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PROJECTIONS IN DUALS TO ASPLUND SPACES MADE WITHOUT SIMONS' LEMMA

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Dedicated to the 70th birthday of Charles Stegall

ABSTRACT. G. Godefroy and the second author of this note proved in 1988 that in duals to Asplund spaces there always exists a projectional resolution of the identity. A few years later, Ch. Stegall succeeded to omit from the original proof a deep lemma of S. Simons. Here, we rewrite the condensed argument of Ch. Stegall in a more transparent and detailed way. We actually show that this technology of Ch. Stegall leads to a bit stronger/richer object—the so-called projectional skeleton—recently constructed by W. Kubiś, via S. Simons' lemma and with the help of elementary submodels from logic.

In 1988, G. Godefroy and the second-named author of this note constructed in [FG] a projectional resolution of the identity in duals to general Asplund spaces; see, e.g., [F, Definitions 1.0.1 and 6.1.5]. A few years later, Ch. Stegall presented a simplified variant of this construction by omitting from the proof the use of S. Simons' lemma [F, Lemma 8.1.3] or any other substitute for it. He published it, in a rather condensed form, in [St]. In this note, we perform his argument in full detail. Thus we believe that Ch. Stegall's approach, so far overlooked, will attract broader attention. Ch. Stegall's technology led us to construct a stronger/richer object, which includes the so-called 1-*projectional skeleton*, introduced and studied recently by W. Kubiś; see [K, pages 765, 766], [KKL, pages 369, 370]. W. Kubiś' construction was based on S. Simons' lemma and was done by using elementary submodels from logic.

Let $(X, \|\cdot\|)$ be any Banach space. If V is a subspace of X and $x^* \in X^*$, then $x^*\restriction_V$ means the restriction of x^* to V; similarly, we put $M\restriction_V := \{x^*\restriction_V\colon\ x^* \in M\}$ for any set M in X^*. Alaoglu's theorem asserts that the closed unit ball B_{X^*} in X^* provided with the weak* topology is a compact space. Let $\mathcal{C}(B_{X^*})$ denote the (Banach) space of all continuous functions on this compact space, endowed with the maximum norm $\|\cdot\|$. Consider the multivalued mapping

$$\mathcal{C}(B_{X^*}) \ni f \longmapsto \{x^* \in B_{X^*}:\ f(x^*) = \max f(B_{X^*})\} =: \partial(f) \subset B_{X^*};$$

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thus $\partial : \mathcal{C}(B_{X^*}) \longrightarrow 2^{X^*}$. Clearly, for every $f \in \mathcal{C}(B_{X^*})$ the set $\partial(f)$ is non-empty and weak* compact. The mapping ∂ is also norm-to-weak* upper semicontinuous. To check this, consider any weak* open set W in X^*. We have to show that $\{f \in \mathcal{C}(B_{X^*}) : \ \partial(f) \subset W\}$ is an open set. Assume that this is not the case. Then there exists a sequence $f_0, f_1, f_2, \dots$ of elements in $\mathcal{C}(B_{X^*})$ such that $\partial(f_0) \subset W$, that $\|f_n - f_0\| \to 0$ as $n \to \infty$, and that for every $n \in \mathbb{N}$ there is $x_n^* \in \partial(f_n) \setminus W$. Recalling that all the x_n^*'s belong to the (weak* compact) set B_{X^*}, the sequence (x_n^*) has a weak* cluster point, $x^* \in B_{X^*}$, say. Also

$$\begin{aligned} \max f_0(B_{X^*}) &\leq \max f_n(B_{X^*}) + \|f_0 - f_n\| \\ &= f_n(x_n^*) - f_0(x_n^*) + \|f_0 - f_n\| + f_0(x_n^*) \leq 2\|f_n - f_0\| + f_0(x_n^*) \end{aligned}$$

for every $n \in \mathbb{N}$. And, since f_0 is weak* continuous, $\big(f_0(x^*) \leq \big)$ $\max f_0(B_{X^*}) \leq f_0(x^*)$. We got that $x^* \in \partial(f_0)$ $(\subset W)$. However, simultaneously, $x^* \notin W$ as $x_n^* \notin W$ for all $n \in \mathbb{N}$; a contradiction.

