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LIMITATIONS ON REPRESENTING $\mathcal{P}(X)$ AS A UNION OF PROPER SUBALGEBRAS

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ABSTRACT. For every integer $\mu \geqslant 3$, there exists a function $f_\mu : \mathbb{N}^+ \to \mathbb{N}^+$ such that the following holds: (1) $f_\mu(k) = 2k - \mu$ for k large enough; (2) if $\mathfrak{A}$ is a finite nonempty collection of subalgebras of $\mathcal{P}(X)$ such that $\bigcap \mathfrak{B}$ is not $f_\mu(\#(\mathfrak{B}))$-saturated, for all nonempty $\mathfrak{B} \subseteq \mathfrak{A}$, then $\bigcup \mathfrak{A} \neq \mathcal{P}(X)$.

1. INTRODUCTION

1.1. Consider a σ-additive measure $\mathfrak{m}$ on a set X, which has the following properties: (1) $\mathfrak{m}(\{x\}) = 0$ for all one-point sets $\{x\}$; (2) $\mathfrak{m}(X) = 1$; (3) for any $\mathfrak{m}$-measurable set M either $\mathfrak{m}(M) = 0$ or $\mathfrak{m}(M) = 1$. Such a measure is called a *two-valued measure*. If the cardinality of X is $\aleph_1$, then the simple and wonderful result of Ulam [13] holds: there exist $\aleph_1$ pairwise disjoint $\mathfrak{m}$-nonmeasurable sets. There is the following known Ulam's problem: *what is the smallest cardinality κ_* so that there should exist κ_* many two-valued measures defined on a set X of cardinality $\aleph_1$, with property that each subset of X is measurable in at least one of the measures?*

The Alaoglu-Erdös theorem [1] claims that $\kappa_* \geqslant \aleph_1$. In [12] Shelah established the consistency (relative to large cardinals) of $\kappa_* = \aleph_1$.

1.2. After measures it is natural to pass to more general objects – algebras.

Definition. By an *algebra* $\mathcal{A}$ on a set X we mean a nonempty system of subsets X possessing the following properties: (1) if $M \in \mathcal{A}$, then $X \backslash M \in \mathcal{A}$; (2) if $M_1, M_2 \in \mathcal{A}$, then $M_1 \cup M_2 \in \mathcal{A}$.

It is clear that if $M_1, M_2 \in \mathcal{A}$, then $M_1 \cap M_2, M_1 \backslash M_2 \in \mathcal{A}$; and $X \in \mathcal{A}$.

1.3. As usual, $\mathcal{P}(M)$ denotes the set of all subsets of the set M. By convention, here we fix in advance a nonempty set X, and consider subalgebras of $\mathcal{P}(X)$. Since any algebra is isomorphic to a subalgebra of $\mathcal{P}(X)$ for a certain set X, this convention does not restrict the generality of our results. Also, we shall consider two-valued measures on the same set X. Let $\mathcal{A}_1,\ldots,\mathcal{A}_k,\ldots$ be a countable sequence of algebras, with X of cardinality $\aleph_1$, and for any k the following holds: $\mathcal{A}_k$ is the algebra of all $\mathfrak{m}_k$-measurable sets of a certain two-valued measure $\mathfrak{m}_k$. In the language of algebras the Alaoglu-Erdös theorem claims that $\bigcup \mathcal{A}_k \neq \mathcal{P}(X)$.

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1.4. Let $\mathfrak{A}$ be a family of algebras, and for any $\mathcal{A} \in \mathfrak{A}$ we have $\mathcal{A} \neq \mathcal{P}(X)$. The question we would like to answer in a general form is: under what conditions does $\bigcup \mathfrak{A} \neq \mathcal{P}(X)$? If we are given two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, and each of them does not coincide with $\mathcal{P}(X)$, then it is easy to show that $\mathcal{A}_1 \cup \mathcal{A}_2 \neq \mathcal{P}(X)$. However, if we take not two but three algebras, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, each of which is not equal to $\mathcal{P}(X)$, then it can happen that $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 = \mathcal{P}(X)$.

1.5. The question formulated in the previous subsection was already studied. Let us cite the result of Grzegorek [10]. Here and from now on we use the following notation: $\#(M)$ denotes the cardinality of the set M. *Let $\mathfrak{A}$ be a family of σ-algebras on the real line $\mathbb{R}$ such that for every $\mathcal{A} \in \mathfrak{A}$ all one-element subsets of $\mathbb{R}$ belong to $\mathcal{A}$ and $\mathcal{A} \neq \mathcal{P}(\mathbb{R})$; then any of the conditions (i), (ii), (iii) implies $\bigcup \mathfrak{A} \neq \mathcal{P}(\mathbb{R})$, where (i) $\#(\mathfrak{A}) < \aleph_0$; (ii) $\#(\mathfrak{A}) < 2^{\aleph_0}$ and $2^{\aleph_0} = \aleph_1$; (iii) $\#(\mathfrak{A}) < 2^{2^{\aleph_0}}$ and Gödel's axiom of constructibility holds.*

Grzegorek notes that his result is a generalization of certain results of Ulam, Alaoglu-Erdös, Jensen, Prikry and Taylor related to Ulam's problem about sets of measures.

The present article is a further development of the theory formulated in [4–9]. Notice that all these works provide a generalization of Grzegorek's result under the conditions (i) and (ii). Also Theorems 1.1 and 1.3 of the present article are a generalization of Grzegorek's result under the condition (i), since from the formulation of Grzegorek's result, together with the condition (i), it easily follows that for any algebra $\mathcal{A} \in \mathfrak{A}$ there exist $\aleph_0$ pairwise disjoint subsets of $\mathbb{R}$, which do not belong to $\mathcal{A}$. Also the Theorems 3.1 and 3.2 of the present article are a generalization of Grzegorek's result under the condition (ii), since all algebras from $\mathfrak{A}$ are σ-algebras; and from the formulation of Grzegorek's result, together with the condition (ii), it easily follows that for any algebra $\mathcal{A} \in \mathfrak{A}$ there exist $\aleph_0$ pairwise disjoint subsets of $\mathbb{R}$, which do not belong to $\mathcal{A}$.

