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F-INVARIANTS OF DIAGONAL HYPERSURFACES

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ABSTRACT. In this note, we derive formulas for the F-pure threshold, higher jumping numbers, and test ideals of diagonal and Fermat hypersurfaces. For these hypersurfaces, we answer a question of Schwede regarding the denominators of F-pure thresholds, and obtain tight upper bounds for the number of higher jumping numbers. Our results are valid over all (or all but finitely many) characteristics, and therefore allow us to construct examples in which the characteristic p setting is drastically different than that over $\mathbb{C}$.

1. INTRODUCTION

When working in characteristic $p > 0$, one often uses the Frobenius endomorphism (or p^{th} power map) to measure singularities. In particular, if f is a polynomial over a field of characteristic p, one may use Frobenius to construct the F-pure threshold and test ideals of f, invariants which measure the singularities of the hypersurface $\{f = 0\}$. The F-pure threshold of f, denoted $\mathbf{fpt}(f)$, was introduced in [TW04], and is closely related to the classical notion of F-purity. The test ideals of f, a family of ideals $\{\boldsymbol{\tau}(\lambda \bullet f)\}_{\lambda \in \mathbb{R}_{\geq 0}}$, were defined in [HY03], and are variants of the test ideals originally introduced in [HH90] in the context of tight closure.

The relationship between these invariants is based on the behavior of the test ideals $\boldsymbol{\tau}(\lambda \bullet f)$ as a function of λ: If $\lambda > \xi$, then $\boldsymbol{\tau}(\lambda \bullet f) \subseteq \boldsymbol{\tau}(\xi \bullet f)$; in fact, equality holds whenever λ is sufficiently close to ξ (note that the condition of being "sufficiently close to ξ" typically depends on ξ). Furthermore, for every bounded subset $I \subseteq \mathbb{R}_{\geq 0}$, the set of test ideals $\{\boldsymbol{\tau}(\lambda \bullet f)\}_{\lambda \in I}$ (*a priori*, an infinite set) is in fact finite [BMS08, Theorem 3.1]. The ability to choose a finite set of representatives for $\{\boldsymbol{\tau}(\lambda \bullet f)\}_{\lambda \in I}$ allows one to extract from f a sequence of numerical invariants called F-jumping numbers, and the F-pure threshold of f coincides with the smallest such number.

Surprisingly, these characteristic p invariants are closely related to invariants of singularities arising in the study of complex hypersurfaces. Given a polynomial $f_{\mathbb{C}}$ with complex coefficients, via resolution of singularities (or alternately, via L^2 methods), one may construct a family of ideals $\{\mathscr{J}(\lambda \bullet f_{\mathbb{C}})\}_{\lambda \in \mathbb{R}_{\geq 0}}$, called the multiplier ideals of $f_{\mathbb{C}}$. As in the characteristic p case with test ideals, the multiplier ideals of $f_{\mathbb{C}}$ can be used to construct a sequence of numerical invariants called the jumping numbers of $f_{\mathbb{C}}$; we call the smallest such number the log canonical threshold of $f_{\mathbb{C}}$, and denote it by $\mathbf{lct}(f_{\mathbb{C}})$. Though we are motivated by the characteristic

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zero setting, we will not define test ideals, nor the log canonical threshold, and instead refer the reader to [BL04] for an introduction to these invariants.

The relationship between test ideals and multiplier ideals was first established in [Smi00], later generalized in [HY03], and can be summarized as follows: Fixing $\lambda \in \mathbb{R}_{\geq 0}$, the multiplier ideal $\mathscr{J}(\lambda \bullet f_{\mathbb{C}})$ agrees with the test ideal $\boldsymbol{\tau}(\lambda \bullet f_p)$ after reduction to characteristic $p \gg 0$ (here, we have used f_p to denote the reduction of $f_{\mathbb{C}}$ mod p). For an introduction to the process of reduction to positive characteristic, see [Smi97]. The relationship between F-pure thresholds and log canonical thresholds is a bit more subtle; for more on this connection (and more generally, on the connection between F-purity and log canonicity), see [HW02, Tak04, Her11].

In this article, we focus on computing test ideals, F-pure thresholds, and higher jumping numbers in the concrete setting of diagonal and Fermat hypersurfaces. Recall that a polynomial f over a field $\mathbb{L}$ is called diagonal if it is an $\mathbb{L}^*$-linear combination of $x_1^{d_1}, \cdots, x_n^{d_n}$, and is called Fermat (of degree d) if it is an $\mathbb{L}^*$-linear combination of $x_1^d, \cdots, x_d^d$. Note that test ideals of diagonal hypersurfaces, in the sense of Hochster and Huneke, were computed by McDermott in [McD01, McD03]. Motivated by the connections mentioned above, we are especially interested understanding the behavior of these invariants for all (or for all but finitely many) possible values of p.

We now briefly summarize our results. Let f_p (respectively, $f_{\mathbb{C}}$) denote an $\mathbb{F}_p^*$ (respectively, $\mathbb{C}^*$)-linear combination of $x_1^{d_1}, \cdots, x_n^{d_n}$. In Theorem 3.4, we provide a formula for $\mathbf{fpt}(f_p)$ that is valid for all values of p. Though we omit this description here, we do mention one interesting consequence: If $\mathbf{fpt}(f_p) \neq \mathbf{lct}(f_{\mathbb{C}})$, then $\mathbf{fpt}(f_p) \in \mathbb{Z}\left[\frac{1}{p}\right]$ (i.e., $\mathbf{fpt}(f_p)$ is a rational number whose denominator is a power of p). This result is closely related to a question of Karl Schwede, who has asked whether or not the denominator of the F-pure threshold (which is always a rational number, by [BMS08]) must always be divisible by p whenever it differs from the log canonical threshold; see Remark 3.7 for more details. In Theorem 4.10, we give explicit conditions on p under which $\boldsymbol{\tau}(\mathbf{fpt}(f_p) \bullet f_p)$, the first non-trivial test ideal of f_p, is equal to $\mathfrak{m}$, the maximal ideal generated by the variables appearing in f_p. We also provide an example in which $\mathbf{fpt}(f_p) = \mathbf{lct}(f_{\mathbb{C}})$, yet $\boldsymbol{\tau}(\mathbf{fpt}(f_p) \bullet f_p)$ is not even monomial; see Example 4.9 and Remark 4.11 for more details.

We now change our focus to Fermat hypersurfaces. In Theorem 5.5, we give a complete description of the set of all jumping numbers of a degree d Fermat hypersurface f_p whenever $p > d$. These formulas, which we omit here, imply the following: If $p > d$, then every F-jumping number of f_p (and not just $\mathbf{fpt}(f_p)$) is a rational number with denominator p. As a corollary, we are also able to obtain bounds on the number of F-jumping numbers of f_p appearing in an arbitrary bounded interval of $\mathbb{R}_{\geq 0}$ as a function of p. Note that these bounds are much smaller than those predicted by [KLZ11], and may provide insight into finding sharper upper bounds for larger classes of polynomials; see Corollaries 5.6 and 5.7 for more details.

We conclude this note by comparing the test ideals and multiplier ideals of Fermat hypersurfaces. In characteristic zero, the situation is simple: for every $\lambda \in (0, 1)$, and for every Fermat hypersurface $f_{\mathbb{C}}$ of arbitrary degree, the multiplier ideal $\mathscr{J}(\lambda \bullet f_{\mathbb{C}})$ is trivial. By Corollary 3.9, the analogous statement holds for test ideals whenever f_p is Fermat of degree d and $p \equiv 1 \bmod d$. Furthermore, Corollary 5.7 shows that $\boldsymbol{\tau}(\lambda \bullet f_p)$ can take at most 2 distinct values as λ varies through $(0,1)$ whenever f_p is Fermat of degree d and $p > d\cdot(d-1)$. However, we see below that the situation can be far more complicated for arbitrary p and d.

Corollary 5.8. *For every $k \in \mathbb{N}$, there exists a prime p and an integer d with the following property: Every Fermat hypersurface f of degree d over any perfect field of characteristic p has at least k distinct jumping numbers in $(0,1)$. Furthermore, if we denote these jumping numbers by $\xi_0 = \mathbf{fpt}(f) < \xi_1 < \cdots < \xi_{k-1}$, then $\boldsymbol{\tau}(\xi_s \bullet f) = \mathfrak{m}^{s+1}$ for every $0 \le s \le k-1$.*

The methods used to prove Corollary 5.8 are explicit, and allow one to easily construct examples of highly singular Fermat hypersurfaces. For example, if f is a Fermat hypersurface of degree 205 over $\mathbb{F}_{409}$, then $\left\{\mathfrak{m}, \mathfrak{m}^2, \cdots, \mathfrak{m}^{100}\right\} \subseteq \{\boldsymbol{\tau}(\lambda \bullet f)\}_{\lambda \in (0,1)}$.

2. Digits and truncations (base p)

Definition 2.1. Fix $\lambda \in (0,1]$, and let p be a prime number. For every $s \ge 1$, let $\lambda^{(s)}$ denote the unique integer $0 \le \lambda^{(s)} \le p-1$ such that $\lambda = \sum_{s\ge 1} \lambda^{(s)} \cdot p^{-s}$, and such that $\lambda^{(s)}$ is not eventually zero as a function of s. We call $\lambda^{(s)}$ the s^{th} digit of λ (base p); we adopt the convention that $\lambda^{(0)} = 0^{(s)} = 0$.

Definition 2.2. Fix $0 \le e \le \infty$ and $\lambda \in [0,1]$. We call $\langle \lambda \rangle_e := \sum_{s=1}^{e} \lambda^{(s)} \cdot p^{-s}$ the e^{th} truncation of λ (base p). Note that $\langle \lambda \rangle_\infty = \lambda$.

Notation 2.3. In this article, we adopt standard decimal notation for base p expansions; we will always use " : " to distinguish between consecutive digits when using decimal notation.

Example 2.4. Consider a rational number $\lambda \in [0,1]$, and positive integers a and b.

