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REPRESENTATIONS WHOSE MINIMAL REDUCTION HAS A TORIC IDENTITY COMPONENT

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Abstract. We classify irreducible representations of connected compact Lie groups whose orbit space is isometric to the orbit space of a representation of a finite extension of a (positive-dimensional) toric group. They turn out to be exactly the non-polar irreducible representations preserving an isoparametric submanifold and acting with cohomogeneity one on it.

1. Introduction

A representation $\rho : G \to \mathbf{O}(V)$ of a compact Lie group on a Euclidean space V is called *polar* if there exists a subspace Σ, called a *section*, that meets all G-orbits and always meets them orthogonally [PT87, BCO03]. In this case, the stabilizer $N(\Sigma)$ acts on Σ with kernel $Z(\Sigma)$, the quotient group $\mathcal{W} = N(\Sigma)/Z(\Sigma)$ is finite, and the inclusion $\Sigma \to V$ induces an isometry between orbit spaces $\Sigma/\mathcal{W} = V/G$ with the quotient metrics. Conversely, assume $\rho : G \to \mathbf{O}(V)$ is a representation whose orbit space $X = V/G$ can also be given as the orbit space of a representation of a finite group. On one hand, the set of regular points X_{reg} is a flat Riemannian manifold. On the other hand, due to O'Neill's formula, the horizontal distribution of the Riemannian submersion $V_{reg} \to X_{reg}$ is integrable. It follows that ρ is polar [Ale06, HLO06]. Thus we have the following characterization: *a representation is polar if and only if its orbit space is isometric to the orbit space of a representation of a finite group.*

This paper follows the program introduced in [GL], namely, to hierarchize the representations of compact Lie groups in terms of the complexity of their orbit spaces, viewed as metric spaces. Dadok [Dad85] has classified polar representations of connected groups and showed that they are orbit-equivalent to the isotropy representations of symmetric spaces. Herein we enlarge the class of polar representations by replacing "finite group" by "finite extension of a toric group" in the above characterization. Namely, we consider a class of representations defined by the condition that the orbit space is isometric to the orbit space of a representation of a finite extension of a toric group, and we classify them under the assumptions that they are irreducible and the group is connected.

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It turns out that such representations have a remarkable connection with isoparametric submanifolds and polar representations. Rather surprisingly, only few families of non-polar representations can occur. Our result can be stated as follows.

Theorem 1.1. *Let $\rho : G \to \mathbf{O}(V)$ be an effective irreducible representation of a connected compact Lie group G on a Euclidean space V. Assume that ρ is not polar. Then the following conditions are equivalent:*

(a) *There is a representation $\tau : H \to \mathbf{O}(W)$ of a compact Lie group with a toric identity component such that the orbit spaces V/G, W/H are isometric.*
(b) *There is a connected subgroup $\hat{G}$ of $\mathbf{O}(V)$ containing $\rho(G)$ which acts polarly on V with cohomogeneity one less than the cohomogeneity of ρ.*
(c) *The action of G leaves an isoparametric submanifold S of V invariant and acts with cohomogeneity one on S.*

Moreover, the representations satisfying any of the above conditions consist of three disjoint families as follows:

(i) *ρ is one of the non-polar irreducible representations of cohomogeneity three:*

G	ρ	*Conditions*
$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	$\mathbf{R}^2 \otimes_{\mathbf{R}} \mathbf{R}^{16}$	–
$\mathbf{U}(2) \times \mathbf{Sp}(n)$	$\mathbf{C}^2 \otimes_{\mathbf{C}} \mathbf{C}^{2n}$	$n \geq 2$
$\mathbf{SU}(2) \times \mathbf{Sp}(n)$	$S^3(\mathbf{C}^2) \otimes_{\mathbf{H}} \mathbf{C}^{2n}$	$n \geq 2$

(ii) *The group G is the semisimple factor of an irreducible polar representation of Hermitian type such that the action of G is not orbit-equivalent to the polar representation:*

G	ρ	*Conditions*
$\mathbf{SU}(n)$	$S^2\mathbf{C}^n$	$n \geq 3$
$\mathbf{SU}(n)$	$\Lambda^2\mathbf{C}^n$	$n = 2p \geq 6$
$\mathbf{SU}(n) \times \mathbf{SU}(n)$	$\mathbf{C}^n \otimes_{\mathbf{C}} \mathbf{C}^n$	$n \geq 3$
$\mathbf{E}_6$	$\mathbf{C}^{27}$	–