If the Banach space $(X, \|\cdot\|)$ is separable, the mapping $\partial : \mathcal{C}(B_{X^*}) \longrightarrow 2^{X^*}$ has the following (crucial) property: For every $\xi \in B_{X^*}$ there is an $f \in \mathcal{C}(B_{X^*})$ such that $\partial(f) = \{\xi\}$. (This is the very point where the approach of Ch. Stegall starts.) Indeed, since the compact space (B_{X^*}, w^*) is then metrizable, ξ is a G_δ point in it and Urysohn's lemma [E, Theorem 1.5.10] easily provides a suitable function f. More specifically (and without using any topological tool), such an f can be constructed by the formula

$$B_{X^*} \ni x^* \longmapsto 1 - \sum_{n=1}^{\infty} 2^{-n} \big| \langle \xi, s_n \rangle - \langle x^*, s_n \rangle \big| =: f(x^*) \tag{1}$$

where $\{s_1, s_2, \dots\}$ is a fixed countable dense subset of the closed unit ball B_X in X. (By [F, Lemma 2.2.1(ii)] we know that the function $C(B_{X^*}) \ni g \longmapsto \max g(B_{X^*})$ is Gateaux differentiable at $g := f$.)

In what follows, let $(X, \|\cdot\|)$ be a Banach space of an arbitrary density. The observation in the latter paragraph leads to introducing the following family of functions from $\mathcal{C}(B_{X^*})$. Let S be any non-empty set in B_X. By $\mathcal{L}(S)$ we denote the family of all functions of the form

$$B_{X^*} \ni x^* \longmapsto 1 - \sum_{n=1}^{k} 2^{-n} \big| a_n - \langle x^*, s_n \rangle \big|,$$

where $k \in \mathbb{N}$, $a_1, a_2, \dots, a_k$ are rational numbers in $[-1, 1]$, and $s_1, s_2, \dots, s_k$ are elements from S; note that $\#\mathcal{L}(S) = \#S + \aleph_0$. It is easy to check that each element of $\mathcal{L}(S)$ is weak* continuous and has maximum norm at most equal to 1. Thus $\mathcal{L}(S)$ is a subset of (the closed unit ball of) $\mathcal{C}(B_{X^*})$. We can easily check that

$$\overline{\mathcal{L}(\overline{S})} = \overline{\mathcal{L}(S)}\,; \tag{2}$$

hence, the set in (2) is separable whenever S is separable. Note that the f defined in (1) belongs to the (norm) closure of $\mathcal{L}(\{s_1, s_2, \dots\})$.

From now on, assume that $(X, \|\cdot\|)$ is an Asplund space, which (equivalently) means that the dual space X^* is weak* dentable [Ph, Theorem 2.32]. Then, we are ready to apply the selection theorem of Jayne and Rogers [JR, Theorem 8], [F, Theorem 8.1.2] to our multivalued mapping ∂ (which is already known to be norm-to-weak* upper semicontinuous and weak* compact valued). Thus we get a sequence $\lambda_j : \mathcal{C}(B_{X^*}) \longrightarrow X^*$, $j \in \mathbb{N}$, of norm-to-norm continuous mappings such that for every $f \in \mathcal{C}(B_{X^*})$ the limit $\lim_{j\to\infty} \lambda_j(f) =: \lambda_0(f)$ exists in the norm topology of X^* and moreover $\lambda_0(f) \in \partial(f)$, that is, $f\big(\lambda_0(f)\big) = \max f(B_{X^*})$. Now, we define the multivalued mapping

$$\mathcal{C}(B_{X^*}) \ni f \longmapsto \{\lambda_1(f), \lambda_2(f), \ldots\} =: \Lambda(f) \subset X^*;$$

thus $\Lambda : \mathcal{C}(B_{X^*}) \longrightarrow 2^{X^*}$. The continuity of the mappings λ_j's and (2) then guarantee that

$$\overline{\Lambda\big(\overline{\mathcal{L}(\overline{S})}\big)} = \overline{\Lambda\big(\mathcal{L}(S)\big)}\,. \tag{3}$$

Proposition 1 (Ch. Stegall). *Let* $(X, \|\cdot\|)$ *be any Asplund space and let* V *be any subspace of it. Then* $B_{V^*} \subset \overline{\Lambda\big(\mathcal{L}(B_V)\big)}\restriction_V$ *and* $\operatorname{dens} V = \operatorname{dens} \overline{\operatorname{sp} \Lambda\big(\mathcal{L}(B_V)\big)} = \operatorname{dens} V^*$*; in particular,* $B_{X^*} \subset \overline{\Lambda\big(\mathcal{L}(B_X)\big)}$.