(1) If $\alpha := \frac{1}{p} = .1 = .0 : \overline{p-1}$ (base p), then $\alpha^{(1)} = 0$, while $\alpha^{(e)} = p-1$ for all $e \ge 2$.
(2) If $p^a \cdot (p^b - 1)\cdot \lambda \in \mathbb{N}$, then $\lambda = .\ \lambda^{(1)} : \cdots : \lambda^{(a)} : \overline{\lambda^{(a+1)} : \cdots : \lambda^{(a+b)}}$ (base p). The proof of this is the same as in the familiar base 10 case, and is left to the reader.
(3) If $(p-1)\cdot\lambda \in \mathbb{N}$, then $\lambda = .\ \overline{(p-1)\cdot\lambda}$ (base p). Indeed, this follows by multiplying both sides of the expansion $1 = .\ \overline{p-1}$ (base p) by λ.

Lemma 2.5. *If $\lambda \in [0,1]$, then $\lceil p^e \lambda \rceil = p^e \langle \lambda \rangle_e + 1$. If $s \in \mathbb{N}$, then $\frac{s}{p^e} < \lambda$ if and only if $\frac{s}{p^e} \le \langle \lambda \rangle_e$.*

Proof. The first claim follows from multiplying the identity $\lambda = \langle \lambda \rangle_e + \sum_{d>e} \lambda^{(d)} \cdot p^{-d}$ by p^e and observing that $\sum_{d>e} \lambda^{(d)} \cdot p^{-d}$ is in $(0, p^{-e}]$.

We now address the second claim. As we are dealing with non-terminating expansions, $\langle \lambda \rangle_e < \lambda$, and the "$\Leftarrow$" implication is clear. Next, suppose that $\frac{s}{p^e} < \lambda$. We have seen above that $\lambda \le \langle \lambda \rangle_e + p^{-e}$, and combining this with our hypothesis shows that $s \cdot p^{-e} < \langle \lambda \rangle_e + p^{-e}$. Multiplying this inequality by p^e and noting that both of the resulting sides are integers shows that $s \le p^e \cdot \langle \lambda \rangle_e$, and the claim follows. ☐

Definition 2.6. Let $(\lambda_1, \cdots, \lambda_n) \in [0,1]^n$, and let p be a prime number. We say the e^{th} digits of $\lambda_1, \cdots, \lambda_n$ add without carrying (base p) if $\sum_{i=1}^{n} \lambda_i^{(e)} \le p-1$, and we say that $\lambda_1, \cdots, \lambda_n$ add without carrying (base p) if all of their digits add without carrying (base p). We say natural numbers $k_1, \cdots, k_n$ add without carrying (base p) if the obvious analogous condition holds.

The notion of adding without carrying is relevant in light of the following classical result.

Lemma 2.7 ([Dic02, Luc78])**.** *Fix $(k_1, \cdots k_n) \in \mathbb{N}^n$ and set $N = \sum k_i$. Then $\binom{N}{\boldsymbol{k}} := \frac{N!}{k_1! \cdots k_n!} \not\equiv 0 \mod p$ if and only if $k_1, \cdots, k_n$ add without carrying (base p).*

Remark 2.8 (On the verification of adding without carrying)*.* By definition, to verify that rational numbers $\lambda_1, \cdots, \lambda_n \in [0,1]$ add without carrying, one must check infinitely many conditions. On the other hand, it is not very hard to see that there exists an integer M (which depends on p and the rational numbers $\lambda_1, \cdots, \lambda_n$) with the following property: $\lambda_1, \cdots, \lambda_n$ add without carrying (base p) if and only if the first M digits add without carrying (base p). Indeed, write $\lambda_i = \frac{\alpha_i}{p^{a_i} \beta_i}$, with $p \nmid \beta_i$. As p is prime, $p \nmid \beta_1 \cdots \beta_n$, and thus defines a unit in $(\mathbb{Z}/\beta_1 \cdots \beta_n \mathbb{Z})^*$; consequently, there exists an integer b such that p^b is congruent to 1 modulo $\beta_1 \cdots \beta_n$. If $a := \max \{a_1, \cdots, a_n\}$, then $p^a(p^b - 1) \cdot \lambda_i \in \mathbb{N}$, and by Example 2.4, $\lambda_i = .\ \lambda_i^{(1)} : \cdots : \lambda_i^{(a)} : \overline{\lambda_i^{(a+1)} : \cdots : \lambda_i^{(a+b)}}$ (base p) for each i. In light of these base p expansions, it is clear that one may set $M = a + b$.

3. F-pure thresholds

Let $\mathbb{L}$ be a a perfect of characteristic $p > 0$, and let $R = \mathbb{L}[x_1, \cdots, x_n]$. We will use $\mathfrak{m}$ to denote the ideal $(x_1, \cdots, x_n)$. As $\mathbb{L}$ is perfect, we have that $R^{p^e} := \mathbb{L}[x_1^{p^e}, \cdots, x_n^{p^e}]$ is the subring of $(p^e)^{\text{th}}$ powers of R. For every ideal $I \subseteq R$, let $I^{[p^e]}$ denote the ideal generated by the set $\{g^{p^e} : g \in I\}$. We call $I^{[p^e]}$ the e^{th} *Frobenius power* of I.

Definition 3.1 ([MTW05])**.** Fix $f \in \mathfrak{m}$, and set $\nu_f(p^e) := \max \{N \in \mathbb{N} : f^N \notin \mathfrak{m}^{[p^e]}\}$. The limit $\mathbf{fpt}(f) := \lim_{e \to \infty} \frac{\nu_f(p^e)}{p^e}$ exists and is called the F-pure threshold of f at $\mathfrak{m}$.

Remark 3.2. As $f \in \mathfrak{m}$, we have that $f^{p^e} \in \mathfrak{m}^{[p^e]}$. Thus, $\nu_f(p^e) < p^e$ for all e, and so $\mathbf{fpt}(f) \in (0, 1]$.

The following illustrates an important property satisfied by F-pure thresholds. For a proof of Proposition 3.3, see [BMS09, Proposition 4.3] or [Her12, Proposition 4.2].

Proposition 3.3. *If $\mathbf{fpt}(f) > \frac{N}{p}$ for some integer N, then $\mathbf{fpt}(f) \geq \frac{N}{p-1}$.*

3.1. F-pure thresholds of diagonal hypersurfaces.

Below, we characterize the F-pure threshold of a diagonal hypersurface. In what follows, given $0 \neq d \in \mathbb{N}$, we use d^{-1} to denote $\frac{1}{d} \in \mathbb{Q}$.

Theorem 3.4. *Let $\mathbb{L}$ be a perfect field of characteristic p, and f an $\mathbb{L}^*$-linear combination of $x_1^{d_1}, \cdots, x_n^{d_n}$. If $L := \inf \left\{ e \geq 0 : \sum_{i=1}^{n} \left({d_i}^{-1}\right)^{(e+1)} \geq p \right\}$, then $\mathbf{fpt}(f) = \sum_{i=1}^{n} \left\langle {d_i}^{-1} \right\rangle_L + p^{-L}$.*

Below, we point out some consequences of Theorem 3.4 (the proof itself will be postponed until the next subsection).

Remark 3.5 ($L = \infty$). By definition, $L = \infty$ if and only if ${d_1}^{-1}, \cdots, {d_n}^{-1}$ add without carrying (base p). We adopt the convention that $p^{-\infty} = 0$; consequently, if $L = \infty$, Theorem 3.4 states that $\mathbf{fpt}(f) = \sum_{i=1}^{n} \left\langle {d_i}^{-1} \right\rangle_\infty + p^{-\infty} = \sum_{i=1}^{n} {d_i}^{-1}$.

Remark 3.6 (An effective algorithm). *A priori*, calculating L requires verifying infinitely many conditions. However, by Remark 2.8, there exists an integer M (depending only on p and $d_1, \cdots, d_n$) such that ${d_1}^{-1}, \cdots, {d_n}^{-1}$ add without carrying (base p) if and only if the first M digits add without carrying. In light of this, it follows that either $L \leq M$, or $L = \infty$. Moreover, as illustrated in Remark 2.8, the integer M may be explicitly computed as a function of p and $d_1, \cdots, d_n$, and consequently, L can always be determined in a well-understood, finite number of steps. Thus, Theorem 3.4 provides an algorithm, recently implemented in `Macaulay 2` by Sara Malec, Karl Schwede, and Emily Witt, for computing the F-pure threshold of a diagonal hypersurface.

Remark 3.7 ($\mathbf{fpt}(f)$ vs. $\mathbf{lct}(f_{\mathbb{Q}})$). Let $f_{\mathbb{Q}}$ denote a polynomial over $\mathbb{Q}$, and for every prime $p \gg 0$, let f_p denote the polynomial over $\mathbb{F}_p$ obtained by reducing the coefficients of $f_{\mathbb{Q}}$ modulo p. As discussed in the introduction, the log canonical threshold of $f_{\mathbb{Q}}$, denoted $\mathbf{lct}(f_{\mathbb{Q}})$, is an invariant of the hypersurface $\{f_{\mathbb{Q}} = 0\}$; like the F-pure threshold, $\mathbf{lct}(f_{\mathbb{Q}}) \in (0, 1] \cap \mathbb{Q}$, and it is conjectured that $\mathbf{fpt}(f_p) = \mathbf{lct}(f_{\mathbb{Q}})$ for infinitely many p [MTW05, Conjecture 3.6]. We also recall the following question, due to Karl Schwede: If $\mathbf{fpt}(f_p) \neq \mathbf{lct}(f_{\mathbb{Q}})$, must p divide the denominator of $\mathbf{fpt}(f_p)$?