(iii) *ρ is one of the two exceptions: $\mathbf{SO}(3) \otimes \mathbf{G}_2$, $\mathbf{SO}(4) \otimes \mathbf{Spin}(7)$.*

A further motivation for Theorem 1.1 comes from one of the main results of [GL], which gives another charaterization of the above class. Namely, if $\rho : G \to \mathbf{O}(V)$, $\rho' : G' \to \mathbf{O}(V')$ are two non-polar representations of compact Lie groups with isometric orbit spaces $V/G = V'/G'$ such that the restriction of ρ to the identity component G° is irreducible but that of ρ' to $(G')^\circ$ is reducible, then the restriction of ρ to G° satisfies condition (a) for some representation $\tau : H \to \mathbf{O}(W)$.

2. Non-classifying arguments

Since principal orbits of polar representations are isoparametric submanifolds of Euclidean space [PT87], (b) implies (c). Conversely, assume (c) holds. Then S is full and irreducible, and thus is either homogeneous or has codimension two in V, due to the theorem of Thorbergsson [Tho91]. In the former case, the maximal connected subgroup of $\mathbf{O}(V)$ preserving S acts polarly on V and has S as a principal orbit [PT87]. In the latter case, the cohomogeneity of G on V is 3. Such actions are listed in (i) above, and we can directly check that (b) holds in each case by finding a larger connected group acting as the isotropy representation of a symmetric space of rank two, namely, $(\mathbf{SO}(2)\times\mathbf{SO}(16),\mathbf{R}^2\otimes_{\mathbf{R}}\mathbf{R}^{16})$, $(\mathbf{U}(2)\times\mathbf{SU}(2n),\mathbf{C}^2\otimes_{\mathbf{C}}\mathbf{C}^{2n})$, $(\mathbf{Sp}(2)\times\mathbf{Sp}(n),\mathbf{C}^4\otimes_{\mathbf{H}}\mathbf{C}^{2n})$, respectively. Hence (b) and (c) are equivalent.

Next we prove that (a) implies (b). If τ is as in (a), then the induced representation of H° is reducible, since H° is a torus. Using [GL, Theorem 1.7], we deduce that the action of H° on W can be identified with that of the maximal torus T^k of $\mathbf{SU}(k+1)$ on $\mathbf{C}^{k+1}$, for some $k\geq 1$ (possibly after replacing H and τ). Of course, H is contained in the normalizer N of H° in $\mathbf{O}(W)$. Moreover, the connected component N° is the maximal torus (of rank $k+1$) of the unitary group of W, and it acts polarly on W.

Let $X=V/G=W/H$, let $Y=X/(N/H)=W/N$ and denote by π the composite map

$$V\to X\to Y.$$

This composite map π is a *submetry* (see [Lyt02, GW11]) whose fibers are smooth equidistant submanifolds of V. Denote by Y_{reg} the subset of principal N/H-orbits in X, denote by V_{reg} the set of G-regular points in V and put $V'=\pi^{-1}(Y_{reg})\cap V_{reg}$. Then V' is an open dense subset of V and $\pi: V'\to Y_{reg}$ is a Riemannian submersion. Since the action of N on W is polar, Y_{reg} is flat, so O'Neill's formula implies that the π-horizontal distribution in V' is integrable. Hence the regular fibers of π have flat normal bundles. Moreover, each regular fiber is equifocal (the focal points are given by the intersection of a horizontal geodesic with singular fibers). It follows that the connected components of the fibers of π in V yield an isoparametric foliation $\mathcal{F}$ by full irreducible submanifolds of V.

We have $\operatorname{codim}\mathcal{F}=\dim Y=\dim X-1=k+1$. In case $k=1$, the cohomogeneity of ρ is 3 and the result follows as above, so we may assume $k\geq 2$. By the theorem of Thorbergsson [Tho91], $\mathcal{F}$ is homogeneous, and the maximal connected subgroup $\hat{G}\subset\mathbf{O}(V)$ which preserves the foliations acts transitively on the leaves. By definition, $\hat{G}$ is closed, acts polarly and contains G. By construction, $\dim(V/\hat{G})=\dim(X)-1$. Hence we have (b).