Proof. First, assume that V is separable. Consider any $v^* \in B_{V^*}$. Find a countable dense subset $\{s_1, s_2, \ldots\}$ of B_V. Define $f := 1 - \sum_{n=1}^\infty 2^{-n}\big|\langle v^*, s_n\rangle - \langle \cdot, s_n\rangle\big|$; thus $f \in \overline{\mathcal{L}(B_V)}$. Then $f(x^*) = 1$ for every $x^* \in B_{X^*}$, with $x^*\restriction_V = v^*$, and $-1 \le f(y^*) < 1$ whenever $y^* \in B_{X^*}$ and $y^*\restriction_V \ne v^*$. Thus $\partial(f)\restriction_V = \{v^*\}$, and, since $\lambda_0(f) \in \partial(f)$, we have that $\lambda_0(f)\restriction_V = v^*$. Recalling that $\|\lambda_j(f) - \lambda_0(f)\| \longrightarrow 0$ as $j \to \infty$, we can conclude that

$$\lambda_0(f) \in \overline{\Lambda(f)} \subset \overline{\Lambda\big(\overline{\mathcal{L}(B_V)}\,\big)} = \overline{\Lambda(\mathcal{L}(B_V))}$$

by (3). Therefore, $v^* = \lambda_0(f)\restriction_V \in \overline{\Lambda\big(\mathcal{L}(B_V)\big)}\restriction_V$.

Second, assume that V is non-separable (provided that X is non-separable). Consider any $v^* \in B_{V^*}$. Find $\xi \in B_{X^*}$ so that $\xi\restriction_V = v^*$. We shall construct countable sets $S_0 \subset S_1 \subset S_2 \subset \cdots \subset B_V$ and separable subspaces $Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset V$ as follows. Pick any countable subset S_0 of B_V and any separable (rather infinite-dimensional) subspace $Z_0 \subset V$. Let $m \in \mathbb{N}$ be given and assume that we have found S_{m-1} and Z_{m-1}. Clearly, $\Lambda\big(\mathcal{L}(S_{m-1})\big)$ is a countable subset of X^*. Find then a countable set $S_{m-1} \subset S_m \subset B_V$ such that $\overline{S_m} \supset B_X \cap Z_{m-1}$ and that

$$\|\xi - x^*\| = \sup\big\{\langle \xi, s\rangle - \langle x^*, s\rangle :\ s \in S_m\big\} \ \text{ for every } x^* \in \Lambda\big(\mathcal{L}(S_{m-1})\big)\,. \tag{4}$$

Then put $Z_m := \overline{\operatorname{sp}(Z_{m-1} \cup S_m)}$. Do so for every $m \in \mathbb{N}$ and finally put $S := S_0 \cup S_1 \cup S_2 \cup \cdots$ and $Z := \overline{Z_0 \cup Z_1 \cup Z_2 \cup \cdots}$. Clearly, S is a countable set, Z is a separable subspace of V, and $\overline{S} = B_Z$. The "separable" case says that $\xi\restriction_Z \in \overline{\Lambda\big(\mathcal{L}(B_Z)\big)}\restriction_Z$. By (3) we get that $\xi\restriction_Z$ actually belongs to the set $\overline{\Lambda(\mathcal{L}(S))}\restriction_Z$. Pick $x^* \in \overline{\Lambda(\mathcal{L}(S))}$ so that $\xi\restriction_Z = x^*\restriction_Z$. For every $i \in \mathbb{N}$ find $x_i^* \in \Lambda(\mathcal{L}(S))$ so that $\|x^* - x_i^*\| < \frac{1}{i}$. A moment reflection reveals that for every $i \in \mathbb{N}$ there is $m_i \in \mathbb{N}$ so that $x_i^* \in \Lambda(\mathcal{L}(S_{m_i-1}))$. Then, by (4),