Note that Theorem 3.4 allows us to address these questions in the case of diagonal hypersurfaces: Suppose that $f_{\mathbb{Q}}$ is a $\mathbb{Q}^*$-linear combination of $x_1^{d_1}, \cdots, x_n^{d_n}$ such that $\sum_i {d_i}^{-1} \leq 1$. It is well known that $\mathbf{lct}(f_{\mathbb{Q}}) = \sum_i {d_i}^{-1}$, and we claim that $\mathbf{fpt}(f_p) = \sum_i {d_i}^{-1}$ whenever $p \equiv 1$ modulo $d_1 \cdots d_n$ (note that there exist infinitely many such primes by Dirichlet's theorem). Indeed, by Theorem 3.4, it suffices to show that ${d_1}^{-1}, \cdots, {d_n}^{-1}$ add without carrying (base p); by our choice of p, Example 2.4 implies that $\left({d_i}^{-1}\right)^{(e)} = (p-1) \cdot {d_i}^{-1}$, and hence $\sum \left({d_i}^{-1}\right)^{(e)} = (p-1) \sum {d_i}^{-1} \leq (p-1)$ for all $e \geq 1$. Theorem 3.4 also gives a description of the denominator of $\mathbf{fpt}(f_p)$: either $\mathbf{fpt}(f_p) = \mathbf{lct}(f_{\mathbb{Q}})$, or $\mathbf{fpt}(f_p) \in \mathbb{Z}\left[\frac{1}{p}\right]$.

Explicit formulas for the F-pure threshold of $x^2 + y^3$ and $x^2 + y^7$ are given in [MTW05, Example 4.3 and 4.4]. At first glance, these formulas appear to be quite different from the expressions appearing above. Below, we give an example of how Theorem 3.4 may be used to recover the well-known formula for $\mathbf{fpt}(x^2 + y^3)$.

Example 3.8. Let f be a $\mathbb{L}^*$-linear combination of x^2 and y^3. If $p = 3$, then

$$\frac{1}{2} = .\overline{1} \text{ (base 3) and } \frac{1}{3} = .1 = .0\,\overline{2} \text{ (base 3)}.$$

We see that carrying is required to add the second digits of $\frac{1}{2}$ and $\frac{1}{3}$ (but not the first), and Theorem 3.4 implies $\mathbf{fpt}(f) = \left\langle \frac{1}{2} \right\rangle_1 + \left\langle \frac{1}{3} \right\rangle_1 + \frac{1}{3} = 0 + \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Similarly, one can show that $\mathbf{fpt}(f) = \frac{1}{2}$ if $p = 2$. If $p = 6\omega + 1$ for some $\omega \geq 1$, then

$$\frac{1}{2} = .\overline{3\omega} \text{ (base } p\text{) and } \frac{1}{3} = .\overline{2\omega} \text{ (base } p\text{)}.$$

We notice that $\frac{1}{2}$ and $\frac{1}{3}$ add without carrying (base p), and Theorem 3.4 implies $\mathbf{fpt}(f) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Finally, if $p = 6\omega + 5$ for some $\omega \geq 0$, then

$$\frac{1}{2} = .\overline{3\omega + 2} \text{ (base } p) \text{ and } \frac{1}{3} = .\overline{2\omega + 1 \quad 4\omega + 3} \text{ (base } p).$$

Once more, we see that carrying is needed to add the second digits of $\frac{1}{2}$ and $\frac{1}{3}$ (but not the first) and Theorem 3.4 implies

$$\mathbf{fpt}(f) = \left\langle \frac{1}{2} \right\rangle_1 + \left\langle \frac{1}{3} \right\rangle_1 + \frac{1}{p} = \frac{3\omega + 2}{p} + \frac{2\omega + 1}{p} + \frac{1}{p} = \frac{5\omega + 4}{p}. \tag{3.1}$$

The reader may verify that $\frac{5\omega+4}{p} + \frac{1}{6p} = \frac{5}{6}$, so we may rewrite (3.1) as $\mathbf{fpt}(f) = \frac{5}{6} - \frac{1}{6p}$. Thus, we recover the following formula from [MTW05, Example 4.3]:

$$\mathbf{fpt}(x^2 + y^3) = \begin{cases} 1/2 & \text{if } p = 2, \\ 2/3 & \text{if } p = 3, \\ 5/6 & \text{if } p \equiv 1 \bmod 6, \\ \frac{5}{6} - \frac{1}{6p} & \text{if } p \equiv 5 \bmod 6. \end{cases}$$

3.2. **The proof.** Before proceeding with the proof of Theorem 3.4, we will deduce a useful characterization of $\nu_f(p^e)$: Let $f \in \mathbb{L}[x_1, \cdots x_n]$ be as in Theorem 3.4, and fix elements $u_i \in \mathbb{L}^*$ such that $f = u_1 x_1^{d_1} + \cdots + u_n x_n^{d_n}$. By the multinomial theorem,

$$f^N = \sum \binom{N}{s_1, \cdots, s_n} u_1^{s_1} \cdots u_n^{s_n} \cdot x_1^{d_1 s_1} \cdots x_n^{d_n s_n},$$

where the sum extends over all partitions $\boldsymbol{s} = (s_1, \cdots, s_n) \in \mathbb{N}^n$ of N. As each summand above is distinct, we see that $f^N \notin \mathfrak{m}^{[p^e]} = (x_1^{p^e}, \cdots, x_n^{p^e})$ if and only if some summand is not contained in $\mathfrak{m}^{[p^e]}$. Furthermore, the summand $\binom{N}{s_1, \cdots, s_n} \cdot u_1^{s_1} \cdots u_n^{s_n} \cdot x_1^{d_1 s_1} \cdots x_n^{d_n s_n}$ is not contained in $\mathfrak{m}^{[p^e]}$ if and only if the multinomial coefficient $\binom{N}{s_1, \cdots, s_n} \neq 0 \bmod p$ and $d_i \cdot s_i < p^e$ for all $1 \leq i \leq n$; this last condition can be rewritten as $\frac{s_i}{p^e} < {d_i}^{-1}$, which by Lemma 2.5 is equivalent to the condition that $s_i \leq p^e \cdot \langle {d_i}^{-1} \rangle_e$ for all $1 \leq i \leq n$. Hence,

$$\nu_f(p^e) = \max \left\{ |\boldsymbol{s}| : \boldsymbol{s} \in \mathbb{N}^n, \boldsymbol{s} \preccurlyeq \left(p^e \cdot \langle {d_1}^{-1} \rangle_e, \cdots, p^e \cdot \langle {d_n}^{-1} \rangle_e\right), \text{ and } \binom{|\boldsymbol{s}|}{\boldsymbol{s}} \not\equiv 0 \bmod p \right\}, \tag{3.2}$$

where we have used $|\boldsymbol{s}|$ to denote the coordinate sum $s_1 + \cdots + s_n$, $\binom{|\boldsymbol{s}|}{\boldsymbol{s}}$ to denote the multinomial coefficient $\binom{s_1 + \cdots + s_n}{s_1, \cdots, s_n}$, and $\preccurlyeq$ to denote component-wise inequality.

Proof of Theorem 3.4. We separate the proof into two cases.

Case 1 ($L = \infty$). It follows from the bounds in (3.2) that $\nu_f(p^e) \leq p^e \cdot \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_e$ for all $e \geq 1$; we will now show that equality must hold. Fix $e \geq 1$, and set

$$(s_1, \cdots, s_n) = \left(p^e \cdot \langle {d_1}^{-1} \rangle_e, \cdots, p^e \cdot \langle {d_n}^{-1} \rangle_e\right) \in \mathbb{N}^d.$$

By assumption, ${d_1}^{-1}, \cdots, {d_n}^{-1}$ add without carrying (base p), and it is apparent that this property is inherited by $s_1, \cdots, s_n$ as well. Lemma 2.7 shows that $\binom{|\boldsymbol{s}|}{\boldsymbol{s}} \neq 0 \bmod p$, and it follows from (3.2) that $\nu_f(p^e) \geq s_1 + \cdots + s_n = p^e \cdot \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_e$. Thus, we conclude that $\nu_f(p^e) = \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_e$ for all $e \geq 1$, and it follows that $\mathbf{fpt}(f) = \lim_{e \to \infty} \frac{\nu_f(p^e)}{p^e} = \lim_{e \to \infty} \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_e = \sum_{i=1}^{n} {d_i}^{-1}$.

Case 2 ($L < \infty$). By definition of L, the entries of $\left(p^L \cdot \langle {d_1}^{-1} \rangle_L, \cdots p^L \cdot \langle {d_n}^{-1} \rangle_L\right)$ add without carrying (base p), and the preceding argument shows that $\nu_f(p^L) = p^L \cdot \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_L$. By definition of L, we also know that $\left({d_1}^{-1}\right)^{(L+1)} + \cdots + \left({d_n}^{-1}\right)^{(L+1)} \geq p$, and so we may choose a partition $(\gamma_1, \cdots, \gamma_n) \in \mathbb{N}^n$ of $p-1$ such that $\gamma_i \leq \left({d_i}^{-1}\right)^{(L+1)}$ for all i, with at least one of these inequalities being strict. Without loss of generality, we will suppose that $\gamma_1 < \left({d_1}^{-1}\right)^{(L+1)}$. Fix $e \geq 1$, and set

$$s_1 = p^{L+1+e} \cdot \left(\frac{\left({d_1}^{-1}\right)^{(1)}}{p} + \cdots + \frac{\left({d_1}^{-1}\right)^{(L)}}{p^L} + \frac{\gamma_1}{p^{L+1}} + \frac{p-1}{p^{L+2}} + \cdots + \frac{p-1}{p^{L+1+e}} \right) \text{ and}$$

$$s_i = p^{L+1+e} \cdot \left(\frac{\left({d_i}^{-1}\right)^{(1)}}{p} + \cdots + \frac{\left({d_i}^{-1}\right)^{(L)}}{p^L} + \frac{\gamma_i}{p^{L+1}} + \frac{0}{p^{L+2}} + \cdots + \frac{0}{p^{L+1+e}} \right) \text{ for } 2 \leq i \leq n.$$

Observe the following:

(1) $(s_1, \cdots, s_n) \in \mathbb{N}^n$
(2) $s_i \leq p^{L+1+e} \cdot \langle {d_i}^{-1} \rangle_{L+1+e}$ for all $1 \leq i \leq n$.
(3) $s_1, \cdots, s_n$ add without carrying (base p), so that $\binom{s_1 + \cdots + s_n}{s_1, \cdots, s_n} \not\equiv 0 \bmod p$.
(4) $s_1 + \cdots + s_n = p^{L+1+e} \cdot \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_L + p^{e+1} - 1$.