It was remarked in [GL] that the representations in (i), (ii), and (iii) all satisfy (a). More precisely, representations of cohomogeneity three have copolarity 1 ([GL, Example 1.9]; see also [Str94]). The representations listed in (iii) have copolarity 2 and 3, respectively [GL, Theorem 1.11]. Thus, due to Theorem 1.5 in the same paper, the representations listed in (i) and (iii) satisfy (a). The fact that the representations coming from the Hermitian symmetric spaces and listed in (ii) satisfy (a) has been observed in [GL, Example 1.10]; see also [GOT04].

We shall finish the proof of Theorem 1.1 in the next section by proving that the representations satisfying (b) are listed in (i), (ii) and (iii) above.

3. The classification

Let $\hat{G}$ be as in (b). Consider the maximal connected subgroup K of $\mathbf{O}(V)$ with the same orbits as $\hat{G}$. It is known that the action of K on V is the isotropy representation of an irreducible symmetric space [BCO03, Prop. 4.3.9], say, of compact type. Let (L,K) and $M=L/K$ be the corresponding symmetric pair and symmetric space, respectively. The cohomogeneity $c(K,V)$ is just the rank $r=\operatorname{rk}(M)$ of the symmetric space M. We have $G\subset K$ and $c(G,V)=c(K,V)+1=r+1$. We run over the cases of irreducible symmetric spaces of compact type [Hel78, Wol84].

3.1. **Adjoint representations.** These are associated to type II symmetric spaces. For the adjoint representation of K on its Lie algebra $\mathfrak{k}$, we have that $\mathfrak{g}$, the Lie algebra of G, is a proper invariant subspace of $\mathfrak{k}$ for the G-action; hence the action of G on $\mathfrak{k}$ is not irreducible.

3.2. **List of symmetric spaces of type I.** One says that the symmetric space $M = L/K$ has maximal rank if $\mathrm{rk}(M) = \mathrm{rk}(L)$. Such spaces are marked with $*$ below. The space marked $**$ has maximal rank if and only if $|p - q| \leq 1$.

L/K	*dimension*	*rank*
$\mathbf{SU}(n)/\mathbf{SO}(n)*$	$\frac{1}{2}(n-1)(n+2)$	$n-1$
$\mathbf{SU}(2n)/\mathbf{Sp}(n)$	$(n-1)(2n+1)$	$n-1$
$\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$	$2pq$	$\min\{p,q\}$
$\mathbf{SO}(p+q)/(\mathbf{SO}(p)\times\mathbf{SO}(q)) **$	pq	$\min\{p,q\}$
$\mathbf{SO}(2n)/\mathbf{U}(n)$	$n(n-1)$	$[\frac{n}{2}]$
$\mathbf{Sp}(n)/\mathbf{U}(n)*$	$n(n+1)$	n
$\mathbf{Sp}(p+q)/(\mathbf{Sp}(p)\times\mathbf{Sp}(q))$	$4pq$	$\min\{p,q\}$
$\mathbf{E}_6/\mathbf{Sp}(4)*$	42	6
$\mathbf{E}_6/\mathbf{SU}(6)\mathbf{SU}(2)$	40	4
$\mathbf{E}_6/\mathbf{Spin}(10)\mathbf{U}(1)$	32	2
$\mathbf{E}_6/\mathbf{F}_4$	26	2
$\mathbf{E}_7/\mathbf{SU}(8)*$	70	7
$\mathbf{E}_7/\mathbf{Spin}(12)\mathbf{SU}(2)$	64	4
$\mathbf{E}_7/\mathbf{E}_6\mathbf{U}(1)$	54	3
$\mathbf{E}_8/\mathbf{Spin}(16)*$	128	8
$\mathbf{E}_8/\mathbf{E}_7\mathbf{SU}(2)$	112	4
$\mathbf{F}_4/\mathbf{Sp}(3)\mathbf{Sp}(1)*$	28	4
$\mathbf{F}_4/\mathbf{Spin}(9)$	16	1
$\mathbf{G}_2/\mathbf{SO}(4)*$	8	2

3.3. **Spaces of low rank.** Here we deal with the case $r \leq 4$. We rely on the classification of irreducible representations of connected groups of cohomogeneity at most five ([Str96] and [GL, Theorem 1.11]).