$$\begin{aligned} \|\xi - x_i^*\| &= \sup\big\{\langle \xi, s\rangle - \langle x_i^*, s\rangle :\ s \in S_{m_i}\big\} \\ &\le \big\|\xi\restriction_Z - x_i^*\restriction_Z\big\| = \big\|x^*\restriction_Z - x_i^*\restriction_Z\big\| \le \|x^* - x_i^*\| < \tfrac{1}{i} \end{aligned}$$

for every $i \in \mathbb{N}$. Therefore $\xi \in \overline{\Lambda(\mathcal{L}(S))} \subset \overline{\Lambda(\mathcal{L}(B_V))}$, and so, $v^* = \xi\restriction_V \in \overline{\Lambda\big(\mathcal{L}(B_V)\big)}\restriction_V$.

Finally, consider any subspace V of X. Choosing a dense subset M of B_V, with $\#M = \operatorname{dens} V$, we have by (3)

$$\begin{aligned} \operatorname{dens} V \le \operatorname{dens} V^* &\le \operatorname{dens} \overline{\Lambda\big(\mathcal{L}(B_V)\big)}\restriction_V \le \operatorname{dens} \overline{\Lambda\big(\mathcal{L}(B_V)\big)} = \operatorname{dens} \overline{\operatorname{sp}\Lambda\big(\mathcal{L}(B_V)\big)} \\ &= \operatorname{dens} \overline{\Lambda(\mathcal{L}(M))} \le \#\Lambda\big(\mathcal{L}(M)\big) = \#M = \operatorname{dens} V. \end{aligned}$$

□

Remark 2. Proposition 1 is crucial for dropping S. Simons' lemma from the original construction of a projectional resolution of the identity in duals to Asplund spaces; see [FG] modulo [F1]. Indeed, in [FG], instead of the mapping $\partial : \mathcal{C}(B_{X^*}) \longrightarrow 2^{B_{X^*}}$, there is considered the (so-called duality) mapping $X \ni x \longmapsto \{x^* \in S_{X^*} : \langle x^*, x\rangle = \|x\|\} =: J(x)$. This $J : X \longrightarrow 2^{S_{X^*}}$ is also norm-to-weak* upper semicontinuous and weak* compact valued. Hence, the Jayne-Rogers theorem [F, Theorem 8.1.2] yields norm-to-norm continuous mappings $D_1, D_2, \ldots$ (now) from X into X^* such that for every $x \in X$ the limit $\lim_{n\to\infty} D_n(x) =: D_0(x)$ exists in the norm topology of X^* and moreover $\langle D_0(x), x\rangle = \|x\|$. Yet, there is no obvious guarantee that the set $D_0(X)$ is equal to, or at least norm-dense in the unit sphere S_{X^*}. Here, S. Simons' lemma enters the argument and remedies the situation by showing that $D_0(X)$ is linearly dense in all of X^*.

Proposition 3. *Let $(X, \|\cdot\|)$ be a non-separable Asplund space and let $Z \subset X$ be an infinite-dimensional subspace with* $\operatorname{dens} Z < \operatorname{dens} X$. *Then there exists an overspace $Z \subset V \subset X$, with* $\operatorname{dens} V = \operatorname{dens} Z$, *such that the restriction mapping* $\overline{\operatorname{sp}\Lambda\big(\mathcal{L}(B_V)\big)} \ni x^* \longmapsto x^*\!\restriction_V =: R(x^*) \in V^*$ *is a (surjective) isometry.*

Proof. Put $\aleph := \operatorname{dens} Z$. By induction, we shall construct sets $S_0 \subset S_1 \subset S_2 \subset \cdots \subset B_X$, all of cardinality $\aleph$, as follows. Let S_0 be a dense subset of B_Z, with $\#S_0 = \aleph$. Let $m \in \mathbb{N}$ be fixed and assume that we have already found S_{m-1}. We know that $\#\mathcal{L}(S_{m-1}) = \aleph$. Hence $\overline{\operatorname{sp}\Lambda\big(\mathcal{L}(S_{m-1})\big)}$ is a subspace of X^*, with density $\aleph$; let M be a dense set in it, with $\#M = \aleph$. Find a set $S_{m-1} \subset S_m \subset B_X$, with $\#S_m = \aleph$, so big that $\operatorname{sp} S_{m-1} \cap B_X \subset \overline{S_m}$ and that $\|\xi\| = \sup\langle \xi, S_m\rangle$ for every $\xi \in M$. Then, of course, $\|x^*\| = \sup\langle x^*, S_m\rangle$ for every $x^* \in \operatorname{sp}\Lambda\big(\mathcal{L}(S_{m-1})\big)$. This finishes the induction (step).