1.6. *Some notation and names.* By $\mathbb{N}^+$ we denote the set of all natural numbers. If $n_1, n_2 \in \mathbb{N}^+$ and $n_1 \leq n_2$, then $[n_1, n_2] = \{k \in \mathbb{N}^+ | \, n_1 \leq k \leq n_2\}$. By $\lfloor \rho \rfloor$ we denote the maximum integer $\leq \rho$. We shall frequently deal with the following situation: we consider an algebra $\mathcal{A}$, $k \in \mathbb{N}^+$, and there exist k pairwise disjoint sets belonging to $\mathcal{P}(X) \setminus \mathcal{A}$; then, as usual, we say that $\mathcal{A}$ *is not k-saturated.*

1.7. In [5], Chapter 9, we proved the following result. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be a finite sequence of algebras, for each $k \in [1, n]$ let the algebra $\mathcal{A}_k$ be not m_k-saturated, and let $\sum_{k=1}^n 2^{-\lfloor \frac{m_k+1}{2} \rfloor} \leq 1$; then $\bigcup_{k=1}^n \mathcal{A}_k \neq \mathcal{P}(X)$.* The following theorem is being published for the first time. It is implied by the previous result.

Theorem 1.1. *For any real number $\zeta > 2$ there exists a nondecreasing function $\psi_\zeta : \mathbb{N}^+ \to \mathbb{N}^+$ such that:* (1) *$\psi_\zeta(k) = \lfloor 2 \log_2 k + \zeta \cdot \log_2(\ln k) \rfloor$ starting from a certain number k;* (2) *if $\mathcal{A}_1 \dots, \mathcal{A}_n$ is a finite sequence of algebras and for each $k \in [1, n]$ the algebra $\mathcal{A}_k$ is not $\psi_\zeta(k)$-saturated, then $\bigcup_{k=1}^n \mathcal{A}_k \neq \mathcal{P}(X)$.*

It is clear that for the fixed real number $\zeta > 2$ there exist many functions $\psi_\zeta(k)$. The estimate $\lfloor 2 \log_2 k + \zeta \cdot \log_2(\ln k) \rfloor$ is rather strong. It follows from Statement 2.11 in [5]: if instead of a function ψ_ζ we take a nondecreasing function, which equals $4 \cdot \lfloor \log_5 k \rfloor$ starting from a certain number k, then our theorem is not true. The conditions of our theorem are not uniform, since each algebra $\mathcal{A}_k$ is not $\psi_\zeta(k)$-saturated, where the number $\psi_\zeta(k)$ depends on k.

1.8. In [9] we proved the following theorem with uniform conditions. Define the function f_2 on $\mathbb{N}^+ : f_2(1) = f_2(2) = 1; f_2(k) = 2k - 2$ if $k \geq 3$.[1]

Theorem 1.2. *Let $\mathfrak{A}$ be a finite nonempty family of algebras, and for every nonempty $\mathfrak{B} \subseteq \mathfrak{A}$, the algebra $\bigcap \mathfrak{B}$ is not $f_2(\#(\mathfrak{B}))$-saturated. Then $\bigcup \mathfrak{A} \neq \mathcal{P}(X)$.*

1.9. Take $\mu \in \mathbb{N}^+$ and $\mu \geq 3$. Define the function f_μ on $\mathbb{N}^+$: $f_\mu(1) = f_\mu(2) = 1; f_\mu(k) = 3\mu - 5$ if $k \in [3, 2\mu - 3]$; $f_\mu(k) = 2k - \mu$ if $k > 2\mu - 3$. Our aim is to prove the following theorem, which is stronger than Theorem 1.2 in a certain sense.

Theorem 1.3. *Let $\mathfrak{A}$ be a finite nonempty family of algebras, and the integer $\mu \geqslant 3$ is fixed. If for every nonempty $\mathfrak{B} \subseteq \mathfrak{A}$ the algebra $\bigcap \mathfrak{B}$ is not $f_\mu(\#(\mathfrak{B}))$-saturated, then $\bigcup \mathfrak{A} \neq \mathcal{P}(X)$.*

1.10. *Remark.* The proof of Theorem 1.2 as well as the proof of Theorem 1.3 uses the "combinatorics of ultrafilters" and Hall's theorem. However, the proof of Theorem 1.2 immediately follows from the simple Lemma 4.1 in [9]. Our situation is different: in order to prove Theorem 1.3 we introduce the notion of a system (Section 2.3) and the notion of a cycle (Section 2.5), then we prove the Lemma, and afterwards we introduce the notion of a chain in Section 2.7 and construct the corresponding algorithm.

2. The proof of Theorem 1.3

2.1. The theorem of this section is used below in Section 2.4. Let $\Delta_1, \ldots, \Delta_n$ be a finite sequence of subsets of a certain set Δ. Let a set $J \subseteq [1, n]$. We denote $b_J = \#(\bigcup_{j \in J} \Delta_j)$. The following theorem is well known and is a clear generalization of the classical Hall's theorem on systems of distinct representatives [11].

Theorem 2.1. *Let $\gamma \in \mathbb{N}^+$, $\gamma < n$, and for $J \subseteq [1, n]$, when $\#(J) = k$, we have: (1) $b_J \geqslant k$ if $k \leqslant \gamma$; (2) $b_J \geqslant 2k - \gamma$ if $k > \gamma$. Then it is possible to choose pairwise distinct elements $\delta_{1,1},\ldots,\delta_{n,1}$, $\delta_{\beta_1,2},\ldots,\delta_{\beta_{n-\gamma},2}$, where $1 \leqslant \beta_1 < \beta_2 < \cdots < \beta_{n-\gamma} \leqslant n$, and $\delta_{k,i} \in \Delta_k$.*

2.2. In this section we present the notions that are used in our previous publications, and those notions we shall use in the present article. We shall consider ultrafilters on X. Each ultrafilter is a point βX and vice-versa – each point βX is an ultrafilter on X. (Here, as usual, βX is the Stone-Čech compactification of X with discrete topology.) Consider a subalgebra $\mathcal{A}$ of $\mathcal{P}(X)$. We say that $a, b \in \beta X$ are $\mathcal{A}$-*equivalent* iff $a \cap \mathcal{A} = b \cap \mathcal{A}$. The simple and crucial fact is: $U \in \mathcal{P}(X) \setminus \mathcal{A}$ iff there exist $\mathcal{A}$-equivalent ultrafilters a, b such that $U \in a$, $U \notin b$. Let $[b]_\mathcal{A}$ denote the $\mathcal{A}$-equivalence class of b, and define the *kernel* of the algebra $\mathcal{A}$: $\ker \mathcal{A} = \{b \in \beta X \mid \#([b]_\mathcal{A}) > 1\}$. If $\mathcal{A} = \mathcal{P}(X)$, then $\ker \mathcal{A} = \emptyset$. If $\mathcal{A} \neq \mathcal{P}(X)$, then $\#(\ker \mathcal{A}) \geqslant 2$. If $\mathcal{A}$ is not k-saturated, then there exist $b_1,\ldots,b_k$ in $\ker \mathcal{A}$ which are pairwise not $\mathcal{A}$-equivalent. In particular, $\#(\ker \mathcal{A}) \geqslant k$.

