

FREE BUT NOT RECURSIVELY FREE ARRANGEMENTS

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(Communicated by Lev Borisov)

ABSTRACT. We construct counterexamples to the conjecture that every free arrangement is recursively free in characteristic zero. The intersection lattice of our smallest example has a realization over a finite field which is recursively free, thus recursive freeness is not a combinatorial property of the intersection lattice of an arrangement.

1. INTRODUCTION

Motivated by his famous Addition-Deletion-Theorem (see [9] or Theorem 2.3 below), Terao introduced the notion of *inductive freeness* of an arrangement \mathcal{A} (Definition 2.4), a property which implies the freeness of the module of derivations $D(\mathcal{A})$. Inductive freeness is a purely combinatorial property of the intersection lattice of an arrangement; see for example Lemma 2.5. Terao's longstanding conjecture states that for a fixed field, freeness of the module of derivations of an arrangement of hyperplanes is a combinatorial property of its intersection lattice (see [9] or [10]).

Although inductive freeness is a powerful tool to verify the freeness of many interesting arrangements, there are probably even more arrangements, which are free but not inductively free (see [6, Example 4.59] for one of the oldest examples). But in fact, the Addition-Deletion-Theorem allows to prove the freeness of a much larger class of arrangements introduced in [13, Def. 3.6.4],¹ the *recursively free* arrangements (see also Definition 2.6). An attempt to settle Terao's conjecture is to answer the question whether every free arrangement is recursively free (see [13, 3.6], [6, 4.3], [7, 5]). However, this is not the case: In this note we present free but not recursively free arrangements in characteristic zero, and complete the picture:

$$\text{inductively free} \subsetneq \text{recursively free} \subsetneq \text{free}.$$

We first found our counterexample \mathcal{A} with 27 hyperplanes in \mathbb{C}^3 by using the enumeration techniques introduced in [3] and the realization algorithm from [2].

The intersection lattice of \mathcal{A} has a further realization \mathcal{B} over \mathbb{F}_{11} . It turns out that \mathcal{B} is free and recursively free although it has the same intersection lattice as \mathcal{A} . Hence recursive freeness is not a purely combinatorial property of the intersection lattice, and should thus be perceived in a different way from the notion of inductive

Received by the editors February 28, 2013.

2010 *Mathematics Subject Classification*. Primary 13N15, 14N20, 20F55, 52C35.

¹The original definition by Ziegler is slightly different from [6, Def. 4.60], but the two definitions coincide in dimension three.

freeness. This observation is implicit in the examples in [14, 4] for arrangements over finite fields: Ziegler defines a matroid which has a realization which is free or not free depending on the chosen field, but is of course never inductively free. However, all his arrangements are recursively free when they are free.

Last but not least, the intersection lattice of \mathcal{A} has realizations over \mathbb{F}_q for q large enough (for example $q = 251$) which are free but not recursively free. Thus the inclusion “recursively free \subsetneq free” also holds for (at least certain) positive characteristics.

2. PRELIMINARIES

We briefly review the required notions, cf. [6].

Definition 2.1. Let \mathcal{A} be an arrangement of hyperplanes, i.e. a finite set of hyperplanes in a fixed vector space V over a field K . Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V . We choose a basis x_1, \dots, x_r for V^* and identify S with $K[x_1, \dots, x_r]$ via the natural isomorphism $S \cong K[x_1, \dots, x_r]$. We write $\text{Der}(S)$ for the set of derivations of S over K . It is a free S -module with basis D_1, \dots, D_r where D_i is the usual derivation $\partial/\partial x_i$.

A nonzero element $\theta \in \text{Der}(S)$ is *homogeneous of polynomial degree p* if $\theta = \sum_{k=1}^r f_k D_k$ and $f_k \in S_p$ for $a \leq k \leq r$. In this case we write $\text{pdeg } \theta = p$.

Let \mathcal{A} be an arrangement in V with defining polynomial

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

where $H = \ker \alpha_H$, $\alpha_H \in V^*$. Define the *module of \mathcal{A} -derivations* by

$$D(\mathcal{A}) = \{\theta \in \text{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

An arrangement \mathcal{A} is called a *free arrangement* if $D(\mathcal{A})$ is a free module over S .

If \mathcal{A} is free and $\{\theta_1, \dots, \theta_r\}$ is a homogeneous basis for $D(\mathcal{A})$, then $\text{pdeg } \theta_1, \dots, \text{pdeg } \theta_r$ are called the *exponents* of \mathcal{A} and we write

$$\exp \mathcal{A} = \{\text{pdeg } \theta_1, \dots, \text{pdeg } \theta_r\},$$

where we use the notation $\{*\}$ to emphasize the fact that it is a multiset. Remark that the exponents depend only on \mathcal{A} .