Indeed, the first point follows by definition, while the second holds by the defining properties of the partition $(\gamma_1, \cdots, \gamma_n)$ of $p-1$. The third point follows by definition of L, while the last point follows from a routine calculation. Thus, we may apply (3.2) to conclude that

$$\nu_f(p^{L+1+e}) \geq p^{L+1+e} \cdot \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_L + p^{e+1} - 1 \text{ for all } e \geq 1.$$

We claim that this inequality is an equality: Otherwise, $\nu_f(p^{L+1+e}) \geq p^{L+1+e} \cdot \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_L + p^{e+1}$, so that

$$\left(f^{p^L \cdot \sum_i \langle {d_i}^{-1} \rangle_L + 1} \right)^{p^{e+1}} = f^{p^{L+1+e} \cdot \sum_i \langle {d_i}^{-1} \rangle_L + p^{e+1}} \notin \mathfrak{m}^{[p^{L+1+e}]} = \left(\mathfrak{m}^{[p^L]} \right)^{[p^{e+1}]}.$$

As a consequence, we see that $f^{p^L \cdot \sum_i \langle {d_i}^{-1} \rangle_L + 1} \notin \mathfrak{m}^{[p^L]}$, which contradicts our earlier computation of $\nu_f(p^L) = p^L \cdot \sum_i \langle {d_i}^{-1} \rangle_L$. Thus, $\nu_f(p^{L+1+e}) = p^{L+1+e} \cdot \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_L + p^{e+1} - 1$ for all $e \geq 1$, and so $\mathbf{fpt}(f) = \lim_{e \to \infty} \frac{\nu_f(p^{L+1+e})}{p^{L+1+e}} = \sum_{i=1}^{n} \langle {d_i}^{-1} \rangle_L + \frac{1}{p^L}$. □

Corollary 3.9. *If f is an $\mathbb{L}^*$-linear combination of $x_1^d, \cdots, x_d^d$, then*

$$\mathbf{fpt}(f) = \begin{cases} \frac{1}{p^s} & \text{if } p^s \leq d < p^{s+1} \text{ for some } s \geq 1, \\ 1 - \frac{a-1}{p} & \text{if } p > d \text{ and } a \text{ is the least positive residue of } p \text{ modulo } d. \end{cases}$$

Proof. If $p^s \leq d < p^{s+1}$, then $\frac{1}{p^{s+1}} < d^{-1} \leq \frac{1}{p^s}$: this upper bound implies that the first s digits of d^{-1} (base p) must be zero, while the lower bound implies that $\left(d^{-1}\right)^{(s+1)} \geq 1$. Consequently, we see that adding d copies of $\left(d^{-1}\right)^{(e)}$ produces zero for all $1 \leq e \leq s$, while adding d copies of $\left(d^{-1}\right)^{(s+1)}$ produces $d \cdot \left(d^{-1}\right)^{(s+1)} \geq d \geq p^s \geq p$. In the notation of Theorem 3.4, we have that $L = s$, and so $\mathbf{fpt}(f) = d \cdot \left\langle d^{-1} \right\rangle_s + p^{-s} = p^{-s}$.

We now assume that $p > d$, and use a to denote the least positive residue of p modulo d. Dividing the identity $p = \left(\frac{p-a}{d}\right) \cdot d + a$ by $p \cdot d$ shows that

$$d^{-1} = \left(\frac{p-a}{d}\right) \cdot \frac{1}{p} + \left(\frac{a}{d}\right) \cdot \frac{1}{p},$$

from which we deduce that $\left(d^{-1}\right)^{(1)} = \frac{p-a}{d}$ and $\left(d^{-1}\right)^{(e+1)} = \left(\frac{a}{d}\right)^{(e)}$ for all $e \geq 1$. If $a = 1$, it follows from these identities that, in the notation of Theorem 3.4, $L = \infty$, so that $\mathbf{fpt}(f) = d \cdot d^{-1} = 1$, which agrees with our formula. If $a \geq 2$, then these identities show that adding d copies of the first digit of d^{-1} (base p) produces $p - a$, and it is left to the reader to verify that adding d copies of $\left(d^{-1}\right)^{(2)} = \left(\frac{a}{d}\right)^{(1)}$ produces an integer strictly greater than p (the reader may also set $\boldsymbol{m} = \mathbf{0}$ in Lemma 5.3 to verify this claim). Thus, we see that $L = 1$, and so $\mathbf{fpt}(f) = d \cdot \left(\frac{p-a}{d} \cdot \frac{1}{p}\right) + \frac{1}{p}$. □

4. Test ideals and F-pure thresholds

$\mathbb{L}$ will continue to denote a perfect field of characteristic $p > 0$, R will denote a polynomial ring over $\mathbb{L}$, and $\mathfrak{m}$ will denote the ideal of R generated by the variables.

Definition 4.1. Let $\mathbb{B}_e$ denote the set of monomials $\left\{\mu \in R : \mu \notin \mathfrak{m}^{[p^e]}\right\}$. The reader may verify that $\mathbb{B}_e$ is a free basis for R as an R^{p^e}-module, so that for every $f \in R$, there exist unique elements $\Theta_\mu^e(f)$ such that $f = \sum_{\mu \in \mathbb{B}_e} \Theta_\mu^e(f)^{p^e} \cdot \mu$. The ideal generated by the set $\left\{\Theta_\mu^e(f) : \mu \in \mathbb{B}_e\right\}$ will be denoted by $(f)^{\left[p^{-e}\right]}$.

Definition 4.2. For every $\lambda \geq 0$, the set $\left\{\left(f^{\lceil p^e \lambda \rceil}\right)^{\left[p^{-e}\right]}\right\}_{e \geq 1}$ defines an increasing sequence of ideals [BMS08, Lemma 2.8]. We call the stabilizing ideal the *test ideal* of f (with respect to the parameter λ), and denote it by $\boldsymbol{\tau}(\lambda \bullet f)$. In other words,

$$\boldsymbol{\tau}(\lambda \bullet f) = \bigcup_{e \geq 1} \left(f^{\lceil p^e \lambda \rceil}\right)^{\left[p^{-e}\right]} = \left(f^{\lceil p^e \lambda \rceil}\right)^{\left[p^{-e}\right]} \text{ for all } e \gg 0.$$

The following lemma, whose proof we omit, allows us to identify when the test ideal stabilizes in an important special case.

Lemma 4.3 ([BMS09, Lemma 2.1]). *If $\lambda \in p^{-e} \cdot \mathbb{N}$, then $\boldsymbol{\tau}(\lambda \bullet f) = \left(f^{p^e \lambda}\right)^{\left[p^{-e}\right]}$.*

Test ideals form a decreasing sequence of ideals and are stable to the right [BMS08, Proposition 2.11, Corollary 2.16]. That is, $\boldsymbol{\tau}(\lambda \bullet f) \subseteq \boldsymbol{\tau}(\xi \bullet f)$ if $\lambda \geq \xi$. Additionally, for every $\lambda \geq 0$ there exists $\epsilon > 0$ such that $\boldsymbol{\tau}(\lambda \bullet f) = \boldsymbol{\tau}((\lambda + \delta) \bullet f)$ whenever $0 \leq \delta < \epsilon$. This behavior motivates the following definition.

Definition 4.4. We say that $\xi > 0$ is an *F-jumping number* of f if

$$\boldsymbol{\tau}(\xi \bullet f) \neq \boldsymbol{\tau}((\xi - \epsilon) \bullet f) \text{ for all } 0 < \epsilon < \xi.$$

The following proposition shows that the set of all F-jumping numbers of f is completely determined by its intersection with $(0, 1)$.

Proposition 4.5 ([BMS08, Proposition 3.4])**.** *1 is an F-jumping number of f, and $\gamma > 1$ is an F-jumping number if and only if $\gamma - 1$ is an F-jumping number.*

We now recall the relationship between test ideals and the F-pure threshold. The F-pure threshold, as defined in Definition 3.1, is a member of a general family of invariants called F-thresholds. In light of this, [MTW05, Proposition 2.7] shows the following:

$$\text{If } f \in \mathfrak{m}, \text{ then } \mathbf{fpt}(f) = \min\{\lambda > 0 : \boldsymbol{\tau}(\lambda \bullet f) \subseteq \mathfrak{m}\}. \tag{4.1}$$

It follows that $\boldsymbol{\tau}(\lambda \bullet f) \nsubseteq \mathfrak{m}$ for all $0 < \lambda < \mathbf{fpt}(f)$, which shows that $\mathbf{fpt}(f)$ must be an F-jumping number of f.

Remark 4.6. In this article, we only consider diagonal hypersurfaces. If f is diagonal and p does not divide any exponent in f, then Corollary 4.8 below implies that $\boldsymbol{\tau}(\lambda \bullet f)$ is monomial for all $\lambda \in [0, 1)$. In this setting, the monomial ideal $\boldsymbol{\tau}(\lambda \bullet f)$ is contained in $\mathfrak{m}$ if and only if it is not trivial, and so (4.1) shows that $\mathbf{fpt}(f) = \min\{\lambda > 0 : \boldsymbol{\tau}(\lambda \bullet f) \neq R\}$. This characterization is often used to define F-pure thresholds, while Definition 3.1 (which gives rise to (4.1)) is sometimes called the F-pure threshold of f (at $\mathfrak{m}$). As we have just shown, these two invariants coincide for the polynomials being considered here, which justifies our choice of terminology.

4.1. Test ideals of diagonal hypersurfaces.

We now turn our attention to the test ideals of diagonal hypersurfaces. Throughout this subsection, f will denote an $\mathbb{L}^*$-linear combination of $x_1^{d_1}, \cdots, x_n^{d_n}$ in $R = \mathbb{L}[x_1, \cdots, x_n]$.