2.3. Let us describe such a situation. **I.** Let $\mathfrak{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_\mathfrak{n}\}$ be a finite sequence of subalgebras of $\mathcal{P}(X)$. **II.** Let $\lambda \in [1, \mathfrak{n}]$ be fixed, and suppose that there exists a

[1]We denote the function by $f_2(k)$, since in its definition there is the estimate $2k - 2$. In Section 1.9 we denote the function by $f_\mu(k)$, since in its definition there is the estimate $2k - \mu$.

sparse matrix

$$\mathfrak{S} = \begin{pmatrix} s_{1,1} & \\ s_{2,1} & \\ \cdot & \\ \cdot & \\ \cdot & \\ s_{\lambda,1} & \\ s_{\lambda+1,1} & s_{\lambda+1,2} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ s_{\mathfrak{n},1} & s_{\mathfrak{n},2} \end{pmatrix},$$

of size $\mathfrak{n} \times 2$, consisting of pairwise distinct elements of βX. Suppose also that the components on the k-th row are elements of $\ker \mathcal{B}_k$. Each k-th row, where $k \in [1, \lambda]$, contains one element $s_{k,1}$; if $\lambda < \mathfrak{n}$, each row starting with $(\lambda+1)$-th contains two elements $s_{k,1}$, $s_{k,2}$.[2] We use the following natural definition: the ultrafilters $s_{k,1}$ and $s_{k,2}$, where $k > \lambda$, are said to be *neighboring*, since they are situated in the k-th row of the sparse matrix $\mathfrak{S}$. We denote by $realm(\mathfrak{S})$ the set of all nonvoid components of the sparse matrix $\mathfrak{S}$. (It is implied by the existence of the ultrafilters $s_{k,i} \in \ker \mathcal{B}_k$ that $\mathcal{B}_k \neq \mathcal{P}(X)$ for any $k \in [1, \mathfrak{n}]$. It is clear that if one requires only that $\mathcal{B}_k \neq \mathcal{P}(X)$ for any $k \in [1, \mathfrak{n}]$, then $\mathfrak{S}$, in general, may not exist. On the other hand, we do not claim that if such a sparse matrix $\mathfrak{S}$ exists, then it is unique. We consider the situation, when such sparse matrices $\mathfrak{S}$ exist and we fix one of them.) **III.** Since $s_{k,i} \in \ker \mathcal{B}_k$, the $\mathcal{B}_k$-equivalence class $[s_{k,i}]_{\mathcal{B}_k}$ contains more than one element (see Section 2.2). For each k and i such that $s_{k,i}$ is defined, take an arbitrary element from the set $[s_{k,i}]_{\mathcal{B}_k} \setminus \{s_{k,i}\}$, and denote it by $t_{k,i}$. Thus a sparse matrix

$$\mathfrak{T} = \begin{pmatrix} t_{1,1} & \\ t_{2,1} & \\ \cdot & \\ \cdot & \\ \cdot & \\ t_{\lambda,1} & \\ t_{\lambda+1,1} & t_{\lambda+1,2} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ t_{\mathfrak{n},1} & t_{\mathfrak{n},2} \end{pmatrix},$$

of size $\mathfrak{n} \times 2$, consisting of elements of βX, is constructed; each k-th row, where $k \in [1, \lambda]$, contains one element $t_{k,1}$; if $\lambda < \mathfrak{n}$, then each row starting from $(\lambda+1)$-th contains two elements $t_{k,1}, t_{k,2}$. (In general, the elements of the sparse matrix $\mathfrak{T}$ are not necessarily pairwise distinct. It is clear that fixing of $\mathfrak{S}$ does not provide the uniqueness of the sparse matrix $\mathfrak{T}$; we fix $\mathfrak{S}$ and then we choose one such $\mathfrak{T}$.)

If the conditions **I, II, III** hold, we say that there is a *system* $\mathfrak{Z} = (\mathfrak{B}, \mathfrak{n}, \lambda, \mathfrak{S}, \mathfrak{T})$. If $\lambda = \mathfrak{n}$, the system is called *singular*. In a singular system, the sparse matrices $\mathfrak{S}$

[2]As usual, we call $\mathfrak{S}$ a *sparse* matrix, since it has empty entries (in our case, if $k \in [1, \lambda]$, then the element $s_{k,2}$ does not exist; however, if $\lambda < n$ and $k \in [\lambda+1, n]$, then the element $s_{k,2}$ exists).

and $\mathfrak{T}$ essentially contain only one column. If $\lambda < \mathfrak{n}$, the system is called *regular*. If we say that we have a system, then it possibly is singular or regular.

2.4. Let us address a scenario in which a system as in Section 2.3 exists. Suppose that $\mathfrak{A}$ and μ satisfy the hypothesis of Theorem 1.3. Let $\mathfrak{A} = \{\mathcal{A}_1, ..., \mathcal{A}_n\}$ be a certain enumeration of $\mathfrak{A}$. Assume that $\mu < n$. For all $J \subseteq [1, n]$, where $\#(J) = k > 0$, we have that the algebra $\bigcap_{j\in J} \mathcal{A}_j$ is not $f_\mu(k)$-saturated. In particular, $\#\left(\bigcup_{j\in J} \ker \mathcal{A}_j\right) \geqslant f_\mu(k)$; since all $\mathcal{A}_k$ are proper subalgebras, we have $\#(\ker \mathcal{A}_k) \geqslant 2$ for any $k \in [1, n]$. Thus, the sets $\ker \mathcal{A}_1$,...,$\ker \mathcal{A}_k$ satisfy the conditions of Theorem 2.1. Consequently, there exists a regular system $\left(\mathfrak{A}, n, \mu, \tilde{\mathfrak{S}}, \tilde{\mathfrak{T}}\right)$; of course, the indices in the sequence $\mathcal{A}_1$,..., $\mathcal{A}_n$ should be changed if it is needed.