Having constructed the S_m for every $m \in \mathbb{N}$, put $S := S_1 \cup S_2 \cup \cdots$ and $V := \overline{\operatorname{sp} S}$; then $\#S = \aleph$ and V is a subspace of X, with density $\aleph$. We shall show that this V, together with the corresponding R, serve for the conclusion of our proposition. Consider any $x^* \in \overline{\operatorname{sp}\Lambda\big(\mathcal{L}(B_V)\big)}$ and let $\varepsilon > 0$ be arbitrary. It is easy to verify that $\overline{S} = B_V$. By (3), $\overline{\Lambda(\mathcal{L}(B_V))} = \overline{\Lambda(\mathcal{L}(S))}$, and hence $\overline{\operatorname{sp}\Lambda(\mathcal{L}(B_V))} = \overline{\operatorname{sp}\Lambda(\mathcal{L}(S))}$. Thus x^* belongs to $\overline{\operatorname{sp}\Lambda\big(\mathcal{L}(S)\big)}$. Then find $\xi \in \operatorname{sp}\Lambda\big(\mathcal{L}(S)\big)$ so that $\|x^* - \xi\| < \varepsilon$. We remark that ξ belongs even to $\operatorname{sp}\Lambda\big(\mathcal{L}(S_{m-1})\big)$ for some big $m \in \mathbb{N}$. Then

$$\begin{aligned} \|x^*\| - \varepsilon &< \|\xi\| = \sup\langle \xi, S_m\rangle \le \sup\langle \xi, B_V\rangle \\ &= \|R(\xi)\| < \|R(x^*)\| + \varepsilon \le \|x^*\| + \varepsilon. \end{aligned}$$

Thus $\|x^*\| = \|R(x^*)\|$. We proved that R is an isometry. That R is surjective follows immediately from Proposition 1. □

Proposition 4. *Let V be a subspace of an Asplund space $(X, \|\cdot\|)$ such that the restriction mapping* $\overline{\operatorname{sp}\Lambda\big(\mathcal{L}(B_V)\big)} \ni x^* \longmapsto x^*\!\restriction_V =: R(x^*) \in V^*$ *is a (surjective) isometry. Then the mapping* $X^* \ni x^* \longmapsto R^{-1}\big(x^*\!\restriction_V\big) =: P(x^*)$ *is a linear norm-1 projection,* $P(X^*) = \overline{\operatorname{sp}\Lambda\big(\mathcal{L}(B_V)\big)}$, $\operatorname{dens} P(X^*) = \operatorname{dens} V$, *and* $\overline{V}^{w^*} = P^*(X^{**})$.

Proof. The first three statements concerning P immediately follow from the definition of it. The "density" statement is contained in Proposition 1. It remains to prove the last equality. That $V \subset P^*(X^{**})$ follows from the definition of P; hence $\overline{V}^{w^*} \subset P^*(X^{**})$. Assume there exists $x^{**} \in P^*(X^{**}) \setminus \overline{V}^{w^*}$. The Hahn-Banach separation theorem yields an $x^* \in X^*$ such that $\langle x^{**}, x^* \rangle \neq 0$ and $x^*\restriction_V \equiv 0$. But

$$\langle x^{**}, x^* \rangle = \langle P^*(x^{**}), x^* \rangle = \langle x^{**}, P(x^*) \rangle = \big\langle x^{**}, R^{-1}\big(x^*\restriction_V\big)\big\rangle = 0,$$

a contradiction. □

Proposition 5. *Let V_1, V_2 be two subspaces of an Asplund space $(X, \|\cdot\|)$ such that $V_1 \subset V_2$ and that the restriction mappings $\overline{\operatorname{sp}\Lambda\big(\mathcal{L}(B_{V_i})\big)} \ni x^* \longmapsto x^*\restriction_{V_i} =: R_i(x^*) \in V_i^*$, $i = 1, 2$, are (surjective) isometries. Define $P_i : X^* \to X^*$ by $P_i(x^*) = {R_i}^{-1}\big(x^*\restriction_{V_i}\big)$, $x^* \in X^*$, $i = 1, 2$. Then $P_1 \circ P_2 = P_1$ $(= P_2 \circ P_1)$, and $(P_2 - P_1)(X^*)$ is isometrical with $(V_2/V_1)^*$.*

