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HYPERBOLICITY, TRANSITIVITY AND THE TWO-SIDED LIMIT SHADOWING PROPERTY

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Abstract. We explore the notion of the two-sided limit shadowing property introduced by Pilyugin in 2007. Indeed, we characterize the C^1-interior of the set of diffeomorphisms with such a property on closed manifolds as the set of transitive Anosov diffeomorphisms. As a consequence we obtain that all codimension-one Anosov diffeomorphisms have the two-sided limit shadowing property. We also prove that every diffeomorphism f with such a property has neither sinks nor sources and is transitive Anosov (in the Axiom A case). In particular, no Morse-Smale diffeomorphism has the two-sided limit shadowing property. Finally, we prove that C^1-generic diffeomorphisms with the two-sided limit shadowing property are transitive Anosov. All these results allow us to reduce the well-known conjecture about the transitivity of Anosov diffeomorphisms to prove that the set of diffeomorphisms with the two-sided limit shadowing property coincides with the set of Anosov diffeomorphisms.

1. Introduction and statement of results

Many kinds of *shadowing* properties have been intensely studied in the last few years, especially the pseudo-orbit shadowing property (see [14]) due to its relationship with stability and ergodic theories. This property studies the closeness of approximate and exact orbits of dynamical systems on unbounded intervals. We consider (X, d) a compact metric space and $f : X \to X$ a homeomorphism. The orbit of a point $x \in X$ is the set $\{f^i(x); i \in \mathbb{Z}\}$. Fix $\delta > 0$. We say that a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in X is a *δ-pseudo-orbit* if it satisfies

$$d(f(x_i), x_{i+1}) < \delta, \quad i \in \mathbb{Z}.$$

A pseudo-orbit is *ε-shadowed* if there is a point $y \in X$ such that

$$d(f^i(y), x_i) < \varepsilon, \quad i \in \mathbb{Z}.$$

We say that f has the *shadowing property* (usually called the pseudo-orbit tracing property) if it satisfies: given $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed. This property is well studied in [13] and [14]. Often pseudo-orbits are obtained as results of numerical studies of dynamical systems. In this context the shadowing property means that numerically found orbits with uniform small errors are close to real orbits.

Another kind of shadowing property that has been intensely studied is the *limit shadowing property*. It deals with pseudo-orbits with errors tending to zero. More

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precisely, we say that a sequence $(x_i)_{i\in\mathbb{N}}$ of points in X is a *limit pseudo-orbit* if it satisfies

$$d(f(x_i),x_{i+1})\to 0, \quad i\to\infty.$$

A limit pseudo-orbit is *limit-shadowed* if there exists a point $y\in X$ such that

$$d(f^i(y),x_i)\to 0, \quad i\to\infty.$$

We say that f has the *limit shadowing property* if every limit pseudo-orbit is limit-shadowed. In this case the values $d(f(x_i),x_{i+1})$ may be large for small $i\in\mathbb{N}$ but converge to zero as $i\to\infty$. From the numerical viewpoint this property means the following (as observed in [3]): if we apply a numerical method that approximates f with 'improving accuracy' so that one step errors tend to zero as time goes to infinity, then the numerically obtained orbits tend to real ones.

We define a negative limit shadowing property as follows. A sequence $(x_i)_{i\leq 0}$ of points in X is a *negative limit pseudo-orbit* if it satisfies

$$d(f(x_i),x_{i+1})\to 0, \quad i\to-\infty.$$

A negative limit pseudo-orbit is *limit-shadowed* if there exists a point $y\in X$ such that

$$d(f^i(y),x_i)\to 0, \quad i\to-\infty.$$

We say that f has the *negative limit shadowing property* if every negative limit pseudo-orbit is limit-shadowed.

In this work we consider an analogue property that considers two-sided limit pseudo-orbits and two-sided limit shadows. Precisely, we say that a sequence $(x_i)_{i\in\mathbb{Z}}$ of points in X is a *two-sided limit pseudo-orbit* if it satisfies

$$d(f(x_i),x_{i+1})\to 0, \quad |i|\to\infty.$$

A two-sided limit pseudo-orbit is *two-sided limit-shadowed* if there is a point $y\in X$ such that

$$d(f^i(y),x_i)\to 0, \quad |i|\to\infty.$$

We say that f has the *two-sided limit shadowing property* if every two-sided limit pseudo-orbit is two-sided limit-shadowed. Our first aim here is to show that this property is different from the shadowing and limit shadowing properties.