Lemma 4.7. *Suppose $p \nmid d_i$ for all $1 \leq i \leq n$. If $0 \leq N \leq p^e - 1$, then $\left(f^N\right)^{[p^{-e}]}$ is the monomial ideal generated by*

$$\left\{ x_1^{\left\lfloor \frac{d_1 \cdot s_1}{p^e} \right\rfloor} \cdots x_n^{\left\lfloor \frac{d_n \cdot s_n}{p^e} \right\rfloor} : s_1 + \cdots + s_n = N \text{ and } \binom{N}{s_1, \cdots, s_n} \not\equiv 0 \bmod p \right\}.$$

Proof. For every partition $\mathbf{s} = (s_1, \cdots, s_n)$ of N, set $\mu_{\boldsymbol{s}} = x_1^{[d_1 \cdot s_1 \% p^e]} \cdots x_n^{[d_n \cdot s_n \% p^e]}$, where $[d_i \cdot s_i \% p^e]$ denotes the least positive residue of $d_i \cdot s_i$ modulo p^e, and $s_i' = \frac{d_i \cdot s_i - [d_i \cdot s_i \% p^e]}{p^e}$. By definition, $s_i' \in \mathbb{N}$; in fact, the identity $d_i \cdot s_i = s_i' \cdot p^e + [d \cdot s_i \% p^e]$ shows that $s_i' = \left\lfloor \frac{d \cdot s_i}{p^e} \right\rfloor$. Thus, setting $\boldsymbol{s}' = (s_1', \cdots, s_n') \in \mathbb{N}^n$, we have that $x_1^{d_1 \cdot s_1} \cdots x_n^{d_n \cdot s_n} = \left(\boldsymbol{x}^{\boldsymbol{s}'}\right)^{p^e} \cdot \mu_{\boldsymbol{s}}$. Note that $\mu_{\boldsymbol{s}} \in \mathbb{B}_e$, the free basis for R over R^{p^e} described in Definition 4.1. Furthermore, if $\boldsymbol{t} = (t_1, \cdots t_n)$ is another partition of N, then $\mu_{\boldsymbol{s}} = \mu_{\boldsymbol{t}}$ if and only if $\boldsymbol{s} = \boldsymbol{t}$. Indeed, suppose that $[d_i \cdot s_i \% p^e] = [d_i \cdot t_i \% p^e]$. As $p \nmid d_i$, we conclude that $[s_i \% p^e] = [t_i \% p^e]$. However, as $\boldsymbol{s}$ and $\boldsymbol{t}$ are partitions of $N \leq p^e - 1$, s_i and t_i are both bounded above by $p^e - 1$, and hence must be equal.

Fix coefficients $u_i \in \mathbb{L}^*$ such that $f = u_1 x_1^{d_1} + \cdots + u_n x^{d_n}$. We have shown above that

$$f^N = \sum \binom{N}{s_1, \cdots, s_d} u_1^{s_1} \cdots u_d^{s_d} x_1^{d \cdot s_1} \cdots x_d^{d \cdot s_d} = \sum \left(\gamma_{\boldsymbol{s}} \cdot \boldsymbol{x}^{\boldsymbol{s}'} \right)^{p^e} \cdot \mu_{\boldsymbol{s}}$$

is the unique expression of f^N as an R^{p^e}-linear combination of elements from $\mathbb{B}_e$, where the sum extends over all partitions $(s_1, \cdots, s_n) \in \mathbb{N}^n$ of N, and $\gamma_{\boldsymbol{s}}$ denotes the $(p^e)^{\text{th}}$ root of $\binom{N}{s_1, \cdots, s_n} \cdot u_1^{s_1} \cdots u_n^{s_n}$ in $\mathbb{L} = \mathbb{L}^p$. As $\gamma_{\boldsymbol{s}} = 0$ if and only if $\binom{N}{\boldsymbol{s}} \equiv 0 \bmod p$, it follows by definition that $\left(f^N\right)^{[p^{-e}]}$ is generated by the set $\left\{ \boldsymbol{x}^{\boldsymbol{s}'} : \boldsymbol{s} \text{ is a partition of } N \text{ and } \binom{N}{\boldsymbol{s}} \neq 0 \bmod p \right\}$. ☐

Corollary 4.8. *If p does not divide any d_i, then $\boldsymbol{\tau}(\lambda \bullet f)$ is monomial for all $\lambda \in [0, 1)$.*

Proof. For such a parameter, $\lambda < 1 - p^{-e}$ for all $e \gg 0$. In particular, we may choose $e \gg 0$ such that $\lceil p^e \lambda \rceil \leq p^e - 1$ and $\boldsymbol{\tau}(\lambda \bullet f) = \left(f^{\lceil p^e \lambda \rceil}\right)^{[p^{-e}]}$, which is monomial by Lemma 4.7. ☐

Example 4.9 ($\boldsymbol{\tau}(\mathbf{fpt}(f) \bullet f)$ need not be monomial)**.** Let p be an odd prime, so that $a := \frac{p-1}{2}$ and $b := \frac{p^2-1}{2}$ are integers, and set $f = x_1^b + \cdots + x_a^b + y_1^{p \cdot b} + \cdots + y_a^{p \cdot b} \in \mathbb{F}_p[x_1, \cdots, x_a, y_1, \cdots, y_a]$.

Multiplying the expansion $\frac{1}{p^2-1} = .\overline{0\ :1}$ (base p) by 2 shows that

$$\frac{1}{b} = \frac{2}{p^2 - 1} = .\overline{0\ :2} \ (\text{base}\, p) \text{ and } \frac{1}{p \cdot b} = .0\ :\ \overline{0\ :2} \ (\text{base}\, p).$$

Thus, the reciprocals of the exponents of f add without carrying (base p), and it follows from Theorem 3.4 that $\mathbf{fpt}(f) = a \cdot \frac{1}{b} + a \cdot \frac{1}{p \cdot b} = \frac{1}{p}$. By definition, $b = a \cdot p + a$ is the base p expansion of b, and so $f = (x_1^a)^p \cdot x_1^a + \cdots + (x_a^a)^p \cdot x_a^a + \left(y_1^b + \cdots + y_a^b\right)^p \cdot 1$. By Lemma 4.3, we conclude that $\boldsymbol{\tau}(\mathbf{fpt}(f) \bullet f) = (f)^{[p^{-1}]} = \left(x_1^a, \cdots, x_a^a, y_1^b + \cdots + y_a^b\right)$.

Theorem 4.10. *If f is any $\mathbb{L}^*$-linear combination of $x_1^{d_1}, \cdots, x_n^{d_n}$, then*

$$\boldsymbol{\tau}(\mathbf{fpt}(f) \bullet f) = \begin{cases} (f) & \text{if } \mathbf{fpt}(f) = 1 \\ (x_1, \cdots, x_n) & \text{if } \mathbf{fpt}(f) \neq 1 \text{ and } p > \max\{d_1, \cdots, d_n\}. \end{cases}$$

Proof. Lemma 4.3 implies that $\boldsymbol{\tau}(1 \bullet f) = (f^p)^{[p^{-1}]} = (f)$. We now assume that $\mathbf{fpt}(f) \neq 1$. Our first step will be to show that

$$\lceil p \cdot \mathbf{fpt}(f) \rceil = \sum_{i=1}^{n} \left(d_i^{-1}\right)^{(1)} + 1 \leq p - 1. \tag{4.2}$$

Set $L := \inf \left\{ e \geq 0 : \sum_{i=1}^{n} \left(d_i^{-1}\right)^{(e+1)} \geq p \right\}$, so that $\mathbf{fpt}(f) = \sum_{i=1}^{n} \left\langle d_i^{-1} \right\rangle_L + p^{-L}$ by Theorem 3.4. As we are assuming that $\mathbf{fpt}(f) \neq 1$, this formula for $\mathbf{fpt}(f)$ shows that $L \geq 1$. Moreover, it follows from the definition of L that

$$\mathbf{fpt}(f) = \sum_{i=1}^{n} {d_i}^{-1} = .\sum_{i=1}^{n} \left({d_i}^{-1}\right)^{(1)} : \sum_{i=1}^{n} \left({d_i}^{-1}\right)^{(2)} : \cdots : \sum_{i=1}^{n} \left({d_i}^{-1}\right)^{(e)} : \cdots (\text{base}\, p)$$

if $L = \infty$, and

$$\mathbf{fpt}(f) = \sum_{i=1}^{n} \left\langle {d_i}^{-1} \right\rangle_L + p^{-L} = .\sum_{i=1}^{n} \left({d_i}^{-1}\right)^{(1)} : \cdots : \sum_{i=1}^{n} \left({d_i}^{-1}\right)^{(L)} : \overline{p-1} \ (\text{base}\, p)$$

if $L < \infty$. Note that the expansions of $\mathbf{fpt}(f)$ just considered show that, no matter the value of L,

$$\mathbf{fpt}(f)^{(1)} = \sum_{i=1}^{n} \left({d_i}^{-1}\right)^{(1)}. \tag{4.3}$$

Next, we observe that $\mathbf{fpt}(f)^{(1)} \leq p-2$: Otherwise, $\mathbf{fpt}(f) > \langle \mathbf{fpt}(f) \rangle_1 = \frac{p-1}{p}$, and by Proposition 3.3, $\mathbf{fpt}(f) \geq \frac{p-1}{p-1} = 1$, which contradicts our assumption that $\mathbf{fpt}(f) \neq 1$. It follows from (4.3) and Lemma 2.5 that $\lceil p \cdot \mathbf{fpt}(f) \rceil = p \cdot \langle \mathbf{fpt}(f) \rangle_1 + 1 = \sum\limits_{i=1}^{n} \left({d_i}^{-1}\right)^{(1)} + 1 \leq p-1$, establishing (4.2).