2.5. Return to the system $\mathfrak{Z}$. A sequence of ultrafilters s_{k_1,i_1},..., $s_{k_\nu,i_\nu} \in \mathfrak{S}$ is said to be a *cycle of length* ν of the system $\mathfrak{Z}$ if the following conditions hold: **(a)** k_1,..., k_ν are pairwise distinct numbers; **(b)** ν is an odd number $\geqslant 3$; **(c)** $t_{k_1,i_1} = s_{k_2,i_2}$,..., $t_{k_{\nu-1},i_{\nu-1}} = s_{k_\nu,i_\nu}$, $t_{k_\nu,i_\nu} = s_{k_1,i_1}$; **(d)** either all numbers k_j are $\leqslant \lambda$ or among them there is only one $> \lambda$. If all numbers $k_j \leqslant \lambda$, we say that our cycle is of the *first kind*. If among the numbers k_j there is one $> \lambda$, we say that our cycle is of the *second kind*. If we say that the system $\mathfrak{Z}$ has a cycle $\mathfrak{C}$, then it is possible that $\mathfrak{C}$ is of the first kind or of the second kind.

2.6. A *triad* $(\mathfrak{D}, S, T)$ is a triple, where $\mathfrak{D}$ is a finite (possibly empty) set of subalgebras of $\mathcal{P}(X)$; S, T are disjoint finite subsets of βX, and if $\mathfrak{D} \neq \emptyset$, then for each algebra $D \in \mathfrak{D}$ there exist D-equivalent ultrafilters a_D, b_D such that $a_D \in S$, $b_D \in T$. If $\mathfrak{D} = \emptyset$, then we require that $S = T = \emptyset$. The next lemma is about the system $\mathfrak{Z}$ that was introduced in Section 2.3.

Lemma. Assume: **(i)** *given the regular system* $\mathfrak{Z}$*, and the positive integer* $\lambda_0 \geqslant \lambda$ *is fixed;* **(ii)** $\#(\ker \mathcal{B}_{m_1} \cup \ker \mathcal{B}_{m_2} \cup \ker \mathcal{B}_{m_3}) \geqslant 3\lambda_0 - 5$ *for any three pairwise distinct* m_1*,* m_2*,* $m_3 \in [1, \mathfrak{n}]$*;* **(iii)** *given the triad* $(\mathfrak{D}, S, T)$*, and* $\mathfrak{B} \cap \mathfrak{D} = \emptyset$*,* $(S \cup T) \bigcap realm(\mathfrak{S}) = \emptyset$*,* $\#(S \cup T) \leqslant 3(\lambda_0 - \lambda)$*;* **(iv)** *the system* $\mathfrak{Z}$ *has a cycle* $\mathfrak{C}$ *of length* ν*.*

Then: **(i′)** *there exists a system* $\mathfrak{Z}' = (\mathfrak{B}', \mathfrak{n}', \lambda', \mathfrak{S}', \mathfrak{T}')$*, and* $\mathfrak{B}' \subset \mathfrak{B}$*,* $\mathfrak{n}' = \mathfrak{n} - \nu + 1$*,* $\lambda' < \lambda$*;* **(ii′)** *there exists the triad* $(\mathfrak{D}', S', T')$*, and* $\mathfrak{D}' = \mathfrak{D} \cup (\mathfrak{B} \setminus \mathfrak{B}')$*,* $(S' \cup T') \bigcap realm\,(\mathfrak{S}') = \emptyset$*,* $\#(S' \cup T') \leqslant 3(\lambda_0 - \lambda')$*.*

Remark. If the cycle $\mathfrak{C}$ is of the first kind, then $\lambda \geqslant \nu \geqslant 3$. If the cycle $\mathfrak{C}$ is of the second kind, then $\lambda \geqslant \nu - 1 \geqslant 2$. If $\mathfrak{D} \neq \emptyset$, then obviously $\lambda_0 > \lambda$. Let $\mathfrak{B}' = \{\mathcal{B}'_1, ..., \mathcal{B}'_{\mathfrak{n}'}\}$. If $\mathfrak{n}' \geqslant 3$, then obviously $\#\left(\ker \mathcal{B}'_{m_1} \cup \ker \mathcal{B}'_{m_2} \cup \ker \mathcal{B}'_{m_3}\right) \geqslant 3\lambda_0 - 5$ for any three pairwise distinct $m_1, m_2, m_3 \in [1, \mathfrak{n}']$. In the first part of the statement of our lemma we assume that $\mathfrak{B} \cap \mathfrak{D} = \emptyset$, while in the second part it should be $\mathfrak{B}' \cap \mathfrak{D}' = \emptyset$. However, it is obvious, since $\mathfrak{D}' = \mathfrak{D} \cup (\mathfrak{B} \setminus \mathfrak{B}')$. If $n = \lambda + 1$, then it is possible that the system $\mathfrak{Z}'$ is singular. Obviously, it can happen that the system $\mathfrak{Z}'$ does not have a cycle.

Proof. By the conditions of our lemma, there exists the cycle $\mathfrak{C} = \{s_{k_1,i_1}, ..., s_{k_\nu.i_\nu}\}$. The proof is divided into two cases. In Case 1, $\mathfrak{C}$ is of the first kind. In Case 2, $\mathfrak{C}$ is of the second kind. Put $G = \{s_{k,i} \in \mathfrak{S} \mid k \geqslant \lambda + 1\}$. Case 1 is divided into two subcases. In Case 1-1, $\left(\bigcup_{j=1}^{\nu} \ker \mathcal{B}_{k_j}\right) \cap G \neq \emptyset$. In Case 1-2, $\left(\bigcup_{j=1}^{\nu} \ker \mathcal{B}_{k_j}\right) \cap G = \emptyset$.