Let M be a closed C^∞-manifold and let $\text{Diff}^1(M)$ be the set of all diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|.\|$ on the tangent bundle TM. Let $\Lambda\subset M$ be a compact and f-invariant set. A linear subbundle E of the tangent bundle $T_\Lambda M$ is *uniformly contracted* by f if it is Df-invariant and there exists $N\geq 1$ such that for any $x\in K$ and any unit vector $v\in E_x$ we have

$$||Df^N(x)v||<\frac{1}{2}.$$

If E is uniformly contracted for f^{-1} we say that it is *uniformly expanded.* We say that Λ is *hyperbolic* if $T_\Lambda M=E\oplus F$, where E is uniformly contracted and F is uniformly expanded. We call E the *stable bundle* and F the *unstable bundle.* If the whole ambient manifold M is hyperbolic we say that f is *Anosov.* We say that f is *transitive* if there exists a point $x\in M$ whose future orbit $\{f^n(x);n\in\mathbb{N}\}$ is dense in M.

We denote by $\mathcal{S}$ the set of all diffeomorphisms of M with the shadowing property, denote by $\mathcal{LS}$ the set of all diffeomorphisms of M with the limit shadowing property and denote by $\mathcal{TLS}$ the set of all diffeomorphisms of M with the two-sided limit shadowing property.

S. Pilyugin says that it is unreasonable to study the two-sided limit shadowing property without putting any restriction on the values $d(f(x_i), x_{i+1})$ (see Remark 2 [15]). However, we not only study this property without any restriction on the values $d(f(x_i), x_{i+1})$, but we are able to characterize the C^1-interior of $\mathcal{TLS}$ and relate this to a well-known open problem.

Theorem A. *The C^1-interior of $\mathcal{TLS}$ is equal to the set of transitive Anosov diffeomorphisms.*

This theorem can be related with other results concerning the C^1-interior of the other shadowing properties: [8] shows that the C^1-interior of $\mathcal{S}$ is equal to the set of C^1-structurally stable diffeomorphisms and [15] shows that the C^1-interior of $\mathcal{LS}$ is equal to the set of Ω-stable diffeomorphisms.

This theorem can also be related to a very old open problem (see [7] section 18) of whether Anosov diffeomorphisms are transitive. Many authors have tried to prove this but have gotten only partial results. J. Franks [4] and S. Newhouse [11] proved it for codimension-one Anosov diffeomorphisms (with dimension of stable or unstable space equal to one). If we could prove that Anosov diffeomorphisms belong to $\mathcal{TLS}$, then they belong to the C^1-interior of $\mathcal{TLS}$ (the set of Anosov diffeomorphisms is open in $\text{Diff}^1(M)$) and Theorem A assures that they are transitive. On the other hand, if an Anosov diffeomorphism is transitive, then Theorem A shows that it belongs to $\mathcal{TLS}$. This allow us to state the following:

Conjecture 1.1. *$\mathcal{TLS}$ is equal to the set of Anosov diffeomorphisms.*

As already mentioned, this reformulates the problem of the transitivity of Anosov diffeomorphisms in terms of the two-sided limit shadowing property. Using the results of Franks and Newhouse and also Theorem A, we obtain a particular case of this conjecture:

Corollary 1.2. *Codimension-one Anosov diffeomorphisms have the two-sided limit shadowing property.*

Another problem that is included in Conjecture 1.1 is when diffeomorphisms in $\mathcal{TLS}$ are Anosov. Theorem A shows that diffeomorphisms in the C^1-interior of $\mathcal{TLS}$ are transitive Anosov, but unfortunately we do not know if $\mathcal{TLS}$ is open in $\text{Diff}^1(M)$.

We observe that the sets $\mathcal{S}$ and $\mathcal{LS}$ are not open in $\text{Diff}^1(M)$. J. Lewowicz [9] gives an example of a diffeomorphism that is expansive (see Section 2 for a definition) and has the shadowing property but is not Ω-stable. In particular, it is not C^1-structurally stable. Then [8] and [15] imply that it does not belong to the C^1-interior of $\mathcal{S}$ or $\mathcal{LS}$. It is a consequence of Lemma 2.1 in section 2 that such a system has the limit shadowing property.