We now compute

$$\left(f^{\lceil p \cdot \mathbf{fpt}(f) \rceil}\right)^{[p^{-1}]} : \text{Set } (s_1, \cdots, s_n) := \left(\left\lceil p \cdot {d_1}^{-1} \right\rceil, \left({d_2}^{-1}\right)^{(1)}, \cdots, \left({d_n}^{-1}\right)^{(1)}\right).$$

It follows from (4.2), combined with the observation that $\left\lceil p \cdot {d_1}^{-1} \right\rceil = \left({d_1}^{-1}\right)^{(1)} + 1$, that $(s_1, \cdots, s_n)$ is a partition of $\lceil p \cdot \mathbf{fpt}(f) \rceil$, and that the multinomial coefficient $\binom{\lceil p \cdot \mathbf{fpt}(f) \rceil}{s_1, \cdots, s_n} \not\equiv 0 \bmod p$. Applying the inequalities $\gamma \leq \lceil \gamma \rceil < \gamma + 1$ for all $\gamma \in \mathbb{R}$ to $d_1 s_1 = d_1 \left\lceil p \cdot {d_1}^{-1} \right\rceil$ shows that

$$p \leq d_1 \cdot s_1 < p + d_1 < 2p,$$

where the last inequality follows from our assumption that $p > \max\{d_1, \cdots, d_n\}$. Similarly, if $i \neq 1$, $d_i \cdot s_i = d_i \cdot \left({d_i}^{-1}\right)^{(1)} = d_i \cdot p \left\langle {d_i}^{-1} \right\rangle_1 < p \cdot d_i \cdot {d_i}^{-1} = p$, and Lemma 4.7 then shows that

$$x_1^{\left\lfloor \frac{d_1 s_1}{p} \right\rfloor} \cdots x_n^{\left\lfloor \frac{d_n \cdot s_n}{p} \right\rfloor} = x_1$$

is a generator for $\left(f^{\lceil p \cdot \mathbf{fpt}(f) \rceil}\right)^{[p^{-1}]}$. By symmetry, the same holds for every variable, and thus $\left(f^{\lceil p \cdot \mathbf{fpt}(f) \rceil}\right)^{[p^{-1}]} = \mathfrak{m}$. By definition, $\boldsymbol{\tau}(\mathbf{fpt}(f) \bullet f)$ is the ascending union of the ideals $\left(f^{\lceil p^e \cdot \mathbf{fpt}(f) \rceil}\right)^{[p^{-e}]}$, and hence must also be equal to $\mathfrak{m}$. ☐

Remark 4.11. We saw in Example 4.9 that $\boldsymbol{\tau}(\mathbf{fpt}(f) \bullet f)$ need not be monomial, even in the case that $L = \infty$ (so that, in the notation of Remark 3.5, $\mathbf{fpt}(f) = \mathbf{lct}(f_{\mathbb{Q}})$). For examples in which $L < \infty$ (i.e., $\mathbf{fpt}(f) \neq \mathbf{lct}(f_{\mathbb{Q}})$) and $\boldsymbol{\tau}(\mathbf{fpt}(f) \bullet f)$ is monomial, but not equal to $\mathfrak{m}$, see [MY09, Proposition 4.2].

5. The higher jumping numbers of Fermat hypersurfaces

Notation 5.1. For the remainder of this article, p will denote a prime number, a will denote the least positive residue of p modulo d, and ω will denote the integer $\frac{p-a}{d}$. Finally, f will denote a degree d Fermat hypersurface over a perfect field of characteristic p.

Before examining the higher jumping numbers of f, we establish several key lemmas.

Lemma 5.2. *Fix $m \in \mathbb{N}$, and suppose that $\omega \cdot (m+1) + \left\lfloor \frac{a(m+1)-1}{d} \right\rfloor \leq p-1$. Then*

(1) *the rational number $\sigma := \frac{m+1}{d}$ is in $(0,1]$, and*
(2) $\sigma^{(1)} = \omega \cdot (m+1) + \left\lfloor \frac{a(m+1)-1}{d} \right\rfloor$.

Proof. First, suppose that $\frac{a(m+1)}{d} \in \mathbb{N}$. Multiplying $p = d \cdot \omega + a$ by $m+1$ shows that $p \cdot (m+1) = d\omega(m+1) + a(m+1) = d\left(\omega(m_i+1) + \frac{a(m+1)}{d}\right)$, so that

$$\sigma = \frac{m+1}{d} = \frac{1}{p} \cdot \left(\omega \cdot (m+1) + \frac{a(m+1)}{d}\right). \tag{5.1}$$

As $\frac{a(m+1)}{d} \in \mathbb{N}$, we have that $\frac{a(m+1)}{d} - 1 = \left\lfloor \frac{a(m+1)-1}{d} \right\rfloor$, and (5.1) shows that

$$\sigma = \frac{1}{p} \cdot \left(\omega \cdot (m+1) + \left\lfloor \frac{a(m+1)-1}{d} \right\rfloor + 1\right). \tag{5.2}$$

As we are assuming that $\omega \cdot (m+1) + \left\lfloor \frac{a(m+1)-1}{d} \right\rfloor \leq p-1$, (5.2) shows that $\sigma \leq 1$. Furthermore, (5.2) also shows that $\sigma = .\ \omega \cdot (m+1) + \left\lfloor \frac{a(m+1)-1}{d} \right\rfloor \ :\ \overline{p-1}$ (base p), from which we conclude that $\sigma^{(1)} = \omega \cdot (m+1) + \left\lfloor \frac{a(m+1)-1}{d} \right\rfloor$. The case that $\frac{a(m+1)}{d} \notin \mathbb{N}$ is dealt with similarly and is thus left to the reader. □

Lemma 5.3. *Fix $(m_1, \cdots, m_d) \in \mathbb{N}^d$, and suppose $\sigma_i := \frac{m_i+1}{d} \in (0,1]$ for every $1 \leq i \leq d$. Suppose $p > d$, and that $p \not\equiv 1 \bmod d$. If $\sum_{i=1}^{d} \sigma_i^{(1)} \leq p-1$, then $\sum_{i=1}^{d} \sigma_i^{(2)} > p$.*

Proof. If $(m_1, \cdots, m_d) = 0$, then $\sigma_i = \frac{1}{d}$ for all $1 \leq i \leq d$. Dividing the equation $p = d \cdot \omega + a$ by pd shows that $\sigma_i = \frac{1}{d} = \omega \cdot p^{-1} + \left(\frac{a}{d}\right) p^{-1}$, so that $\sigma_i^{(2)} = \left(\frac{a}{d}\right)^{(1)}$ for all $1 \leq i \leq d$. It follows that

$$\sum \sigma_i^{(2)} = d \cdot \left(\frac{a}{d}\right)^{(1)} = d \cdot p \cdot \left\langle \frac{a}{d} \right\rangle_1 \geq d \cdot p \cdot \left(\frac{a}{d} - \frac{1}{p}\right) = ap - d.$$

Notice that our assumption that $p \neq 1$ implies that $a \geq 2$, and substituting this into the preceding inequality shows that $\sum \sigma_i^{(2)} \geq 2p - d > p$.

If $(m_1, \cdots, m_d) \neq 0$, then $\sum \sigma_i \geq 1 + \frac{1}{d}$. Furthermore, as $\sigma_i \leq \sigma_i^{(1)} \cdot p^{-1} + \left(\sigma_i^{(2)} + 1\right) \cdot p^{-2}$, we see that $\sigma_i^{(2)} \geq \sigma_i \cdot p^2 - \sigma_i^{(1)} \cdot p - 1$. It follows from these inequalities that

$$\sum \sigma_i^{(2)} \geq \left(\sum \sigma_i\right) \cdot p^2 - \left(\sum \sigma_i^{(1)}\right) \cdot p - d \geq \left(1 + \frac{1}{d}\right) \cdot p^2 - (p-1) \cdot p - d$$
$$= \frac{p^2}{d} + p - d > 2p - d > p. \qquad \square$$

Lemma 5.4. *Fix* $(m_1, \cdots, m_d) \in \mathbb{N}^d$, *and set*

$$\mathscr{N} := \sum_{i=1}^{d} \left(\omega \cdot (m_i + 1) + \left\lfloor \frac{a(m_i+1) - 1}{d} \right\rfloor \right).$$

1. *Suppose* $p \nmid d$. *If* $x_1^{m_1} \cdots x_d^{m_d} \in \boldsymbol{\tau}\left(\frac{N}{p} \bullet f\right)$ *for some* $N \in \mathbb{N}$, *then* $N \leq \mathscr{N}$.
2. *Suppose* $p \nmid d$. *If* $\mathscr{N} \geq p$, *then* $x_1^{m_1} \cdots x_d^{m_d} \in \boldsymbol{\tau}(\lambda \bullet f)$ *for all* $\lambda \in [0,1)$.
3. *Suppose* $p > d$ *and that* $p \not\equiv 1 \bmod d$. *If* $\mathscr{N} \leq p - 1$, *then* $x_1^{m_1} \cdots x_d^{m_d} \in \boldsymbol{\tau}\left(\left(\frac{\mathscr{N}}{p} + \frac{\lambda}{p}\right) \bullet f\right)$ *for all* $\lambda \in [0,1)$.

Proof. If $x_1^{m_1} \cdots x_d^{m_d} \in \boldsymbol{\tau}\left(\frac{N}{p} \bullet f\right)$ for some $N \in \mathbb{N}$, then by Lemma 4.7, there exists a partition $(s_1, \cdots, s_d) \in \mathbb{N}^d$ of N such $\left\lfloor \frac{d \cdot s_i}{p^e} \right\rfloor \leq m_i$. Applying the inequality $\gamma < \lfloor \gamma \rfloor + 1$ for all $\gamma \in \mathbb{R}$ shows that $d \cdot s_i < p(m_i+1) = d \cdot \omega(m_i+1) + a(m_i+1)$ for all $1 \leq i \leq d$, where the last equality follows from the identity $p = d \cdot \omega + a$. From this, we deduce that $s_i \leq \omega \cdot (m_i+1) + \left\lfloor \frac{a(m_i+1)-1}{d} \right\rfloor$, and as $(s_1, \cdots, s_d)$ is a partition of N, we see that $N = \sum_{i=1}^{d} s_i \leq \sum_{i=1}^{d} \left(\omega \cdot (m_i+1) + \left\lfloor \frac{a(m_i+1)-1}{d} \right\rfloor\right) = \mathscr{N}$.