Case 1-1. Let $s_{\lambda+1,1} \in \ker \mathcal{B}_{k_\nu}$. Put $\mathfrak{B}' = \mathfrak{B} \setminus \{\mathcal{B}_{k_1}, ..., \mathcal{B}_{k_{\nu-1}}\}$. The sparse matrix $\mathfrak{S}'$ is obtained from $\mathfrak{S}$ by removing the rows $k_1, ..., k_\nu$, and instead of the $(\lambda+1)$-th row we take two rows, each of which contains one ultrafilter; the first row contains $s_{\lambda+1,1}$ (this row corresponds to the algebra $\mathcal{B}_{k_\nu}$); the second row contains $s_{\lambda+1,2}$ (this row corresponds to the algebra $\mathcal{B}_{\lambda+1}$). It is clear that $\mathfrak{n}' = \mathfrak{n} - \nu + 1$, $\lambda' = \lambda - \nu + 2$. We have $\mathfrak{D}' = \mathfrak{D} \cup (\mathfrak{B} \setminus \mathfrak{B}') = \mathfrak{D} \cup \{\mathcal{B}_{k_1}, ..., \mathcal{B}_{k_{\nu-1}}\}$. Put $S' = S \cup \{s_{k_j,i_j} \mid j \text{ is odd and } j \leqslant \nu\}$, $T' = T \cup \{s_{k_j,i_j} \mid j \text{ is even and } j < \nu\}$. For the algebra $\mathcal{B}_{k_1}$ we consider $\mathcal{B}_{k_1}$-equivalent ultrafilters s_{k_1,i_1}, s_{k_2,i_2} ($s_{k_1,i_1} \in S'$, $s_{k_2,i_2} \in T'$). For the algebra $\mathcal{B}_{k_2}$ we consider $\mathcal{B}_{k_2}$-equivalent ultrafilters s_{k_2,i_2}, s_{k_3,i_3} ($s_{k_2,i_2} \in T'$, $s_{k_3,i_3} \in S'$), etc. For the algebra $\mathcal{B}_{k_{\nu-1}}$ we consider $\mathcal{B}_{k_{\nu-1}}$-equivalent ultrafilters $s_{k_{\nu-1},i_{\nu-1}}$, s_{k_ν,i_ν} ($s_{k_{\nu-1},i_{\nu-1}} \in T'$, $s_{k_\nu,i_\nu} \in S'$). If $\nu = 3$, then $\#(S' \cup T') \leqslant 3(\lambda_0 - \lambda')$. If $\nu \geqslant 5$, then $\#(S' \cup T') < 3(\lambda_0 - \lambda')$. It is clear that if $\mathfrak{n} = \lambda + 1$, then S' essentially contains only one column.

Remark. In order to construct the system $\mathfrak{Z}'$ we present $\mathfrak{B}'$ in a natural way as a sequence of algebras $\{\mathcal{B}'_1, ..., \mathcal{B}'_{\mathfrak{n}'}\}$, and correspondingly we order the sparse matrix $\mathfrak{S}'$; denote the elements of $\mathfrak{S}'$ by $s'_{k,i}$. Fix the sparse matrix $\mathfrak{T}'$ (see Section 2.3); denote the elements of $\mathfrak{T}'$ by $t'_{k,i}$. It is clear that $s'_{k,i}$, $t'_{k,i}$ are $\mathcal{B}'_k$-equivalent ultrafilters for each $k \in [1, \mathfrak{n}']$.

Case 1-2. We have $\#\left(\left(\bigcup_{j=1}^{\nu} \ker \mathcal{B}_{k_j}\right) \setminus (realm(\mathfrak{S}) \cup S \cup T)\right) \geqslant 3\lambda_0 - 5 - (\lambda + 3(\lambda_0 - \lambda)) = 2\lambda - 5 > 0$, since the cycle $\mathfrak{C}$ is of the first kind and therefore $\lambda \geqslant 3$. Let $s_* \in \ker \mathcal{B}_{k_\nu} \setminus (realm(\mathfrak{S}) \cup S \cup T)$. We define $\mathfrak{B}'$ as above. The sparse matrix $\mathfrak{S}'$ is obtained from $\mathfrak{S}$ by removing the rows $k_1, ..., k_\nu$, and instead of the k_ν-th row we take the row which contains only one ultrafilter s_* (this row corresponds to the algebra $\mathcal{B}_{k_\nu}$). It is clear that $\lambda' = \lambda - \nu + 1$. We define $\mathfrak{D}'$, S', T' as above. In this case $(S' \cup T') < 3(\lambda_0 - \lambda')$. Here we have to repeat what we said in the Remark in Case 1-1.

Case 2. Let $s_{k_\nu,i_\nu} = s_{\lambda+1,2}$. Again we define $\mathfrak{B}'$ as above. The sparse matrix $\mathfrak{S}'$ is obtained from $\mathfrak{S}$ by removing the rows $k_1, ..., k_\nu$, and instead of the $(\lambda+1)$-th row we take the row which contains only one ultrafilter $s_{\lambda+1,1}$ (this row corresponds to the algebra $\mathcal{B}_{\lambda+1}$). It is clear that $\lambda' = \lambda - \nu + 1$. Again we define $\mathfrak{D}'$, S', T' as above. Here we have $(S' \cup T') < 3(\lambda_0 - \lambda')$. It is clear that if $\mathfrak{n} = \lambda + 1$, then $\mathfrak{S}'$ contains only one column. Here we have to repeat what we said in the Remark in Case 1-1.

Let us sum up. The required system $\mathfrak{Z}' = (\mathfrak{B}', \mathfrak{n}', \lambda', \mathfrak{S}', \mathfrak{T}')$ and the triad $(\mathfrak{D}', S', T')$ are constructed. □

Remark. If $\mathfrak{S}'$ essentially contains only one column (it can occur in Cases 1-1 and 2), then the system $\mathfrak{Z}'$ is singular.

2.7. We are now ready to prove Theorem 1.3. Suppose that $\mathfrak{A} = \{\mathcal{A}_1, ..., \mathcal{A}_n\}$ and μ are as in the hypothesis of Theorem 1.3. Without loss of generality suppose that $n > \mu$. Then, as is shown in Section 2.4, we have a regular system $\left(\mathfrak{A}, n, \mu, \tilde{\mathfrak{S}}, \tilde{\mathfrak{T}}\right)$, which we denote as $\mathfrak{Z}_1$.

