ind}_{2d}(\Gamma)$ . Let  $F$  be a face of  $T$  of codimension  $\text{Ind}_{d+k}(\Gamma)$ . Then  $\text{Ind}_{d+k}(\Gamma)$  linearly independent defining inequalities of  $T$  are tight on  $F$ . Therefore for any  $u$  in the relative interior of  $F$ , the coordinates of  $u$  are zero in at least  $\text{Ind}_{d+k}(\Gamma)$  entries corresponding to the tight inequalities on  $F$ , and are nonzero in at least  $\text{Ind}_{2d}(\Gamma) - \text{Ind}_{d+k}(\Gamma)$  entries corresponding to the points where evaluation is linearly independent from the zeroes of  $u$ . By relabeling, if necessary, let  $B = \{v_1, \dots, v_{\text{Ind}_{d+k}}\}$  be the collection points corresponding to the tight linearly independent inequalities of  $u$ .

We know from above that there exists  $y \in \Sigma_{2k}(S)$  such that  $u \cdot y \in \Sigma_{2d+2k}(S)$ . By genericity of  $\Gamma$  we know that any  $f \in \mathbb{R}[x]_{n,d+k}$  that vanishes on  $B$  must vanish on all of  $\Gamma$ . Therefore we see that  $u \cdot y = 0$ . Therefore  $y$  comes from a form  $r \in \Sigma_{n,2k}$  that vanishes on  $\text{Ind}_{2d}(\Gamma) - \text{Ind}_{d+k}(\Gamma)$  points in  $\Gamma$ . Let  $r = \sum r_i^2$ . Then each  $r_i \in \mathbb{R}[x]_{n,k}$  must vanish on  $\Gamma$ . Since we have  $k < d$ , it follows that all  $r_i$  are identically zero, which is a contradiction.  $\square$

We now use Theorem 7.3 to establish Corollary 2.8.

*Proof of Corollary 2.8.* First let  $\Gamma$  be a fully real generic intersection of two curves  $f, g \in \mathbb{R}[x]_{3,2k+3}$ . Then, since no form of degree  $k$  vanishes on  $\Gamma$  we have

$$\text{Ind}_k(\Gamma) = \dim \mathbb{R}[x]_{3,k} = \binom{k+2}{2},$$

and since the ideal generated by  $f$  and  $g$  is radical and  $f$  and  $g$  have no syzygies until degree  $4k+6$  we have

$$\text{Ind}_{3k+3}(\Gamma) = \dim \mathbb{R}[x]_{3,3k+3} - 2 \dim \mathbb{R}[x]_{3,k} = \binom{3k+5}{2} - 2 \binom{k+2}{2}.$$

Thus  $\text{Ind}_k(\Gamma) + \text{Ind}_{3k+3}(\Gamma) = 4k^2 + 12k + 9 = (2k+3)^2$ . Also,

$$\begin{aligned} \text{Ind}_{4k+6}(\Gamma) &= \dim \mathbb{R}[x]_{3,4k+6} - 2 \dim \mathbb{R}[x]_{3,2k+3} + 1 = \binom{4k+8}{2} - 2 \binom{2k+5}{2} + 1 \\ &= (2k+3)^2. \end{aligned}$$

Therefore we can apply Theorem 7.3 with  $d = 2k+3$ .

Next let  $\Gamma$  be a fully real generic intersection of three cubics from  $\mathbb{R}[x]_{4,3}$ . Then  $\text{Ind}_1(\Gamma) = 4$  and  $\text{Ind}_4(\Gamma) = \dim \mathbb{R}[x]_{4,4} - 3 \cdot 4 = 23$ . Also

$$\text{Ind}_6(\Gamma) = \dim \mathbb{R}[x]_{4,6} - 3 \dim \mathbb{R}[x]_{4,3} + \binom{3}{2} = 27,$$

since the three cubics have exactly  $\binom{3}{2}$  (Koszul) syzygies in degree 6. Thus we can apply Theorem 7.3 with  $d = 3$  and  $k = 1$ .

Finally let  $\Gamma$  be a fully real generic intersection of six quadrics from  $\mathbb{R}[x]_{7,2}$ . Then  $\text{Ind}_1(\Gamma) = 7$  and  $\text{Ind}_3(\Gamma) = \dim \mathbb{R}[x]_{7,3} - 6 \cdot 7 = 42$ . Also  $\text{Ind}_4(\Gamma) = \dim \mathbb{R}[x]_{7,4} - 6 \dim \mathbb{R}[x]_{7,2} + \binom{6}{2} = 57$ . Thus we can apply Theorem 7.3 with  $d = 2$  and  $k = 1$ .  $\square$

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