Another partial answer to Conjecture 1.1 is as follows. A point $p \in M$ is *periodic* for f if there exists $n \in \mathbb{N}$ such that $f^n(p) = p$. The smallest $n \in \mathbb{N}$ satisfying $f^n(p) = p$ is called the period of p and will be denoted by $\pi(p)$. We denote by $Per(f)$ the set of all periodic points of f. A periodic point is called *hyperbolic* if its orbit $\mathcal{O}(p)$ is a hyperbolic set.

We define the *index* of a hyperbolic periodic point p as the dimension of the stable bundle and denote it by $ind(p)$. If $ind(p) = dim(M)$ we say that p is a *sink*, and if $ind(p) = 0$ we say that p is a *source*.

A point $p \in M$ is called *non-wandering* if for every neighborhood U of p there exists $n \in \mathbb{N}$ satisfying $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of all non-wandering points of f. This set is compact, f-invariant and contains all the asymptotic dynamics of f. If $\Omega(f) = \overline{Per(f)}$ and is a hyperbolic set, we say that f is *Axiom A*.

We say that a diffeomorphism f is *Morse-Smale* if its non-wandering set is a finite union of hyperbolic periodic points all of whose invariant manifolds are transversal. We prove the following:

Theorem B. *If $f \in \mathcal{TLS}$, then f has neither sinks nor sources. If, in addition, f is Axiom A, then f is transitive Anosov. In particular, Morse-Smale diffeomorphisms do not belong to $\mathcal{TLS}$.*

More generally this shows that Axiom A diffeomorphisms that are not transitive Anosov do not have the two-sided limit shadowing property. It remains to know the relation of the two-sided limit shadowing and non-Axiom A diffeomorphisms.

It is also interesting to observe that Morse-Smale diffeomorphisms belong to the C^1-interior of $\mathcal{S}$ since they are C^1-structurally stable (see [12]), but they do not belong to $\mathcal{TLS}$.

We also study C^1-generic diffeomorphisms with the two-sided limit shadowing property. F. Abdenur and L. Diaz in [1] conjectured that C^1-generic diffeomorphisms with the shadowing property are C^1-structurally stable. We prove an anologue of this conjecture for the two-sided limit shadowing property. We say that a subset $\mathcal{R}$ of $\text{Diff}^1(M)$ is *residual* if it contains a countable intersection of open and dense subsets of $\text{Diff}^1(M)$.

Theorem C. *There exists a residual subset $\mathcal{R}$ of $\text{Diff}^1(M)$ such that every $f \in \mathcal{R} \cap \mathcal{TLS}$ is a transitive Anosov (and so structurally stable) diffeomorphism.*

We observe that D. Todorov proved some results in the stability theory [18] using a property called *Lipschtiz two-sided limit shadowing property with exponent γ*. He proved that for all $\gamma \geq 0$ this property is equivalent to structural stability (Theorem 4 in [18]). As did Pilyugin, he put restrictions on the values $d(f(x_i), x_{i+1})$. The biggest difference of this paper is that we consider the two-sided limit shadowing property without any restriction on these values.

This paper is organized as follows. In section 2 we discuss some topological properties, their consequences and relate them to the two-sided limit shadowing property. We also prove some consequences of the two-sided limit shadowing property and prove Theorem B. Then in section 3 we prove Theorems A and C.

2. Some topological properties and proof of Theorem B

Consider (X, d) a compact metric space. We say that a homeomorphism $f : X \to X$ is *expansive* if there exists $\varepsilon > 0$ such that if $x, y \in X$ satisfy $d(f^i(x), f^i(y)) < \varepsilon$ for all $i \in \mathbb{Z}$, then $x = y$. This says that two different orbits move away from each other.

For any $y \in X$ and $\varepsilon > 0$ we define

$$W^s_\varepsilon(y) = \{x \in X; d(f^n(x), f^n(y)) < \varepsilon \text{ for every } n \in \mathbb{N}\},$$

$$W^u_\varepsilon(y) = \{x \in X; d(f^{-n}(x), f^{-n}(y)) < \varepsilon \text{ for every } n \in \mathbb{N}\}.$$

These sets are called ε-stable and ε-unstable sets of y respectively. More generally we define the stable and unstable sets of $y \in X$ as

$$W^s(y) = \{x \in X; d(f^n(x), f^n(y)) \to 0, \text{ if } n \to \infty\},$$

$$W^u(y) = \{x \in X; d(f^{-n}(x), f^{-n}(y)) \to 0, \text{ if } n \to \infty\}.$$

It is well known that if f is expansive, then there exists $\varepsilon > 0$ such that $W^s_\varepsilon(y) \subset W^s(y)$ and $W^u_\varepsilon(y) \subset W^u(y)$ for all $y \in X$ (see [10] Lemma 1).