Recall that $c(G,V) = r+1$. If $r = 1$, then $c(G,V) = 2$ and ρ is polar.

If $r = 2$, then $c(G,V) = 3$ and ρ is listed in (i) [Str96].

If $r = 3$, then $c(G,V) = 4$ and ρ is given in [GL, Theorem 1.11, Table 1]. The only cases in the table not listed in (ii) or (iii) are the first two, for which $\dim M = \dim V \leq 8$ and $\mathrm{rk}(M) = 3$. But there are no symmetric spaces under these conditions.

Finally, if $r = 4$, then $c(G,V) = 5$ and ρ is given in [GL, Theorem 1.11, Table 2]. The only cases in the table not listed in (ii) or (iii) are the first two and the last. In those cases the dimension of V is 8, 12 or 24. Going through the list of symmetric spaces of type I and rank 4, we see that $\dim(V) = \dim(M)$ cannot be 8 or 12 and we are left with the 24-dimensional case $M = \mathbf{SO}(10)/(\mathbf{SO}(6)\times\mathbf{SO}(4))$. Thus we only have to exclude the case $K = \mathbf{SO}(6)\times\mathbf{SO}(4)$ and $G = \mathbf{U}(3)\times\mathbf{Sp}(2)$. However, $\mathbf{U}(3)\times\mathbf{Sp}(2)$ cannot be a subgroup of $\mathbf{SO}(6)\times\mathbf{SO}(4)$.

Henceforth we shall assume $r \geq 5$.

3.4. **Spaces of maximal rank.** In this case, the principal isotropy group K_{princ} is finite, so G_{princ} is also finite. Therefore G has codimension one in K, so it is a normal subgroup of K. Since K has a normal subgroup of codimension one, it has a normal (hence central) subgroup of dimension 1. Thus K has a circle factor and the corresponding symmetric space M is Hermitian. Since the rank is maximal and at least 5, we deduce that $M = \mathbf{Sp}(n)/\mathbf{U}(n)$. In this case, there is a unique codimension one subgroup of K, and we get one example $(\mathbf{SU}(n), S^2\mathbf{C}^n)$ listed in (ii).

3.5. **Reformulation.** There remain five classical families of symmetric spaces that we are going to analyse in the following. For easy reference, we list their isotropy representations together with the identity components of their principal isotropy groups.

K	V	$H := (K_{\text{princ}})^\circ$	*Conditions*
$\mathbf{Sp}(n)$	$[\Lambda^2\mathbf{C}^{2n} \ominus \mathbf{C}]_\mathbf{R}$	$\mathbf{Sp}(1)^n$	$n \geq 6$
$\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$ $(p \geq q)$	$\mathbf{C}^p \otimes_\mathbf{C} \mathbf{C}^{q*}$	$\mathbf{S}(\mathbf{U}(p-q) \times \mathbf{U}(1)^q)$	$p \geq q \geq 5$
$\mathbf{SO}(p) \times \mathbf{SO}(q)$ $(p \geq q)$	$\mathbf{R}^p \otimes_\mathbf{R} \mathbf{R}^q$	$\mathbf{SO}(p-q)$	$p \geq q+2 \geq 7$
$\mathbf{U}(n)$	$\Lambda^2\mathbf{C}^n$	$\mathbf{SU}(2)^{\frac{n}{2}}$ if n is even $\mathbf{SU}(2)^{\frac{n-1}{2}}\mathbf{U}(1)$ if n is odd	$n \geq 10$
$\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ $(p \geq q)$	$\mathbf{H}^p \otimes_\mathbf{H} \mathbf{H}^{q*}$	$\mathbf{Sp}(p-q) \times \mathbf{Sp}(1)^q$	$p \geq q \geq 5$

3.6. **Relation with the work of Kollross.** Since G acts on the Euclidean space V with cohomogeneity one less than K, the action of G on the principal orbits of K has cohomogeneity one. Since those orbits are of type K/K_{princ}, this means that $G \times K_{\text{princ}}$, and therefore $G \times H$ where $H := K^\circ_{\text{princ}}$, acts on the Lie group K with cohomogeneity one, where the first factor acts from the left and the second from the right. Such (and more general) cohomogeneity one actions have been extensively studied by Kollross. He has classified such actions in the case of simple groups K, unfortunately, only *up to orbit equivalence*. More precisely, he has shown that for any such action, one finds larger connected groups $G \subset \tilde{G} \subset K$ and $H \subset \tilde{H} \subset K$ such that the triple $(\tilde{G}, K, \tilde{H})$ is listed in [Kol02, Theorem B].