We now address the second assertion. Suppose that

$$\mathscr{N} = \sum_{i=1}^{d} \left(\omega \cdot (m_i + 1) + \left\lfloor \frac{a(m_i+1) - 1}{d} \right\rfloor \right) \geq p.$$

This inequality guarantees the existence of a partition $(\gamma_1, \cdots \gamma_d) \in \mathbb{N}^d$ of $p-1$ such that $\gamma_i \leq \omega \cdot (m_i+1) + \left\lfloor \frac{a(m_i+1)-1}{d} \right\rfloor$ for all i, with one of these inequalities being strict; without loss of generality, assume that $\gamma_1 \leq \omega \cdot (m_i+1) + \left\lfloor \frac{a(m_i+1)-1}{d} \right\rfloor - 1$. It follows that

1. $d \cdot \gamma_1 \leq p \cdot (m_i+1) - 1 - d$, and similarly,
2. $d \cdot \gamma_i \leq p \cdot (m_i+1) - 1$ for $2 \leq i \leq d$.

Indeed, both of these follow from the identity $p = d \cdot \omega + a$, combined with a straightforward manipulation of roundings. Fix $e \geq 1$, and $(s_1, \cdots, s_d) \in \mathbb{N}^d$ defined by

$$s_1 := p^{e-1} \cdot \gamma_1 + p^{e-1} - 1 \text{ and } s_i := p^{e-1} \cdot \gamma_i \text{ for } 2 \leq i \leq d.$$

As $(\gamma_1, \cdots, \gamma_d)$ is a partition of $p-1$, we see that $(s_1, \cdots, s_d)$ is a partition of $p^e - 1$. Furthermore, it is apparent from looking at the base p expansion of s_1 that the entries of $(s_1, \cdots, s_d)$ sum to $p^e - 1$ without carrying (base p); Lemma 2.7 then implies that the multinomial coefficient $\binom{p^e-1}{s_1, \cdots, s_d}$ is non-zero modulo p. Finally, it follows from the definition of s_i and the inequalities in (1) and (2) above that

3. $d \cdot s_1 \leq p^e \cdot (m_i+1) - p^{e-1} - d$, and
4. $d \cdot s_i \leq p^e \cdot (m_i+1) - p^{e-1}$ for $2 \leq i \leq d$.

The bounds in (3) and (4) imply that $\left\lfloor \frac{d \cdot s_i}{p^e} \right\rfloor \leq m_i$, and it follows from Lemma 4.7 that $x_1^{m_1} \cdots x_d^{m_d}$ is contained in $\boldsymbol{\tau}\left(\frac{p^e-1}{p^e} \bullet f\right)$. As e was chosen arbitrarily, the claim follows.

It remains to establish the last claim. Suppose $\mathscr{N} \leq p-1$, and set $\sigma_i := \frac{m_i+1}{d}$. By Lemma 5.2, $(\sigma_1, \cdots, \sigma_d) \in (0,1]^d$, and $\left(\sigma_1^{(1)}, \cdots, \sigma_d^{(1)}\right)$ is a partition of $\mathscr{N}$, and by Lemma 5.3, $\sum_{i=1}^{n} \sigma_i^{(2)} \geq p$. As a consequence, there exists a partition $(\beta_i, \cdots, \beta_d)$ of $p-1$ such that $\beta_i \leq \sigma_i^{(2)}$ for all i, with at least one inequality being strict; we assume that $\beta_1 < \sigma_1^{(2)}$. Fix $e \geq 2$, and set

$t_1 := \sigma_1^{(1)} \cdot p^{e-1} + \beta_1 \cdot p^{e-2} + p^{e-2} - 1$ and $t_i := \sigma_i^{(1)} \cdot p^{e-1} + \beta_i \cdot p^{e-2}$ for $2 \leq i \leq d$.

As $\left(\sigma_1^{(1)}, \cdots, \sigma_d^{(1)}\right)$ partitions $\mathscr{N}$ and $(\beta_1, \cdots, \beta_d)$ partitions $p-1$, we see that $(t_1, \cdots, t_d)$ is a partition of $\mathscr{N} \cdot p^{e-1} + p^{e-1} - 1$, and it is apparent from the base p expansion of t_1 that the integers $(t_1, \cdots, t_n)$ add without carrying (base p), so that $\binom{\mathscr{N} \cdot p^{e-1} + p^{e-1} - 1}{t_1, \cdots, t_d} \neq 0 \mod p$. We claim that

$$d \cdot t_i < p^e \cdot (m_i + 1) \tag{5.3}$$

for all $1 \leq i \leq d$. Indeed, for $i = 1$, we have the identity

$$d \cdot t_1 = d \cdot p^e \cdot \left(\frac{\sigma_1^{(1)}}{p} + \frac{\beta_1}{p^2} + \frac{p-1}{p^3} + \cdots + \frac{p-1}{p^e}\right),$$

and as $\beta_1 < \sigma_1^{(2)}$, we see that

$$d \cdot t_i \leq d \cdot p^e \cdot \left(\frac{\sigma_1^{(1)}}{p} + \frac{\sigma_1^{(2)}}{p^2}\right) = p^e \cdot d \cdot \langle \sigma_1 \rangle_2 < p^e \cdot d \cdot \sigma_1 = p^e \cdot (m_1 + 1),$$

and similar estimates show that the analogous bound holds for each $d \cdot t_i$. The bound in (5.3) implies that $\left\lfloor \frac{d \cdot t_i}{p^e} \right\rfloor \leq m_i$, and it follows from Lemma 4.7 that the monomial $x_1^{m_1} \cdots x_d^{m_d}$ is in $\boldsymbol{\tau}\left(\left(\frac{\mathscr{N} \cdot p^{e-1} + p^{e-1} - 1}{p^e}\right) \bullet f\right) = \boldsymbol{\tau}\left(\left(\frac{\mathscr{N}}{p} + \frac{1}{p} \cdot \frac{p^{e-1}-1}{p^{e-1}}\right) \bullet f\right)$ for all $e \geq 2$. Letting e go to infinity shows that $x_1^{m_1} \cdots x_d^{m_d}$ is contained in $\boldsymbol{\tau}\left(\frac{\mathscr{N}+\lambda}{p} \bullet f\right)$ for every $\lambda \in (0,1]$. □

5.1. A formula for higher jumping numbers, and some final observations.

We conclude this article by presenting a formula for the higher jumping numbers of a Fermat hypersurface, and examining some immediate consequences.

Setup for Theorem 5.5: We continue to use $\mathbb{L}$ to denote a perfect field of characteristic p, and f to denote an $\mathbb{L}^*$-linear combination of $x_1^d, \cdots, x_d^d$ in $\mathbb{L}[x_1, \cdots, x_d]$. We will also continue to suppose that $p > d$, and that a is the least positive residue of p modulo d.

Theorem 5.5. *For every* $\boldsymbol{m} := (m_1, \cdots, m_d) \in \mathbb{N}^d$, *set*

$$\mathscr{A}_{\boldsymbol{m}} := \sum_{i=1}^{d} \left(m_i \cdot \left(\frac{p-a}{d}\right) + \left\lfloor \frac{a(m_i+1)-1}{d} \right\rfloor\right).$$

The set of jumping exponents of f contained in $(0,1)$ *is* $\left\{1 - \frac{a-1-\mathscr{A}_{\boldsymbol{m}}}{p} : \boldsymbol{m} \in \mathbb{N}^d\right\} \cap (0,1)$.

Proof. Let ω denote the natural number $\frac{p-a}{d}$, and for every $\boldsymbol{m} := (m_1, \cdots, m_d) \in \mathbb{N}^d$, set

$$\mathscr{N}_{\boldsymbol{m}} := \sum_{i=1}^{d} \left(\omega \cdot (m_i + 1) + \left\lfloor \frac{a(m_i+1)-1}{d} \right\rfloor \right).$$

Let $\mathbf{JN}_f^\circ$ denote the set $\left\{\frac{\mathscr{N}_{\boldsymbol{m}}+1}{p} : \boldsymbol{m} \in \mathbb{N}^d\right\} \cap (0,1)$, and let $\mathbb{JN}_f$ denote the set of all jumping exponents of f contained in $(0,1)$. Note that $\mathscr{N}_{\boldsymbol{m}} = p - a + \mathscr{A}_{\boldsymbol{m}}$, so to establish the theorem, we must show that $\mathbb{JN}_f = \mathbf{JN}_f^\circ$.

We first suppose that $p \equiv 1 \bmod d$, i.e., that $a = 1$. In this case, Corollary 3.9 shows that $\mathbf{fpt}(f) = 1$, and so $\mathbb{JN}_f$ is empty. Substituting $a = 1$ into the equation for $\mathscr{N}_{\boldsymbol{0}}$ shows that $\mathscr{N}_{\boldsymbol{0}} = \sum_{i=1}^{d} \omega = d \cdot \omega = p - a = p - 1$. Furthermore, it is apparent that $\frac{\mathscr{N}_{\boldsymbol{m}}+1}{p} \geq \frac{\mathscr{N}_{\boldsymbol{0}}+1}{p} = 1$ for all $\boldsymbol{m} \in \mathbb{N}^d$, so that $\mathbf{JN}_f^\circ$ is empty as well.

For the remainder of the proof, assume that $p \not\equiv 1 \bmod d$. Choose $\boldsymbol{m} = (m_1, \cdots, m_d) \in \mathbb{N}^d$ so that $\frac{\mathscr{N}_{\boldsymbol{m}}+1}{p} \in \mathbf{JN}_f^\circ$. In particular $\mathscr{N}_{\boldsymbol{m}} < p - 1$, and the third point of Lemma 5.4 shows that $x_1^{m_1} \cdots x_d^{m_d} \in \boldsymbol{\tau}\left(\left(\frac{\mathscr{N}_{\boldsymbol{m}}}{p} + \frac{\lambda}{p}\right) \bullet f\right)$ for all $\lambda \in (0,1]$. Furthermore, the first point of Lemma 5.4 implies that $x_1^{m_1} \cdots x_d^{m_d} \notin \boldsymbol{\tau}\left(\frac{\mathscr{N}_{\boldsymbol{m}}+1}{p} \bullet f\right)$, so that $\frac{\mathscr{N}_{\boldsymbol{m}}+1}{p}$ is an F-jumping number of f, by definition, and we conclude that $\mathbf{JN}_f^\circ \subseteq \mathbb{JN}_f$.