Following the proof of Theorem 2.1 in [3] we prove the following:

Lemma 2.1. *Every expansive homeomorphism $f : X \to X$ with the shadowing property has both the limit shadowing and negative limit shadowing properties.*

Proof. Since f is expansive there exists $\varepsilon > 0$ such that $W^s_\varepsilon(y) \subset W^s(y)$ and $W^u_\varepsilon(y) \subset W^u(y)$ for all $y \in X$. Choose $0 < \delta < \frac{\varepsilon}{2}$ such that every δ-pseudo-orbit is ε-shadowed. Let $(x_n)_{n\in\mathbb{N}}$ be a limit pseudo-orbit. For each $j \in \mathbb{N}$ choose $n_j \in \mathbb{N}$ such that

$$d(f(x_n), x_{n+1}) < \frac{\delta}{j}, \quad n > n_j.$$

The shadowing property assures the existence of points $y_j \in X$ such that

$$d(f^n(y_j), x_n) < \frac{\delta}{j}, \quad n > n_j.$$

We claim that $(x_n)_{n\in\mathbb{N}}$ is limit-shadowed by y_1. Indeed,

$$d(f^n(y_1), f^n(y_j)) \le d(f^n(y_1), x_n) + d(x_n, f^n(y_j)) < 2\delta < \varepsilon, \quad n > n_j.$$

By the choice of ε we obtain

$$d(f^n(y_1), f^n(y_j)) \to 0, \quad n \to \infty$$

for all $j \in \mathbb{N}$. This implies that

$$d(f^n(y_1), x_n) \to 0, \quad n \to \infty.$$

Thus every limit pseudo-orbit is limit-shadowed and f has the limit shadowing property. We can similarly prove that f has the negative limit shadowing property. □

Now we define the *specification property.* First, we recall what a specification is. A pair (τ, P) is a *specification* if it consists of a finite collection $\tau = \{I_1, \ldots, I_m\}$ of finite intervals $I_i = [a_i, b_i] \subset \mathbb{Z}$ and a map $P : \bigcup_{i=1}^m I_i \to X$ such that for each $t_1, t_2 \in I \in \tau$ we have

$$f^{t_2 - t_1}(P(t_1)) = P(t_2).$$

The specification (τ, P) is said to be L-spaced if $a_{i+1} \ge b_i + L$ for all $i \in \{1, \ldots, m\}$. Moreover, it is ε-shadowed by $y \in X$ if

$$d(f^n(y), P(n)) < \varepsilon \quad \text{for every } n \in \bigcup_{i=1}^m I_i.$$

We say that a homeomorphism $f : X \to X$ has the *specification property* if for every $\varepsilon > 0$ there exists $L \in \mathbb{N}$ such that every L-spaced specification is ε-shadowed. It is well known (see [2]) that diffeomorphisms with the specification property are *topologically mixing*, i.e., for every two open sets U and V there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \ne \emptyset$ for all $n \ge N$. In particular, they are topologically transitive.

Lemma 2.2. *Every expansive homeomorphism $f : X \to X$ with the shadowing and specification properties has the two-sided limit shadowing property.*

Proof. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a two-sided limit pseudo-orbit. We proved in Lemma 2.1 that f has limit shadowing and negative limit shadowing. Thus there exist points $p_1, p_2 \in X$ such that $d(f^n(p_1), x_n) \to 0$ when $n \to -\infty$ and $d(f^n(p_2), x_n) \to 0$ when $n \to \infty$.

The expansiveness ensures the existence of $\varepsilon > 0$ such that $W^s_\varepsilon(y) \subset W^s(y)$ and $W^u_\varepsilon(y) \subset W^u(y)$ for all $y \in X$. As f has the shadowing property and the specification property, we obtain $\delta > 0$ and $L \in \mathbb{N}$ such that every δ-pseudo-orbit is ε-shadowed and every L-spaced specification is δ-shadowed.