Using this theorem of Kollross it will be easy to finish the proof of our theorem. We only need to circumvent some minor difficulties related to the fact that our group K need not be simple and that our pair (G, H) need not be maximal.

We derive from [Kol02] the following:

Lemma 3.1. *Let* $\mathbf{F}$ *be* $\mathbf{R}$, $\mathbf{C}$ *or* $\mathbf{H}$. *Let* K *be respectively* $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$ *acting on* $V = \mathbf{F}^n$, *with* $n \geq 5$. *Let* H *be a subgroup of* K. *Assume that* V *decomposes into* H*-invariant subspaces as* $V = V_1 \oplus V_2 \oplus V_3$, *with* $1 \leq \dim_\mathbf{F}(V_1) \leq \dim_\mathbf{F}(V_2) \leq \dim_\mathbf{F}(V_3)$. *Assume that for a proper subgroup* $G \subset K$ *the group* $G \times H$ *acts with cohomogeneity at most one on* K. *Then* $\mathbf{F} \neq \mathbf{H}$. *If* $\mathbf{F} = \mathbf{C}$, *then* n *is even and* G *is contained in* $\mathbf{Sp}(\frac{n}{2})$. *If* $K = \mathbf{SO}(7)$, *then* $G \subset \mathbf{G}_2$. *Finally, in all cases,* V_3 *has either* $\mathbf{F}$*-dimension or* $\mathbf{F}$*-codimension at most* 3.

Proof. Assume first that G acts reducibly on V, namely, leaves a proper subspace W of V invariant. Let $K_W = \{k \in K | k \cdot W = W\}$. Then $\mathcal{O} = K/K_W$ is the Grassmannian of $\dim(W)$-dimensional subspaces of V, and since $G \subset K_W$, the group H acts with cohomogeneity at most one on $\mathcal{O}$. However, this is impossible, since H preserves the relative positions to the V_i of any subspace $W' \in \mathcal{O}$, and this implies that $\mathcal{O}/H$ is at least two-dimensional.

Thus G acts irreducibly on V. Consider now a maximal connected subgroup $\tilde{G}$ of K containing G. Set $\tilde{H} = (K_{V_3})^\circ$. Then $\tilde{H}$ is a maximal connected subgroup of K containing H. Thus $\tilde{G} \times \tilde{H}$ acts on K with cohomogeneity at most one. If it acts transitively on K, then it must appear in the list of Onishik [Oni62] (cf. [Kol02, Table 4]), and we get $K = \mathbf{SO}(7)$ and $\tilde{G} = G_2$. Otherwise, if it acts with cohomogeneiy one, then it must be in the list of Kollross [Kol02], whence we see the remaining statements. □

3.7. **General consequences.** In case K is a direct product $K = K_1 \times K_2$, we denote by G_i and H_i, respectively, the projections of G and of H to K_i. Then $1 = \dim(K/G \times H) \geq \dim(K_1/G_1 \times H_1) + \dim(K_2/G_2 \times H_2)$.

In any case, observe now that for each one of the groups K appearing in Subsection 3.5, the assumption that the rank is at least 5 and the structure of the principal isotropy group K_{princ} implies the following: for any simple non-Abelian factor K_i of K, the triple (G_i, K_i, H_i) has the form as in Lemma 3.1 above. Moreover, the decomposition of the space $\mathbf{F}^n$ into H_i-irreducible subspaces has at least five summands. Next we proceed case by case.

3.8. **Symplectic cases.** Assume that $K = \mathbf{Sp}(n)$ or $K = \mathbf{Sp}(p) \times \mathbf{Sp}(q)$. From Lemma 3.1 we see that the projection of G to each factor K_i coincides with K_i. We deduce $K = G$, unless possibly $K = \mathbf{Sp}(q) \times \mathbf{Sp}(q)$ and G is the diagonal subgroup of K (up to conjugation), isomorphic to $\mathbf{Sp}(q)$. However, in the last case $H = \mathbf{Sp}(1)^q$, and $G \times H$ cannot act on K with codimension one since $\dim(K) - \dim(G) - \dim(H) > 1$.