Next, let $\xi \in \mathbb{JN}_f$ be an F-jumping number of f. By Corollary 4.8, $\boldsymbol{\tau}(\xi \bullet f)$ is a monomial ideal, and so we may choose a monomial $x_1^{m_1} \cdots x_d^{m_d}$ contained in $\boldsymbol{\tau}((\xi - \epsilon) \bullet f)$ for all $0 < \epsilon \ll 1$ such that $x_1^{m_1} \cdots x_d^{m_d} \notin \boldsymbol{\tau}(\xi \bullet f)$. Setting $\mathscr{N} := \mathscr{N}_{(m_1, \cdots, m_d)}$, we now show that

$$\xi^{(1)} \leq \mathscr{N} \leq p - 1. \tag{5.4}$$

To see that $\xi^{(1)} \leq \mathscr{N}$, choose $0 < \epsilon \ll 1$ such that $\frac{\xi^{(1)}}{p} = \langle \xi \rangle_1 < \xi - \epsilon$. Consequently, $x_1^{m_1} \cdots x_d^{m_d}$ is contained in $\boldsymbol{\tau}((\xi - \epsilon) \bullet f) \subseteq \boldsymbol{\tau}\left(\frac{\xi^{(1)}}{p} \bullet f\right)$, and the first point of Lemma 5.4 then shows that $\xi^{(1)} \leq \mathscr{N}$. To establish the bound on $\mathscr{N}$, suppose that $\mathscr{N} \geq p$. The second part of Lemma 5.4 would then imply that $x_1^{m_1} \cdots x_d^{m_d} \in \boldsymbol{\tau}(\lambda \bullet f)$ for all $\lambda \in [0,1)$, a contradiction. Thus, $\mathscr{N} \leq p - 1$.

Note that this bound on $\mathscr{N}$ allows us to apply the third point of Lemma 5.4, which shows that $x_1^{m_1} \cdots x_d^{m_d}$ is contained in $\boldsymbol{\tau}\left(\frac{\mathscr{N}+\lambda}{p} \bullet f\right)$ for all $\lambda \in [0,1)$. However, by definition, $x_1^{m_1} \cdots x_d^{m_d}$ is not contained in $\boldsymbol{\tau}(\xi \bullet f)$, and so $\xi \geq \frac{\mathscr{N}+1}{p} \geq \frac{\xi^{(1)}+1}{p}$ (we have used (5.4) to obtain the second inequality); as $\xi \leq \frac{\xi^{(1)}+1}{p}$, by definition, we conclude that equality holds throughout. In summary,

$$\xi = \frac{\xi^{(1)}+1}{p} = \frac{\mathscr{N}+1}{p} \in \mathbf{JN}_f^\circ,$$

and thus $\mathbb{JN}_f \subseteq \mathbf{JN}_f^\circ$. ☐

An upper bound for the number of jumping numbers of a polynomial with isolated singularity was given in [KLZ11]; in the case of a Fermat hypersurface of degree d, [KLZ11, Corollary 1.4] states that there can be at most $d \cdot (d-1)+1$ jumping numbers in $[0,1]$. We see below that the exact number of jumping numbers can be much smaller.

Corollary 5.6. *If $p > d$, then the number of jumping numbers of f in $[0,1]$ is bounded above by the least positive residue of p modulo d.*

Proof. Once more, let a denote the least positive reside of p modulo d. Recall that 1 is always a jumping number. By Theorem 5.5, the jumping numbers in $(0,1)$ are the form $1-\frac{a-1-\mathscr{A}_{\boldsymbol{m}}}{p}$, and it is apparent that there are at most $a-1$ such numbers in $(0,1)$. ☐

Corollary 5.7. *If $p > d \cdot (d-1)$, then $\mathbf{fpt}(f)$ and 1 are the only jumping numbers of f contained in $[0,1]$.*

Proof. The hypothesis on p implies that $p > d$. If $p \equiv 1 \bmod d$, then Corollary 3.9 implies that $\mathbf{fpt}(f) = 1$ is the only jumping number in $[0,1]$. Next, suppose that a, the least positive reside of p mod d, is at least 2. In this case, Corollary 3.9 shows that $\mathbf{fpt}(f) = 1-\frac{a-1}{p}$, which in the notation of Theorem 5.5 is equal to $1-\frac{a-1-\mathscr{A}_{\mathbf{0}}}{p}$, is contained in $(0,1)$.

We will now show that $\mathbf{fpt}(f)$ is the only jumping number in $(0,1)$. Indeed, note that for every $\mathbf{0} \neq \boldsymbol{m} \in \mathbb{N}^d$, we have that $\mathscr{A}_{\boldsymbol{m}} \geq \frac{p-a}{d} > d-1-\frac{a}{d}$. As $\mathscr{A}_{\boldsymbol{m}}$ is an integer and $a < d$, we conclude that $\mathscr{A}_{\boldsymbol{m}} \geq d-1 \geq a$, and it follows from Theorem 5.5 that $\mathbf{fpt}(f) = 1-\frac{a-1-\mathscr{A}_{\mathbf{0}}}{p}$ is the only jumping number in $(0,1)$. ☐

Corollary 5.8. *For every $k \in \mathbb{N}$, there exist a prime p and an integer d with the following property: Every degree d Fermat hypersurface f over any perfect field of characteristic p has at least k distinct jumping numbers in $(0,1)$. Furthermore, if we denote these jumping numbers by $\xi_0 = \mathbf{fpt}(f) < \xi_1 < \cdots < \xi_{k-1}$, then $\boldsymbol{\tau}(\xi_s \bullet f) = \mathfrak{m}^{s+1}$ for every $0 \leq s \leq k-1$.*

Proof. Choose $p \gg 0$ such that $p = 2d-1$ for some $d \geq 2 \cdot k+1$, and let f be a degree d Fermat hypersurface over a perfect field of characteristic p. By definition, $p = d+(d-1)$, so that we have the following expression for every $\boldsymbol{m} = (m_1, \cdots, m_d) \in \mathbb{N}^d$:

$$\mathscr{A}_{\boldsymbol{m}} = \sum_{i=1}^{d} \left(m_i + \left\lfloor \frac{(d-1)(m_i+1)-1}{d} \right\rfloor \right) = 2 \cdot \sum_{i=1}^{d} m_i + \sum_{i=1}^{d} \left\lfloor \frac{d-m_i-2}{d} \right\rfloor .$$

In fact, our choice of $d \geq 2 \cdot k+1$ implies that $\mathscr{A}_{\boldsymbol{m}} = 2 \cdot \sum_{i=1}^{d} m_i \leq d-3$ for any $\boldsymbol{m} \in \mathbb{N}^d$ with $\sum_{i=1}^{d} m_i \leq k-1$, and this expression for $\mathscr{A}_{\boldsymbol{m}}$, combined with Theorem 5.5, shows that for every $0 \leq s \leq k-1$,

$$\xi_s := 1 - \frac{d-2-2s}{p} = \frac{d+1+2s}{p}$$

is an F-jumping number of f contained in $(0,1)$. We now show that $\boldsymbol{\tau}(\xi_s \bullet f) = \mathfrak{m}^{s+1}$ for all $0 \leq s \leq k-1$. As the methods involved are similar to others previously used in this article, we will only indicate the main steps and leave some of the details to the reader.

Using the third part of Lemma 5.4 and the assumption that $d \geq 2k+1$, one can show that every monomial $x_1^{v_1} \cdots x_d^{v_d}$ with $\sum v_i = s+1$ is contained in $\boldsymbol{\tau}\left(\frac{d+2+2s}{p} \bullet f\right)$, which itself is contained in $\boldsymbol{\tau}\left(\frac{d+1+2s}{p} \bullet f\right) = \boldsymbol{\tau}(\xi_s \bullet f)$. Thus, we have shown that $\mathfrak{m}^{s+1} \subseteq \boldsymbol{\tau}(\xi_s \bullet f)$ for all $0 \leq s \leq k-1$, and will show equality by inducing on s. Indeed, as $\xi_0 = \mathbf{fpt}(f)$, the case $s = 0$ follows from Theorem 4.10. Next, suppose that $\boldsymbol{\tau}(\xi_s \bullet f) = \mathfrak{m}^{s+1}$ for some $0 \leq s \leq k-2$, and suppose, by means of contradiction, that $\boldsymbol{\tau}(\xi_{s+1} \bullet f) \neq \mathfrak{m}^{s+2}$, so that

$$\mathfrak{m}^{s+2} \subsetneq \boldsymbol{\tau}(\xi_{s+1} \bullet f) \subseteq \boldsymbol{\tau}(\xi_s \bullet f) = \mathfrak{m}^{s+1}.$$

Corollary 4.8 asserts that all of these ideals are monomial, and so there exists a monomial $x_1^{m_1} \cdots x_d^{m_d} \in \boldsymbol{\tau}(\xi_{s+1} \bullet f)$, and thus necessarily contained in $\mathfrak{m}^{s+1}$, not contained in $\mathfrak{m}^{s+2}$. Thus, we must have that $\sum m_i = s+1$, and applying the first part of Lemma 5.4 to the containment $x_1^{m_1} \cdots x_d^{m_d} \in \boldsymbol{\tau}(\xi_{s+1} \bullet f)$ implies that the numerator of ξ_{s+1}, which is equal to $d + 3 + 2s$, is bounded above by $d + 2 + 2s$, a contradiction. We conclude that $\boldsymbol{\tau}(\xi_{s+1} \bullet f) = \mathfrak{m}^{s+2}$, and thus that $\boldsymbol{\tau}(\xi_s \bullet f) = \mathfrak{m}^{s+1}$ for every $0 \leq s \leq k-1$. □

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