Proof. (i) From the definition of P_i's we have immediately that $P_2 \circ P_1 = P_1$. Now, consider any $x^* \in X^*$ and any $x^{**} \in X^{**}$. Since $\overline{V_1}^{w^*} = P_1^*(X^{**})$ by Proposition 4, there is a net $(v_\tau)_{\tau \in T}$ in V_1 which weak* converges to $P_1^*(x^{**})$. Then, using the definition of R_2 and the inclusion $V_1 \subset V_2$, we get

$$\begin{aligned}\langle P_2^* \circ P_1^*(x^{**}), x^* \rangle &= \langle P_1^*(x^{**}), P_2(x^*) \rangle = \lim_{\tau \in T} \langle P_2(x^*), v_\tau \rangle \\ &= \lim_{\tau \in T} \langle {R_2}^{-1}(x^*\restriction_{V_2}), v_\tau \rangle = \lim_{\tau \in T} \langle x^*, v_\tau \rangle = \langle P_1^*(x^{**}), x^* \rangle,\end{aligned}$$

and so $P_2^* \circ P_1^* = P_1^*$, that is, $P_1 \circ P_2 = P_1$. The "isometrical" statement can be shown as in the proof of [F, Proposition 6.1.9(iv)]. □

Now we are armed to construct a projectional resolution of the identity on the dual to every Asplund space. But we prefer to present a bit stronger/richer statement.

Theorem 6. *Let $(X, \|\cdot\|)$ be a non-separable Asplund space. Then there exist a family $\mathcal{V}$ of subspaces of X and a family $\{Y_V : V \in \mathcal{V}\}$ of subspaces of X^* such that*

(i) $\bigcup\big\{V : V \in \mathcal{V}$ *and* $\operatorname{dens} V = \aleph\big\} = X$ *and* $\bigcup\big\{Y_V : V \in \mathcal{V}$ *and* $\operatorname{dens} V = \aleph\big\} = X^*$ *for every infinite cardinal* $\aleph < \operatorname{dens} X$*;*

(ii) *if* $V_1, V_2 \in \mathcal{V}$*, there is* $V \in \mathcal{V}$ *such that* $V \supset V_1 \cup V_2$ *and* $\operatorname{dens} V = \max\{\operatorname{dens} V_1, \operatorname{dens} V_2\}$*;*

(iii) *for every* $V \in \mathcal{V}$ *the assignment* $Y_V \ni x^* \longmapsto x^*\restriction_V =: R_V(x^*) \in V^*$ *is a surjective isometry, and hence the mapping* $X^* \ni x^* \longmapsto {R_V}^{-1}(x^*\restriction_V) =: P_V(x^*)$ *is a norm-1 linear projection on* X^**, with range* Y_V*, and* $\operatorname{dens} P_V(X^*) = \operatorname{dens} V$*;*

(iv) $\overline{V}^{w^*} = {P_V}^*(X^{**})$ *for every* $V \in \mathcal{V}$*;*

(v) $\mathcal{V}$ *is complete in the following sense: if* γ *is a limit ordinal, and* $\{V_\alpha : 1 \leq \alpha < \gamma\}$ *is an increasing long sequence of elements of* $\mathcal{V}$*, then* $V := \overline{\bigcup_{1 \leq \alpha < \gamma} V_\alpha}$ *belongs to* $\mathcal{V}$ *and* $Y_V = \overline{\bigcup_{1 \leq \alpha < \gamma} Y_{V_\alpha}}$*;*

(vi) *if* $V, U \in \mathcal{V}$ *and* $V \subset U$*, then* $Y_V \subset Y_U$*,* $P_V \circ P_U = P_V$ $(= P_U \circ P_V)$*, and* $(P_U - P_V)(X^*)$ *is isometrical with* $(U/V)^*$*.*

Proof. For every subspace V of X we put $Y_V := \overline{\operatorname{sp}\Lambda(\mathcal{L}(B_V))}$ and we consider the assignment $Y_V \ni x^* \longmapsto x^*\restriction_V =: R_V(x^*) \in V^*$. Let $\mathcal{V}$ consist of all subspaces V of X such that $R_V : Y_V \longrightarrow V^*$ is a surjective isometry.