Choose $N \in \mathbb{N}$ such that $2N \geq L$, $d(f^{-n}(p_1), x_{-n}) < \delta$ and $d(f^n(p_2), x_n) < \delta$ for all $n \geq N$. Define $I_1 = \{-N\}$, $I_2 = \{N\}$, $P(-N) = f^{-N}(p_1)$ and $P(N) = f^N(p_2)$. Then $(\{I_1, I_2\}, P)$ is an L-spaced specification and there exists a point $z \in X$ satisfying

$$d(f^{-N}(z), f^{-N}(p_1)) = d(f^{-N}(z), P(-N)) < \delta$$

and

$$d(f^N(z), f^N(p_2)) = d(f^N(z), P(N)) < \delta.$$

This implies that the sequence $(y_n)_{n\in\mathbb{Z}}$ defined by

$$y_n = f^n(p_1), \;\; n \leq -N;$$

$$y_n = f^n(z), \;\; -N < n < N;$$

$$y_n = f^n(p_2), \;\; n \geq N$$

is a δ-pseudo orbit. Then there exists a point $y \in X$ such that $d(f^n(y), f^n(p_1)) < \varepsilon$ for all $n \leq -N$ and $d(f^n(y), f^n(p_2)) < \varepsilon$ for all $n \geq N$. By the choice of ε we obtain that $d(f^n(y), f^n(p_1)) \to 0$ when $n \to -\infty$ and $d(f^n(y), f^n(p_2)) \to 0$ when $n \to \infty$. By the choice of p_1 and p_2 we obtain that $d(f^n(y), x_n) \to 0$ when $|n| \to \infty$. □

It is well known that Anosov diffeomorphisms have the shadowing and expansiveness properties (see [16]). As noted in the introduction it is not known if Anosov diffeomorphisms are transitive and much less if they have the specification property. If we consider a connected and compact manifold M, transitive Anosov diffeomorphisms are topologically mixing and thus have the specification property (see [2]). So transitive Anosov diffeomorphisms have all the topological properties of the hypothesis of Lemma 2.2 and we obtain the following:

Corollary 2.3. *Transitive Anosov diffeomorphisms have the two-sided limit shadowing property.*

Now we obtain some consequences of the two-sided limit shadowing property.

Lemma 2.4. *If $f : X \to X$ is a homeomorphism with the two-sided limit shadowing property, then $W^s(x) \cap W^u(y) \neq \emptyset$ for all $x, y \in X$.*

Proof. Consider the following sequence:

$$x_n = f^n(y), \;\; n \leq 0;$$

$$x_n = f^n(x), \;\; n > 0.$$

As $d(f(x_n), x_{n+1}) = 0$ for $n < 0$ and $n > 0$, the sequence $(x_n)_{n\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit. The two-sided limit shadowing property assures the existence of a point $z \in X$ such that $d(f^n(z), x_n) \to 0$ when $|n| \to \infty$, i.e.,

$$d(f^n(z), f^n(y)) \to 0, \qquad n \to -\infty;$$
$$d(f^n(z), f^n(x)) \to 0, \qquad n \to \infty.$$

This implies that $z \in W^s(x) \cap W^u(y)$. □

We say that a compact invariant set Λ is an *attracting set* of f if there exists a neighborhood U of Λ such that $f(\overline{U}) \subset U$ and $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(\overline{U})$. We say that Λ is a *repelling set* of f if it is an attracting set for f^{-1}. As a corollary of Lemma 2.4 we obtain:

Corollary 2.5. *If a homeomorphism $f : X \to X$ with the two-sided limit shadowing property has an attracting set A, then $A = X$.*

Proof. Take a neighborhood U of A with $f(\overline{U}) \subset U$ and $A = \bigcap_{n\in\mathbb{N}} f^n(U)$. We claim that for all $y \in X$ there exists $n_y \in \mathbb{N}$ such that $f^n(y) \in U$ for all $n \geq n_y$. Indeed, since $A \neq \emptyset$ we can fix $x \in A$. Lemma 2.4 assures the existence of a point $z \in W^u(x) \cap W^s(y)$. Since $x \in A$ and A is an attracting set we have that $W^u(x) \subset A$, so $z \in A$. Since A is invariant and $z \in W^s(y)$ we obtain the desired n_y.