3.9. **Real case.** Assume that $K = \mathbf{SO}(p) \times \mathbf{SO}(q)$, with $p \geq q + 2 \geq 7$. The projection of H to $\mathbf{SO}(q)$ is trivial; thus the projection of G to $\mathbf{SO}(q)$ has codimension at most one in $\mathbf{SO}(q)$. Hence this projection coincides with $\mathbf{SO}(q)$. The projection of H to $\mathbf{SO}(p)$ fixes pointwise a 5-dimensional subspace. If $p \geq 8$ we find an H-invariant decomposition $\mathbf{R}^p = V_1 \oplus V_2 \oplus V_3$ with dimension and codimension of V_3 at least four, and applying Lemma 3.1 we deduce that the projection G_1 of G to $\mathbf{SO}(p)$ is $\mathbf{SO}(p)$. We are left with the case $p = 7, q = 5$. Then the projection H_1 of H to $\mathbf{SO}(p)$ is $\mathbf{SO}(2)$ and, due to Lemma 3.1, the group G_1 must be a subgroup of the 14-dimensional group $\mathbf{G}_2$. Then $G_1 \times H_1$ cannot act on $\mathbf{SO}(7)$ with cohomogeneity one.

It follows that the projections of G to the simple factors of K coincide with the factors. Since by assumption $p \neq q$, we get $G = K$.

3.10. **Complex case.** Assume that $K = \mathbf{U}(n)$ with $n \geq 10$. Then, for the projection H_1 of H to the semisimple part $\mathbf{SU}(n)$, the space $\mathbf{C}^n$ decomposes into a sum of at least five H_1-invariant subspaces of complex dimension two. From Lemma 3.1 we deduce that the projection G_1 of G to $\mathbf{SU}(n)$ coincides with $\mathbf{SU}(n)$. Therefore G has codimension one in K. Hence $G = \mathbf{SU}(n)$, and we get one example $(\mathbf{SU}(n), \Lambda^2\mathbf{C}^n)$, where n is even, listed in (ii) (note that in case n is odd, the action of $\mathbf{SU}(n)$ is orbit-equivalent to that of $\mathbf{U}(n)$, which is polar [EH99]).

3.11. **Complex Grassmannian.** Assume that $K = \mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$ with $p \geq q \geq 5$. Applying Lemma 3.1 to the projection H_2 of H and G_2 of G to $\mathbf{SU}(q)$, we see that $G_2 = \mathbf{SU}(q)$, unless possibly $q = 6$ and $G_2 \subset \mathbf{Sp}(3)$. But in the latter case, $H_2 = \mathbf{S}(\mathbf{U}(1)^6)$ and $G_2 \times H_2$ cannot act with cohomogeneity one on $\mathbf{SU}(6)$.

Similarly, for the projection G_1 of G to $\mathbf{SU}(p)$ we deduce that $G_1 = \mathbf{SU}(p)$, unless possibly $p = 6$ and $G_1 \subset \mathbf{Sp}(3)$. In the latter case the projection H_1 of H to $\mathbf{SU}(p)$ is equal to $\mathbf{S}(\mathbf{U}(1)^6)$, which is again impossible by dimensional reasons.

Thus the projection of G to the simple factors of K is surjective. Now either the projection $\bar{G}$ of G to $\mathbf{SU}(p)\times\mathbf{SU}(q)$ is surjective or $p = q$ and the image $\bar{G}$ is the twisted diagonal subgroup $\{(g, \alpha(g)) : g \in \mathbf{SU}(q)\}$, where α is an automorphism of $\mathbf{SU}(q)$. The last case is again impossible by dimensional reasons. Thus $\bar{G} = \mathbf{SU}(p)\times\mathbf{SU}(q)$.

Therefore, G has codimension one in K. Hence $G = \mathbf{SU}(p)\times\mathbf{SU}(q)$ and we get one example $(\mathbf{SU}(p)\times\mathbf{SU}(p), \mathbf{C}^p\otimes\mathbf{C}^p)$ listed in (ii) (note that in case $p > q$ the action of $\mathbf{SU}(p)\times\mathbf{SU}(q)$ is orbit-equivalent to that of $\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$, which is polar [EH99]).

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