Definition 2.2 ([6, 1.12-1.14]). Let (\mathcal{A}, V) be an arrangement. We denote $L(\mathcal{A})$ as the set of all nonempty intersections of elements of \mathcal{A} including the empty intersection V .

If $\mathcal{B} \subseteq \mathcal{A}$ is a subset, then (\mathcal{B}, V) is called a *subarrangement*. For $X \in L(\mathcal{A})$ define a subarrangement \mathcal{A}_X of \mathcal{A} by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}.$$

Define an arrangement (\mathcal{A}^X, X) in X by

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}.$$

We call \mathcal{A}^X the *restriction* of \mathcal{A} to X .

Let $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ and let $\mathcal{A}'' = \mathcal{A}^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a *triple* of arrangements and H_0 the *distinguished* hyperplane.

We will use the following important theorem:

Theorem 2.3 (Addition-Deletion, [6, Thm. 4.51]). *Suppose $\mathcal{A} \neq \emptyset$. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple. Any two of the following statements imply the third:*

$$\begin{aligned} \mathcal{A} \text{ is free with } \exp \mathcal{A} &= \{b_1, \dots, b_{r-1}, b_r\}, \\ \mathcal{A}' \text{ is free with } \exp \mathcal{A}' &= \{b_1, \dots, b_{r-1}, b_r - 1\}, \\ \mathcal{A}'' \text{ is free with } \exp \mathcal{A}'' &= \{b_1, \dots, b_{r-1}\}. \end{aligned}$$

Inspired by this theorem, one defines:

Definition 2.4 ([6, Def. 4.53]). The class \mathcal{IF} of *inductively free* arrangements is the smallest class of arrangements which satisfies

- (1) the empty arrangement Φ_ℓ of rank ℓ is in \mathcal{IF} for $\ell \geq 0$;
- (2) if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{IF}$, $\mathcal{A}' \in \mathcal{IF}$, and $\exp \mathcal{A}'' \subset \exp \mathcal{A}'$, then $\mathcal{A} \in \mathcal{IF}$.

Lemma 2.5. *The property of an arrangement \mathcal{A} of being inductively free is a combinatorial property of its intersection lattice $L(\mathcal{A})$.*

Proof. All information needed for part (2) in Definition 2.4 is included in the intersection lattice $L(\mathcal{A})$: For $H \in \mathcal{A}$, the intersection lattices $L(\mathcal{A} \setminus \{H\})$ and $L(\mathcal{A}^H)$ can be obtained as sublattices of $L(\mathcal{A})$. \square

A class of arrangements which is larger than the class of inductively free ones is:

Definition 2.6 ([6, Def. 4.60]). The class \mathcal{RF} of *recursively free* arrangements is the smallest class of arrangements which satisfies

- (1) the empty arrangement Φ_ℓ of rank ℓ is in \mathcal{RF} for $\ell \geq 0$;
- (2) if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{RF}$, $\mathcal{A}' \in \mathcal{RF}$, and $\exp \mathcal{A}'' \subset \exp \mathcal{A}'$, then $\mathcal{A} \in \mathcal{RF}$;
- (3) if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{RF}$, $\mathcal{A} \in \mathcal{RF}$, and $\exp \mathcal{A}'' \subset \exp \mathcal{A}$, then $\mathcal{A}' \in \mathcal{RF}$.

For $\alpha \in V^*$, we will write α^\perp for the kernel of α .

3. THE COUNTEREXAMPLES

We will use the following simple lemma (see also [4, Cor. 2.18]):

Lemma 3.1. *Let \mathcal{A} be a free arrangement in K^3 with exponents $\{1, e, f\}$. If $|\mathcal{A}^H| \notin \{e+1, f+1\}$ for all $H \in \mathcal{A}$, then \mathcal{A} is not inductively free.*

Proof. Assume that $|\mathcal{A}^H| \notin \{e+1, f+1\}$ for all $H \in \mathcal{A}$. If \mathcal{A} was inductively free, then there would exist a hyperplane $H \in \mathcal{A}$ such that $(\mathcal{A}, \mathcal{A} \setminus \{H\}, \mathcal{A}^H)$ is a triple of arrangements with $\exp \mathcal{A}^H \subset \exp \mathcal{A}$. But then the exponents of \mathcal{A}^H would be either $\{1, e\}$ or $\{1, f\}$, and hence $|\mathcal{A}^H| \in \{e+1, f+1\}$, which is a contradiction. \square

Definition 3.2. Let ζ be a fifth root of unity in \mathbb{C} and $\omega = -\zeta^2 - \zeta^3$ be the golden ratio. The Coxeter group W of type H_3 may be generated as a reflection group by the reflections (see for example [5])

$$g_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad g_2 := \begin{pmatrix} -1 & 0 & 0 \\ \omega & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 := \begin{pmatrix} 1 & \omega & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

acting on $V^* \cong K^3$, $K = \mathbb{Q}(\zeta)$. Now let $\Phi^+ \in V^*$ be the corresponding set of positive roots of W (the orbit of the standard basis under W) and

$$R := \Phi^+ \dot{\cup} (1, -\zeta^2, 0) \cdot W.$$

Then $\mathcal{A} := \{v^\perp \mid v \in R\}$ is an arrangement with 27 hyperplanes: The set Φ^+ has 15 elements and the other orbit has 12 hyperplanes.