(i) and (ii) are guaranteed by Propositions 3 and 1.

(iii) follows from the definition of $\mathcal{V}$ via Propositions 1 and 4.

(iv) follows from Proposition 4.

(v) Assume the premise here holds. (3) and some elementary reasoning yield

$$\begin{aligned}\Lambda(\mathcal{L}(B_V)) &\subset \overline{\Lambda(\mathcal{L}(B_V))} = \overline{\Lambda\big(\mathcal{L}(\overline{\bigcup_{\alpha<\gamma} B_{V_\alpha}})\big)} = \overline{\Lambda\big(\mathcal{L}(\bigcup_{\alpha<\gamma} B_{V_\alpha})\big)} \\ &= \overline{\bigcup_{\alpha<\gamma} \Lambda\big(\mathcal{L}(B_{V_\alpha})\big)} \subset \overline{\bigcup_{\alpha<\gamma} \overline{\mathrm{sp}\Lambda\big(\mathcal{L}(B_{V_\alpha})\big)}} = \overline{\bigcup_{\alpha<\gamma} Y_{V_\alpha}}.\end{aligned}$$

Hence

$$Y_V = \overline{\mathrm{sp}\Lambda\big(\mathcal{L}(B_V)\big)} = \overline{\bigcup_{\alpha<\gamma} Y_{V_\alpha}}. \tag{5}$$

It remains to show that $V \in \mathcal{V}$, that is, that the mapping $R_V : Y_V \longrightarrow V^*$ is a surjective isometry. By Proposition 1 and (5), R_V is surjective. Further, fix any $x^* \in Y_V$ and let $\varepsilon > 0$ be arbitrary. Find $\xi \in \bigcup_{\alpha<\gamma} Y_\alpha$ such that $\|x^* - \xi\| < \varepsilon$. Then ξ belongs to Y_α for some $\alpha < \gamma$. Now, as $V_\alpha \in \mathcal{V}$, we have

$$\begin{aligned}\|x^*\| - \varepsilon &< \|\xi\| = \|R_{V_\alpha}(\xi)\| = \big\|\xi\restriction_{V_\alpha}\big\| \le \big\|\xi\restriction_V\big\| \\ &< \big\|x^*\restriction_V\big\| + \varepsilon = \|R_V(x^*)\| + \varepsilon \le \|x^*\| + \varepsilon.\end{aligned}$$

Therefore, $\|x^*\| = \|R_V(x^*)\|$ and the mapping R_V is shown to be an isometry. Thus $V \in \mathcal{V}$.

(vi) immediately follows from Proposition 5. ☐

Corollary 7. *Let* $(X, \|\cdot\|)$ *be a non-separable Asplund space. Then* $(X^*, \|\cdot\|)$ *admits*

(i) [FG] *a projectional resolution of the identity, and also*

(ii) [K] *a* 1*-projectional skeleton.*

Proof. (i) Let μ be the first ordinal with $\#\mu = \operatorname{dens} X$. Find a dense subset $\{x_\alpha : \omega < \alpha < \mu,\ \alpha \text{ is non-limit}\}$ in X. By (i) find a separable element $V_\omega \in \mathcal{V}$. Let $\gamma \in (\omega, \mu]$ be any ordinal and assume that we already found $V_\alpha \in \mathcal{V}$ for every $\omega \le \alpha < \gamma$. Assume first that γ is non-limit. By (i) find a $V \in \mathcal{V}$ such that $V \ni x_\gamma$ and $\operatorname{dens} V = \operatorname{dens} V_{\gamma-1}$. By (ii) find a $V_\gamma \in \mathcal{V}$ such that $V_\gamma \supset V \cup V_{\gamma-1}$ and $\operatorname{dens} V_\gamma = V_{\gamma-1}$. Second, assume that γ is a limit ordinal. Then put $V_\gamma = \overline{\bigcup_{\omega<\alpha<\gamma} V_\alpha}$. By (v), we have that $V_\gamma \in \mathcal{V}$ and $Y_{V_\gamma} = \overline{\bigcup_{\omega\le\alpha<\gamma} Y_{V_\alpha}}$. Now we can immediately verify that $\big\{P_{V_\alpha} : \omega \le \alpha \le \mu\big\}$ is a projectional resolution of the identity on $(X^*, \|\cdot\|)$; see, e.g. [F, Definition 6.1.5].