Now we use the claim to conclude the proof of the corollary. Suppose by contradiction that $A \neq X$. Then, $X \setminus U \neq \emptyset$. Take $V = X \setminus U$. It follows that V is a compact non-empty subset of X with $f^{-1}(V) \subset V$. It follows that $f^{-n-1}(V) \subset f^{-n}(V)$ for all $n \in \mathbb{N}$, and so $\bigcap_{n\in\mathbb{N}} f^{-n}(V)$ is non-empty, as it is the intersection of a nested sequence of compact sets. It is easy to see that such an intersection is also invariant under f. Taking y in this intersection we have $y \in X$ satisfying $f^n(y) \notin U$ for all $n \in \mathbb{N}$, which contradicts the claim. This contradiction ends the proof. □

Proof of Theorem B. Let $f \in \mathcal{TLS}$. We will show that the stable manifold of any sink s of f is the whole manifold. This is obviously absurd because there are no contracting diffeomorphisms on a closed manifold. Suppose, by contradiction, that there exists a point $x \in M$ that does not belong to $W^s(s)$. Lemma 2.4 implies the existence of $y \in W^u(s) \cap W^s(x)$. As $x \notin W^s(s)$ we have $y \neq s$. Then we obtain a point $y \neq s$ that belongs to $W^u(s)$. This is absurd because the unstable manifold of any sink $s \in M$ is the singleton $\{s\}$. Thus $W^s(s) = M$. This implies that f does not have any sink. We can similarly prove that f does not have any source.

Now, suppose that f is Axiom A. The Spectral Decomposition Theorem (see [7], Theorem 18.3.1) says that $\Omega(f)$ is decomposed into a finite number of disjoint, transitive, hyperbolic and isolated sets called *basic sets*. It is known that one of the basic sets must be an attracting set and one must be a repelling set. Corollary 2.5 implies that the whole manifold is a basic set. In particular, f is a transitive Anosov diffeomorphism.

Since the Morse-Smale diffeomorphisms are Axiom A but not Anosov, we obtain that they cannot belong to $\mathcal{TLS}$. □

3. Proof of Theorems A and C

In this section we are interested in obtaining hyperbolicity from the two-sided limit shadowing property. We will do this from both robust and generic viewpoints.

We define the stable and unstable sets for a basic set Λ as

$$W^s(\Lambda) = \{y \in M; d(f^n(y), \Lambda) \to 0, \ \ n \to \infty\},$$

$$W^u(\Lambda) = \{y \in M; d(f^n(y), \Lambda) \to 0, \ \ n \to -\infty\}.$$

We define a relation on the basic sets as $\Lambda_i > \Lambda_j$ if

$$(W^s(\Lambda_i) \setminus \Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset.$$

We say that f satisfies the *no-cycle condition* if $\Lambda_{i_0} > \Lambda_{i_1} > \cdots > \Lambda_{i_j} > \Lambda_{i_0}$ is impossible among the basic sets. It is well known that a diffeomorphism is Axiom A and satisfies the no-cycle condition if and only if it is Ω-stable.

Another notion of stability that is equivalent to the Ω-stability is called *star*. We say that $f \in \text{Diff}^1(M)$ is *star* if there exists a C^1-neighborhood $\mathcal{U}$ of f such that every periodic point of every $g \in \mathcal{U}$ is hyperbolic. In [6], S. Hayashi proved that a diffeomorphism is star if and only if it is Axiom A and satisfies the no-cycle condition. Our proofs are based on this fact, and we will be interested in proving that the two-sided limit shadowing property implies the star condition.

We state a well-known lemma in dynamics that bifurcates a non-hyperbolic periodic point into two distinct hyperbolic periodic points with different indices. It is proved in [17], Lemma 2.4.

Lemma 3.1. *Let $f \in \text{Diff}^1(M)$ and p be a non-hyperbolic periodic point of f. Then for every C^1-neighborhood $\mathcal{U}$ of f there are $g \in \mathcal{U}$ and $p_1, p_2 \in Per(g)$ such that $ind(p_1) \neq ind(p_2)$.*

We say that a diffeomorphism f is *Kupka-Smale* if all periodic points of f are hyperbolic and for all pair of periodic points p and q of f the manifolds $W^s(p)$ and $W^u(q)$ are transversal. We denote by $\mathcal{KS}$ the set of all Kupka-Smale diffeomorphisms. It is well known that $\mathcal{KS}$ is a residual subset of $\text{Diff}^1(M)$.