Remark 3.3. The image of W under the canonical map $\mathrm{GL}_3(\mathbb{C}) \rightarrow \mathrm{PGL}_3(\mathbb{C})$ is isomorphic to the alternating group A_5 . Indeed, \mathcal{A} may also be obtained as the union of two orbits under the action of a three dimensional representation of A_5 .

Remark 3.4. The minimal field of definition over \mathbb{Q} for a realization of the intersection lattice of \mathcal{A} is $\mathbb{Q}(\zeta)$: There is no arrangement over $\mathbb{Q}(\omega)$ with the same intersection lattice as \mathcal{A} .

Theorem 3.5. *The arrangement \mathcal{A} is free but not recursively free.*

Proof. Factorizing the characteristic polynomial gives

$$\chi_{\mathcal{A}}(t) = (t-1)(t-11)(t-15).$$

One can now prove the freeness of \mathcal{A} using [12, Thm. 1.39 (ii)] or [11, Cor. 3.3]:² Choose a hyperplane $H \in \mathcal{A}$ and compute the exponents $\{d_1, d_2\}$ of the multiarrangement $(\mathcal{A}^H, \mathfrak{m}^H)$. It turns out that $d_1 d_2 = 165$, thus \mathcal{A} is free.

Inspection of the intersection lattice gives the following multisets of invariants:

$$\begin{aligned} \{|\mathcal{A}^H| \mid H \in \mathcal{A}\} &= \{10^{15}, 11^{12}\}, \\ \{|\mathcal{A}_p| \mid p \in L(\mathcal{A}), \dim p = 1\} &= \{2^{15}, 3^{70}, 7^6\}. \end{aligned}$$

Notice first that each hyperplane in \mathcal{A} contains either 10 or 11 intersection points, thus \mathcal{A} is not inductively free by Lemma 3.1.

Now assume that we include a new hyperplane H to \mathcal{A} , so $\tilde{\mathcal{A}} := \mathcal{A} \dot{\cup} \{H\}$, and assume that $(\tilde{\mathcal{A}}, \mathcal{A}, \tilde{\mathcal{A}}^H)$ is a triple of arrangements with $\exp \tilde{\mathcal{A}}^H \subset \exp \tilde{\mathcal{A}}$; i.e. the exponents of $\tilde{\mathcal{A}}$ are either $\{1, 11, 16\}$ or $\{1, 12, 15\}$ depending on H .

If H contains no intersection point $p \in L(\mathcal{A})$, $\dim p = 1$, then $|\tilde{\mathcal{A}}^H|$ will be equal to $|\mathcal{A}| = 27$, contradicting $\exp \tilde{\mathcal{A}}^H \subset \exp \tilde{\mathcal{A}}$. If it contains exactly one such point, then $|\tilde{\mathcal{A}}^H| \geq 21$ by the above computation of the numbers $|\mathcal{A}_p|$, again a contradiction.

There are only 1186 cases left in which H contains at least two intersection points of $L(\mathcal{A})$. None of these 1186 arrangements is free: This may either be verified by factorizing their characteristic polynomial or by a direct computation. Thus the assumption $\exp \tilde{\mathcal{A}}^H \subset \exp \tilde{\mathcal{A}}$ for the triple $(\tilde{\mathcal{A}}, \mathcal{A}, \tilde{\mathcal{A}}^H)$ is false. It turns out that there is no way up or down via the Addition-Deletion-Theorem starting at \mathcal{A} and staying in the same dimension; i.e. \mathcal{A} is not recursively free. \square

²Alternatively, a direct computation with MAGMA [1], verified once again with SINGULAR [8], tells us that \mathcal{A} is free.