Part 1. We start with the system $\mathfrak{Z}_1$ and aim to end with the system $\mathfrak{Z}_* = (\mathfrak{A}_*, n_*, \mu_*, \mathfrak{S}_*, \mathfrak{T}_*)$ which does not have a cycle. If $\mathfrak{Z}_1$ does not have a cycle, then put $\mathfrak{A}^* = \mathfrak{A}$, $n_* = n$, $\mathfrak{S}_* = \tilde{\mathfrak{S}}$, $\mu_* = \mu$, $\mathfrak{T}_* = \tilde{\mathfrak{T}}$ and $\mathfrak{Z}_1 = \mathfrak{Z}_* = (\mathfrak{A}_*, n_*, \mu_*, \mathfrak{S}_*, \mathfrak{T}_*)$; and there is the triad $(\mathfrak{D}_*, S_*, T_*)$, where $\mathfrak{D}_* = \emptyset$. Otherwise, $\mathfrak{Z}_1$ satisfies the condition of the Lemma if we put $\mathfrak{B} = \mathfrak{A}$, $\mathfrak{n} = n$, $\lambda_0 = \lambda = \mu$, $\mathfrak{S} = \tilde{\mathfrak{S}}$, $\mathfrak{T} = \tilde{\mathfrak{T}}$; here $\mathfrak{D} = \emptyset$. Applying the Lemma, we construct the system $\mathfrak{Z}_2 = (\mathfrak{A}_2, n_2, \mu_2, \mathfrak{S}_2, \mathfrak{T}_2)$, where $\mathfrak{A}_2 \subset \mathfrak{A}$, $n_2 < n$, $\mu_2 < \mu$, and there is the triad $(\mathfrak{D}_2, S_2, T_2)$, where $\mathfrak{D}_2 = \mathfrak{A} \setminus \mathfrak{A}_2$, $(S_2 \cup T_2) \bigcap realm\,(\mathfrak{S}_2) = \emptyset$, $\#\,(S_2 \cup T_2) \leqslant 3\,(\mu - \mu_2)$. Without loss of generality suppose that the system $\mathfrak{Z}_2$ is regular. If it does not have a cycle, replace the index 2 by $*$; i.e. our system will be of the form $\mathfrak{Z}_* = (\mathfrak{A}_*, n_*, \mu_*, \mathfrak{S}_*, \mathfrak{T}_*)$, and our triad will be of the form $(\mathfrak{D}_*, S_*, T_*)$. Otherwise we continue our process. At the end we arrive at the system $\mathfrak{Z}_p = (\mathfrak{A}_p, n_p, \mu_p, \mathfrak{S}_p, \mathfrak{T}_p)$, which does not have a cycle, and the corresponding triad will be $(\mathfrak{D}_p, S_p, T_p)$, replacing the index p by $*$.

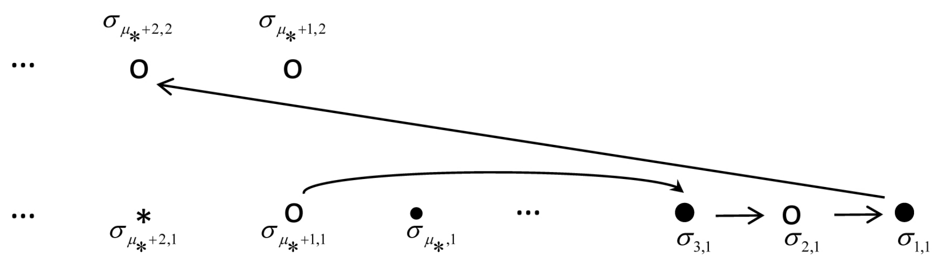
Finally, we consider the system $\mathfrak{Z}_* = (\mathfrak{A}_*, n_*, \mu_*, \mathfrak{S}_*, \mathfrak{T}_*)$, which does not have a cycle, and the triad $(\mathfrak{D}_*, S_*, T_*)$. Let $\mathfrak{A} = \left\{\mathcal{A}_1^*, ..., \mathcal{A}_{n_*}^*\right\}$. Denote the elements of $\mathfrak{S}_*$ by $\sigma_{k,i}$; denote the elements of $\mathfrak{T}_*$ by $\tau_{k,i}$. It is clear that $\sigma_{k,i}$, $\tau_{k,i}$ are $\mathcal{A}_k^*$-equivalent ultrafilters for each $k \in [1, n_*]$. It is clear that $\mathfrak{D}_* = \mathfrak{A} \setminus \mathfrak{A}_*$, $(S_* \cup T_*) \bigcap realm\,(\mathfrak{S}_*) = \emptyset$. Suppose that the system $\mathfrak{Z}_*$ is regular.

Part 2. Consider the finite set $Z = S_* \bigcup realm\,(\mathfrak{S}_*)$. We start the process as a result of which we construct the set $Z_1 \subset Z$, which has the following property: for each algebra $\mathcal{A}_k \in \mathfrak{A}$ there exist $\mathcal{A}_k$-equivalent ultrafilters s_k, t_k such that $s_k \in Z_1$, $t_k \notin Z_1$. Define $Z_2 = Z \setminus Z_1$. We have to determine which of the elements of Z should be inserted into Z_1 and which should be inserted into Z_2. Let us insert all the elements of S_* into Z_1. Now we have to determine which of the elements of $realm\,(\mathfrak{S}_*)$ should be inserted into Z_1 and which should be inserted into Z_2. We shall use the following terminology processing our constructions. If it is not known yet where $\sigma_{k,i}$ is (in Z_1 or in Z_2), then the ultrafilter $\sigma_{k,i}$ is called *indeterminate.* An ultrafilter for which it is known whether it belongs to Z_1 or to Z_2 is called *determinate.* Thus we already know that all the ultrafilters in S_* are determinate.