(ii) We note that the relation "$\subset$" on the family $\mathcal{V}$ from Theorem 6 is a directed partial order, which is moreover σ-complete. Thus, the family $\{P_V : V \in \mathcal{V} \text{ and } \operatorname{dens} V = \aleph_0\}$ is a 1-projectional skeleton on $(X^*, \|\cdot\|)$ in the sense of the definition in [KKL, pages 369, 370]. ☐

Remark 8. 1. A 1-projectional skeleton in the dual to an Asplund space was originally constructed by W. Kubiś [K]. He started from the existence of the so-called *projectional generator* in the dual to an Asplund space [F, Proposition 8.2.1] and then he proceeded using elementary submodels from logic. It should be noted that the proof of [F, Proposition 8.2.1] is based on S. Simons' lemma and on the Jayne-Rogers selection theorem [JR, Theorem 8].

2. Let $\|\cdot\|_1,\ \|\cdot\|_2,\ \dots$ be a sequence of equivalent norms on an Asplund space X. Let $\mathcal{V}_1,\ \mathcal{V}_2,\ \dots$ be corresponding families found in Theorem 6 for these norms. It is not difficult to show that the family $\mathcal{V} := \bigcap_{i=1}^\infty \mathcal{V}_i$ (is not only non-empty but that it even) satisfies all the conditions (i) – (vi) of Theorem 6; see the proof of [BM, Proposition 1.1]. Then Corollary 7 provides one projectional resolution of the

identity and one 1-projectional skeleton on X^* which can be related to any of the norms $\|\cdot\|_1,\ \|\cdot\|_2,\ \ldots$

3. Theorem 6 can also be proved with the help of tools from [FG] (where S. Simons' lemma was used). Indeed, let the mappings $D_n : X \to X^*,\ n \in \mathbb{N}$, be from Remark 2 (they come from the Jayne-Rogers theorem) and define $D : X \to 2^{X^*}$ by $D(x) = \{D_1(x), D_2(x), \ldots\},\ x \in X$. For every subspace W of X put $Y_W := \overline{\operatorname{sp} D(W)}$ and consider the assignment $Y_W \ni x^* \longmapsto x^*\restriction_W =: R_W(x^*) \in W^*$. Let $\mathcal{W}$ consist of all subspaces W of X such that $R_W : Y_W \longrightarrow W^*$ is a surjective isometry. From [FG, pages 145, 146], where S. Simons' lemma and other things were used, we know that

$$\overline{\operatorname{sp} D(W)\restriction_W} = W^* \quad \text{for every subspace} \quad W \subset X. \tag{6}$$

(This is a weakened analogue of our Proposition 1.) By [F1, Lemma 1], for every $Z \subset X$, with $\operatorname{dens} Z < \operatorname{dens} X$ there is a subspace $Z \subset W \subset X$, with $\operatorname{dens} W = \operatorname{dens} Z$, such that R_W is an isometry from Y_W into W^*. (This is an analogue of our Proposition 3.) Using this, we easily get that $\overline{\operatorname{sp} D(W)\restriction_W} = \overline{\operatorname{sp} D(W)}\restriction_W$, and by (6), $R_W : Y_W \to W^*$ is a surjective isometry. Thus $W \in \mathcal{W}$. This way we get the properties (i) – (iv) and (vi) listed in Theorem 6. (v) can be proved as in Theorem 6, now from the norm-to-norm continuity of the D_n's.

4. We use this opportunity to fix some inaccuracy in the book [F]. In the proof of [F, Proposition 8.2.1], on page 152, the equality (2) should read as $\overline{\operatorname{sp} \Phi(B_0)\restriction_Y} = Y^*$ and the set Δ should be defined as $\overline{\Phi(B_0)\restriction_Y} \cap B_{Y^*}$.

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