Remark 3.6. The arrangement \mathcal{A} has the ‘same’ intersection lattice as the arrangement

$$\begin{aligned} \mathcal{B} = & \{(0, 0, 1)^\perp, (0, 1, 1)^\perp, (0, 1, 2)^\perp, (0, 1, 3)^\perp, (0, 1, 4)^\perp, (0, 1, 5)^\perp, \\ & (0, 1, 6)^\perp, (1, 0, 0)^\perp, (1, 0, 1)^\perp, (1, 2, 2)^\perp, (1, 3, 1)^\perp, (1, 3, 10)^\perp, \\ & (1, 4, 3)^\perp, (1, 4, 4)^\perp, (1, 5, 7)^\perp, (1, 6, 4)^\perp, (1, 6, 6)^\perp, (1, 8, 5)^\perp, \\ & (1, 8, 8)^\perp, (1, 9, 0)^\perp, (1, 9, 1)^\perp, (1, 9, 4)^\perp, (1, 9, 5)^\perp, (1, 9, 8)^\perp, \\ & (1, 9, 9)^\perp, (1, 10, 0)^\perp, (1, 10, 5)^\perp\} \end{aligned}$$

in \mathbb{F}_{11}^3 . As for \mathcal{A} , the arrangement \mathcal{B} is a union of two orbits under the action of a subgroup of $\mathrm{PGL}_3(\mathbb{F}_{11})$ isomorphic to A_5 .

Notice that contrary to \mathcal{A} , the arrangement \mathcal{B} is recursively free: Let

$$\begin{aligned} (H_1, \dots, H_7) := & ((0, 1, 0)^\perp, (1, 1, 0)^\perp, (1, 2, 0)^\perp, (1, 3, 0)^\perp, \\ & (1, 4, 0)^\perp, (1, 5, 0)^\perp, (1, 7, 0)^\perp). \end{aligned}$$

Then the arrangements $\mathcal{B}_i := \mathcal{B} \cup \{H_1, \dots, H_i\}$, $i = 1, \dots, 7$ are free, and \mathcal{B}_7 is inductively free.

Remark 3.7. The reflection arrangement \mathcal{A} of the complex reflection group G_{27} has 45 hyperplanes and is also free but not recursively free. It is free with exponents $\{1, 19, 25\}$ for instance by [6, Thm. 6.60]. The proof that it is not recursively free is similar to the proof of Theorem 3.5: It is not inductively free by inspection of the intersection lattice:

$$\begin{aligned} \{|\mathcal{A}^H| \mid H \in \mathcal{A}\} &= \{16^{45}\}, \\ \{|\mathcal{A}_p| \mid p \in L(\mathcal{A}), \dim p = 1\} &= \{3^{120}, 4^{45}, 5^{36}\}. \end{aligned}$$

As in Theorem 3.5, one can then consider all possibilities of including a new hyperplane, although the number of possible new hyperplanes is much larger here.

Remark 3.8. Joining four orbits of sizes 6, 6, 9, and 18 under any subgroup of G_{25} of order 108, one obtains a free but not recursively free arrangement with 39 hyperplanes and with exponents $\{1, 19, 19\}$.

REFERENCES

- [1] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, DOI 10.1006/jsco.1996.0125. Computational algebra and number theory (London, 1993). MR1484478
- [2] Michael Cuntz, *Minimal fields of definition for simplicial arrangements in the real projective plane*, Innov. Incidence Geom. **12** (2011), 12. MR2942717
- [3] Michael Cuntz and David Geis, *Combinatorial simpliciality of arrangements of hyperplanes*, arXiv:1302.2052v1 (2013), 19 pp.
- [4] Torsten Hoge and Gerhard Röhrle, *On inductively free reflection arrangements*, arXiv:1208.3131 (2012), 16 pp.
- [5] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460 (92h:20002)
- [6] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992. MR1217488 (94e:52014)
- [7] Bruce E. Sagan, *Why the characteristic polynomial factors*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), no. 2, 113–133, DOI 10.1090/S0273-0979-99-00775-2. MR1659875 (2000a:06021)
- [8] *Singular – A computer algebra system for polynomial computations*, available at <http://www.singular.uni-kl.de>.

- [9] Hiroaki Terao, *Arrangements of hyperplanes and their freeness. I*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 293–312. MR586451 (84i:32016a)
- [10] Hiroaki Terao, *The exponents of a free hypersurface*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 561–566. MR713280 (85a:32012)
- [11] Masahiko Yoshinaga, *On the freeness of 3-arrangements*, Bull. London Math. Soc. **37** (2005), no. 1, 126–134, DOI 10.1112/S0024609304003704. MR2105827 (2005i:52030)
- [12] Masahiko Yoshinaga, *Freeness of hyperplane arrangements and related topics*, arXiv:1212.3523v1 (2012).
- [13] Gunter Matthias Ziegler, *Algebraic combinatorics of hyperplane arrangements*, ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)—Massachusetts Institute of Technology, 1987. MR2941139
- [14] Günter M. Ziegler, *Matroid representations and free arrangements*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 525–541, DOI 10.2307/2001687. MR986703 (90k:32045)

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