We say that a sequence of ultrafilters $\mathfrak{H} = \{\sigma_{r_1,m_1}, ..., \sigma_{r_\ell,m_\ell}\}$, where $\sigma_{r_i,m_i} \in \mathfrak{S}_*$, form a *chain* if the following hold: **(1)** the numbers r_1,...,r_ℓ are pairwise distinct; **(2)** if $\ell > 1$, then $\tau_{r_1,m_1} = \sigma_{r_2,m_2}$; if $\ell > 2$, then $\tau_{r_1,m_1} = \sigma_{r_2,m_2}$, $\tau_{r_2,m_2} = \sigma_{r_3,m_3}$, etc.; **(3)** if $\ell > 1$, then the numbers r_2,...,r_ℓ are less than $\mu_* + 1$; **(4)** all the elements of $\mathfrak{H}$ are indeterminate; **(5)** the maximality property: there does not exist $\sigma_{r_{\ell+1},m_{\ell+1}}$ such that the sequence of ultrafilters $\sigma_{r_1,m_1}, ..., \sigma_{r_\ell,m_\ell}, \sigma_{r_{\ell+1},m_{\ell+1}}$ satisfies conditions (1)-(4).

Describe the rules according to which all the elements of $\mathfrak{H}$, and possibly other elements of $realm\,(\mathfrak{S}_*)$ (there are at most two such "other elements"), are inserted into Z_1 or into Z_2. All σ_{r_j,m_j}, where j is odd, are contained by one set (or Z_1 or Z_2). All σ_{r_j,m_j}, where j is even, are contained by the other set (or Z_2 or Z_1). Therefore, if we know where the last element $\sigma_{r_\ell,m_\ell} \in \mathfrak{H}$ is, then we know where the other elements of $\mathfrak{H}$ are. Consider all cases for τ_{r_ℓ,m_ℓ}: **(a)** the ultrafilter τ_{r_ℓ,m_ℓ} is determinate; if $\tau_{r_\ell,m_\ell} \in Z_1$, then $\sigma_{r_\ell,m_\ell} \in Z_2$; if $\tau_{r_\ell,m_\ell} \in Z_2$, then $\sigma_{r_\ell,m_\ell} \in Z_1$; **(b)** if $\tau_{r_\ell,m_\ell} \notin Z$, then $\sigma_{r_\ell,m_\ell} \in Z_1$; **(c)** $\tau_{r_\ell,m_\ell} = \sigma_{r_p,m_p} \in \mathfrak{H}$, and $\ell - p$ is odd ($\ell - p$ cannot be even; otherwise the system $\mathfrak{Z}_*$ has a cycle); put $\sigma_{r_\ell,m_\ell} \in Z_1$ (it is clear that $\sigma_{r_p,m_p} \in Z_2$); **(d)** $\tau_{r_\ell,m_\ell} = \sigma_{k',i'}$, $k' > \mu_*$, $\sigma_{k',i'} \neq \sigma_{r_1,m_1}$ and the ultrafilter $\sigma_{k',i'}$ is indeterminate; then $\sigma_{r_\ell,m_\ell} \in Z_1$, $\sigma_{k',i'} \in Z_2$.

If the first element $\sigma_{r_1,m_1} \in \mathfrak{H}$ has the neighboring ultrafilter $\sigma_{r_1,m'}$ (it is possible if $r_1 > \mu_*$) and $\sigma_{r_1,m'}$ is an indeterminate ultrafilter, put $\sigma_{r_1,m'} \in Z_2$. If case (d) takes place, then it could occur that $\sigma_{k',i'} = \sigma_{r_1,m'}$. So, if we have constructed a chain, we know into which of the two sets Z_1 and Z_2 the corresponding ultrafilters must be inserted, i.e. all the ultrafilters of a chain and at most two ultrafilters which do not belong to this chain. Let us construct the first chain, where $\sigma_{r_1,m_1} = \sigma_{\mu_*+1,1}$, and insert the corresponding ultrafilters into Z_1 and Z_2. Anyway, $\sigma_{\mu_*+1,2} \in Z_2$. If condition (d) holds and $k' \neq r_1 = \mu_* + 1$, then for the second chain the first element is the ultrafilter neighboring $\sigma_{k',i'}$. (It is clear that this ultrafilter is indeterminate.) If condition (d) with $k' \neq r_1 = \mu_* + 1$ does not hold, then for the second chain the first element is an arbitrary indeterminate ultrafilter $\sigma_{k_*,1}$, where $k_* > \mu_*$ (if such an ultrafilter $\sigma_{k_*,1}$ exists). It is clear that the neighboring ultrafilter $\sigma_{k_*,2}$ is indeterminate. Insert the corresponding ultrafilters, among which are all the ultrafilters of the second chain, into Z_1 and Z_2, etc. We demonstrate our constructions in the figure. Here the first chain is the sequence of four elements:



$\sigma_{\mu_*+1,1}$, $\sigma_{3,1}$, $\sigma_{2,1}$, $\sigma_{1,1}$; $\tau_{\mu_*+1,1} = \sigma_{3,1}$, $\tau_{3,1} = \sigma_{2,1}$, $\tau_{2,1} = \sigma_{1,1}$, $\tau_{1,1} = \sigma_{\mu_*+2,2}$; and condition (d) with $k' = \mu_* + 2$ holds. For the elements of the first chain we have: $\sigma_{3,1}, \sigma_{1,1} \in Z_1$ and $\sigma_{\mu_*+1,1}, \sigma_{2,1} \in Z_2$. Moreover, $\sigma_{\mu_*+1,2}, \sigma_{\mu_*+2,2} \in Z_2$ and $\sigma_{\mu_*+2,1}$ is the first element of the second chain.

Continuing to construct chains until all the elements of $realm\,(\mathfrak{S})$ become determinate, such a situation can occur: all the ultrafilters $\sigma_{k,1}$, $\sigma_{k,2}$, where $\mu_* < k \leqslant n_*$ are determinate, but some of the ultrafilters $\sigma_{k,1}$, where $k \leqslant \mu_*$, remain indeterminate. Then we shall construct chains whose first elements are indeterminate ultrafilters of the kind $\sigma_{k,1}$, where $k \leqslant \mu_*$, and, using our rules, we determine which of them must be inserted into Z_1 and which of them must be inserted into Z_2. Since the set $realm\,(\mathfrak{S}_*)$ is finite, our process will terminate after the finite number of steps, and the sets Z_1, Z_2 will be constructed. It is easy to check that the goal described above is achieved: for each $k \in [1, n]$ there exist $\mathcal{A}_k$-equivalent ultrafilters s_k, t_k, such that $s_k \in Z_1$, $t_k \notin Z_1$.