We say that two hyperbolic periodic points p and q are *homoclinically related* if $W^s(\mathcal{O}(p)) \pitchfork W^u(\mathcal{O}(q)) \neq \emptyset$ and $W^u(\mathcal{O}(p)) \pitchfork W^s(\mathcal{O}(q)) \neq \emptyset$. It is easy to see that two hyperbolic periodic points homoclinically related have the same index. As another corollary of Lemma 2.4 we obtain:

Corollary 3.2. *If $f \in \mathcal{TLS} \cap \mathcal{KS}$, then all periodic points have the same index.*

Proof. Let p, q be two distinct periodic points of f. Lemma 2.4 assures that the sets $W^s(p) \cap W^u(q)$ and $W^u(p) \cap W^s(q)$ are non-empty. As $f \in \mathcal{KS}$, these intersections are transversal. Thus p and q are homoclinically related and have the same index. □

Proof of Theorem A. Suppose that $f \in int(\mathcal{TLS})$. Let $\mathcal{U}$ be a C^1-neighborhood of f contained in $\mathcal{TLS}$. We claim that f is a star diffeomorphism. Suppose by contradiction that f is not star. Then there exist $g \in \mathcal{U}$ and p a non-hyperbolic periodic point of g. We use Lemma 3.1 and obtain $h \in \mathcal{U}$ and two distinct hyperbolic periodic points p, q of h with different indices. As these points are hyperbolic and $\mathcal{KS}$ is dense in $\text{Diff}^1(M)$ we can perturb h to $\tilde{h} \in \mathcal{U} \cap \mathcal{KS}$ that has two distinct hyperbolic periodic points $p_{\tilde{h}}, q_{\tilde{h}}$ with different indices. This contradicts Corollary 3.2 and proves the claim. Hayashi's Theorem [6] implies that f is Axiom A and Theorem B finishes the proof. □

To prove Theorem C we first need to introduce some generic machinery. Inspired by the work of S. Gan and D. Yang ([19], Lemma 2.1) on C^1-generic expansive homoclinic classes we prove the following:

Lemma 3.3. *There exists an open and dense set $\mathcal{P}$ of $\mathrm{Diff}^1(M)$ such that all $f \in \mathcal{P}$ satisfies: if for any C^1-neighborhood $\mathcal{U}$ of f some $g \in \mathcal{U}$ has two distinct hyperbolic periodic points with different indices, then f has two distinct hyperbolic periodic points with different indices.*

Proof. Let $\mathcal{A}$ be the set of C^1-diffeomorphisms that has two distinct hyperbolic periodic points with different indices. The hyperbolicity of these periodic points implies that $\mathcal{A}$ is open in $\mathrm{Diff}^1(M)$. Let $\mathcal{B} = \mathrm{Diff}^1(M) \setminus \overline{\mathcal{A}}$. Note that $\mathcal{B}$ is also open in $\mathrm{Diff}^1(M)$. Thus $\mathcal{P} = \mathcal{A} \cup \mathcal{B}$ is open and dense in $\mathrm{Diff}^1(M)$.

Let $f \in \mathcal{P}$ and suppose that there is a sequence $(f_n)_{n \in \mathbb{N}}$ of diffeomorphisms converging to f in the C^1-topology such that each f_n has two distinct hyperbolic periodic points with different indices. Then $f_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ implies that f cannot belong to $\mathcal{B}$ and must belong to $\mathcal{A}$; i.e., f has two distinct hyperbolic periodic points with different indices. □

Proof of Theorem C. Let $f \in \mathcal{TLS} \cap \mathcal{KS} \cap \mathcal{P}$. Suppose by contradiction that f is not a star diffeomorphism; that is, for every neighborhood $\mathcal{U}$ of f there exists some $g \in \mathcal{U}$ that has a non-hyperbolic periodic point. We perturb g using Lemma 3.1 and obtain $h \in \mathcal{U}$ such that h has two distinct hyperbolic periodic points with different indices. Since $f \in \mathcal{P}$ it has two distinct hyperbolic periodic points with different indices. But this contradicts Corollary 3.2 and proves that f is a star diffeomorphism. As above Theorem B finishes the proof. □

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