Part 3. The aim of this part is to state and prove the claim which is crucial for our area of research. This claim first appeared in [4] and then was used in our works [5], [6], [7], [8], [9]. It is the "ideological foundation" of the proof of our theorem and of theorems from the next section. Let $M \subseteq \beta X$. Denote by $\overline{M}$ the closure of M in βX.

Claim. Let $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$, $\Lambda \neq \emptyset$, be a family of subalgebras of $\mathcal{P}(X)$.[3] Then $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda \neq \mathcal{P}(X)$ if and only if there exist disjoint closed sets $S, T \subset \beta X$ such that for each $\lambda \in \Lambda$ there exist $\mathcal{A}_\lambda$-equivalent ultrafilters s_λ, t_λ, and $s_\lambda \in S$, $t_\lambda \in T$.

For completeness, let us give the proof of this Claim. Let $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda \neq \mathcal{P}(X)$; i.e. there exists $Q \in \mathcal{P}(X) \setminus \mathcal{A}_\lambda$ for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ take $\mathcal{A}_\lambda$-equivalent ultrafilters s_λ, t_λ, and $s_\lambda \ni Q$, $t_\lambda \ni X \setminus Q$. Let $S_* = \{s_\lambda \mid \lambda \in \Lambda\}$, $T_* = \{t_\lambda \mid \lambda \in \Lambda\}$,

[3] Although in this paper we consider only finite and countable families of algebras, in this Claim it is possible that $\#\,(\Lambda) > \aleph_0$.

$S = \overline{S_*}$, $T = \overline{T_*}$. Obviously, $S \subseteq \overline{Q}$, $T \subseteq \overline{X \setminus Q}$. Since $\overline{Q} \cap \overline{X \setminus Q} = \emptyset$ (X is equipped with the discrete topology), $S \cap T = \emptyset$.

Now suppose that there exist sets S, T which satisfy the condition of our Claim. Take $Q \in \mathcal{P}(X)$, such that $S \subseteq \overline{Q}$, $T \subseteq \overline{X \setminus Q}$. Clearly, $Q \in \mathcal{P}(X) \setminus \mathcal{A}_\lambda$ for all $\lambda \in \Lambda$. The proof of the Claim is completed. □

Part 4. Consider the finite set Z_1 and the ultarfiliters s_k, t_k from part 2. Put $Z' = \{t_k |\ k \in [1, n]\}$. Obviously, Z' is a finite set and $Z_1 \cap Z' = \emptyset$. The sets Z_1, Z' are closed since they are finite, and for each $k \in [1, n]$, $s_k \in Z_1, t_k \in Z'$. By the Claim from part 3, we have $\bigcup \mathfrak{A} \neq \mathcal{P}(X)$. The proof of Theorem 1.3 is complete. □

3. Generalizations

3.1. In [9] there is given a natural generalization of Theorem 1.2 for a countable set of σ-algebras. For the proof of this generalization we used the proof of Theorem 1.2 with the techniques which are developed in [9]. Using the proof of Theorem 1.3 and considerations from [9], Sections 5.12 and 5.13, we can prove a natural generalization of Theorem 1.3 for a countable set of σ-algebras. Let us formulate the theorem which is a generalization of Theorem 1.2 as well as of Theorem 1.3 for a countable set of σ-algebras. Here we used the functions f_2 and f_μ, $\mu \geqslant 3$, from Theorems 1.2 and 1.3.

Theorem 3.1. *Let $\mathfrak{A}$ be a family of σ-algebras, $\#(\mathfrak{A}) = \aleph_0$, and the integer $\mu \geqslant 2$ is fixed. Suppose that for any nonempty finite $\mathfrak{E} \subset \mathfrak{A}$, the algebra $\bigcap \mathfrak{E}$ is not $f_\mu(\#(\mathfrak{E}))$-saturated. Then $\bigcup \mathfrak{A} \neq \mathcal{P}(X)$.*

3.2. The following definition was introduced in [4]: an algebra $\mathcal{A}$ is said to be *almost σ-algebra* if for any family of sets $\{M_k\}_{k \in \mathbb{N}^+}$ such that $\mathcal{P}(M_k) \subset \mathcal{A}$ for each k, we have $\mathcal{A} \ni \bigcup_{k \in \mathbb{N}^+} M_k$. By consideration from [5], Chapter 11, the following natural generalization of Theorem 1.1 for a countable sequence of almost σ-algebras is true. Here we use a function ψ_ζ from Theorem 1.1.

Theorem 3.2. *Let $\{\mathcal{A}_k\}_{k \in \mathbb{N}^+}$ be a sequence of almost σ-algebras, and a real number $\zeta > 2$ fixed. Suppose that for any k the algebra $\mathcal{A}_k$ is not $\psi_\zeta(k)$-saturated. Then $\bigcup_{k \in \mathbb{N}^+} \mathcal{A}_k \neq \mathcal{P}(X)$.*

3.3. *Remark.* Using the notion of *absolute* introduced by Gleason in [3], we can construct the family of algebras $\mathfrak{B}$, $\#(\mathfrak{B}) = \aleph_0$, and each algebra $\mathcal{B} \in \mathfrak{B}$ has the following properties: 1) it is not an almost σ-algebra; 2) the family $\mathcal{P}(X) \setminus \mathcal{B}$ contains $\aleph_0$-sized subfamily of pairwise disjoint sets; 3) $\bigcup \mathfrak{B} = \mathcal{P}(X)$ (see [5], Chapter 5). Therefore, Theorem 3.1 is not true if one omits the assumption that algebras of $\mathfrak{A}$ are σ-algebras. Analogously, Theorem 3.2 is not true if one omits the assumption that algebras $\mathcal{A}_k$ are almost σ-algebras.

3.4. *Remark.* If in the Erdös-Alaoglu theorem (see [1] and Sections 1.1, 1.3) we suppose that $\#(X) = 2^{\aleph_0}$ instead of $\#(X) = \aleph_1$, then its statement remains true. It is the Gitik-Shelah theorem [2]. It is essentially used in the proofs of Theorems 3.1 and 3.2. Therefore, assuming the continuum-hypothesis to be true, in the proofs of Theorems 3.1 and 3.2 we can use the simple theorem of Erdös-Alaoglu instead of the nontrivial and complicated theorem of Gitik-Shelah.

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