SIGNATURES, HEEGAARD FLOER CORRECTION TERMS AND QUASI–ALTERNATING LINKS

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Abstract. Turaev showed that there is a well–defined map assigning to an oriented link \( L \) in the three–sphere a Spin structure \( t_0 \) on \( \Sigma(L) \), the two–fold cover of \( S^3 \) branched along \( L \). We prove, generalizing results of Manolescu–Owens and Donald–Owens, that for an oriented quasi–alternating link \( L \) the signature of \( L \) equals minus four times the Heegaard Floer correction term of \( (\Sigma(L), t_0) \).

1. Introduction

Vladimir Turaev \cite{[21]} §2.2] proved that there is a surjective map which associates to a link \( L \subset S^3 \) decorated with an orientation \( o \) a Spin structure \( t_{(L,o)} \) on \( \Sigma(L) \), the double cover of \( S^3 \) branched along \( L \). Moreover, he showed that the only other orientation on \( L \) which maps to \( t_{(L,o)} \) is \( -o \), the overall reversed orientation. In other words, Turaev described a bijection between the set of quasi–orientations on \( L \) (i.e. orientations up to overall reversal) and the set \( \text{Spin}(\Sigma(L)) \) of Spin structures on \( \Sigma(L) \). Each element \( t \in \text{Spin}(\Sigma(L)) \) can be viewed as a Spin\(^c\) structure on \( \Sigma(L) \), so if \( \Sigma(L) \) is a rational homology sphere, then it makes sense to consider the rational number \( d(\Sigma(L), t) \), where \( d \) is the correction term invariant defined by Ozsváth and Szabó \cite{[13]}. Under the assumption that \( L \) is nonsplit alternating it was proved — in \cite{[10]} when \( L \) is a knot and in \cite{[3]} for any number of components of \( L \) — that

\[
\sigma(L, o) = -4d(\Sigma(L), t_{(L,o)}) \quad \text{for every orientation } o \text{ on } L,
\]

where \( \sigma(L, o) \) is the link signature. For an alternating link associated to a plumbing graph with no bad vertices, this follows from a combination of earlier results of Saveliev \cite{[19]} and Stipsicz \cite{[20]}, each of whom showed that one of the quantities in \( \text{(*)} \) is equal to the Neumann–Siebenmann \( \bar{\mu} \)-invariant of the plumbing tree. The main purpose of this paper is to prove property \( \text{(*)} \) for the family of quasi–alternating links introduced in \cite{[14]}. 

Definition 1. The quasi–alternating links are the links in \( S^3 \) with nonzero determinant defined recursively as follows:

1. the unknot is quasi–alternating;

2. if \( L_1, L_2 \) are quasi–alternating, then so is \( L_1 \# L_2 \),

3. if \( L_1 \) is quasi–alternating and \( L_2 \) is a plumbing link associated to a plumbing tree with no bad vertices, then \( L_1 \# L_2 \) is quasi–alternating,

4. if \( L_1 \) is quasi–alternating and \( L_2 \) is a plumbing link associated to a plumbing tree with no bad vertices, then \( L_1 \# L_2 \) is quasi–alternating.

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(2) If \( L_0, L_1 \) are quasi–alternating, \( L \subset S^3 \) is a link such that \( \det L = \det L_0 + \det L_1 \) and \( L, L_0, L_1 \) differ only inside a 3–ball as illustrated in Figure 1, then \( L \) is quasi–alternating.

Quasi–alternating links have recently been the object of considerable attention [1, 2, 4–6, 11, 16, 17, 22, 23]. Alternating links are quasi–alternating [14, Lemma 3.2], but (as shown in e.g. [1]) there exist infinitely many quasi–alternating, nonalternating links. Our main result is the following:

**Theorem 1.** Let \((L, o)\) be an oriented link. If \( L \) is quasi–alternating, then

\[
\sigma(L, o) = -4d(\Sigma(L), t(L, o)).
\]

The contents of the paper are as follows. In Section 2 we first recall some basic facts on Spin structures and the existence of two natural 4–dimensional cobordisms, one from \( \Sigma(L_1) \) to \( \Sigma(L) \), the other from \( \Sigma(L) \) to \( \Sigma(L_0) \). Then, in Proposition 1 we show that for an orientation \( o \) on \( L \) for which the crossing in Figure 1 is positive, the Spin structure \( t(L, o) \) extends to the first cobordism but not to the second one. In Section 3 we use this information together with the Heegaard Floer surgery exact triangle to prove Proposition 2 which relates the value of the correction term \( d(\Sigma(L), t(L, o)) \) with the value of an analogous correction term for \( \Sigma(L_1) \). In Section 4 we restate and prove our main result, Theorem 1. The proof consists of an inductive argument based on Proposition 2 and the known relationship between the signatures of \( L \) and \( L_1 \). The use of Proposition 2 is made possible by the fact that up to mirroring \( L \) one may always assume the crossing of Figure 1 to be positive. We close Section 4 with Corollary 3 which uses results of Rustamov and Mullins to relate Turaev’s torsion function for the two–fold branched cover of a quasi–alternating link \( L \) with the Jones polynomial of \( L \).

2. **Triads and Spin structures**

A Spin structure on an \( n \)–manifold \( M^n \) is a double cover of the oriented frame bundle of \( M \) with the added condition that if \( n > 1 \), then it restricts to the nontrivial double cover on fibres. A Spin structure on a manifold restricts to give a Spin structure on a codimension–one submanifold, or on a framed submanifold of codimension higher than one. As mentioned in the introduction, an orientation \( o \) on a link \( L \) in \( S^3 \) induces a Spin structure \( t(L, o) \) on the double–branched cover \( \Sigma(L) \), as in [21]. Recall also that there are two Spin structures on \( S^1 = \partial D^2 \): the nontrivial or **bounding** Spin structure, which is the restriction of the unique Spin structure on \( D^2 \), and the trivial or **Lie** Spin structure, which does not extend over the disk. The restriction map from Spin structures on a solid torus to Spin structures on its boundary is injective; thus if two Spin structures on a closed 3–manifold agree
outside a solid torus, then they are the same. For more details on Spin structures see for example \[7\].

If \( Y \) is a 3–manifold with a Spin structure \( t \) and \( K \) is a knot in \( Y \) with framing \( \lambda \), we may attach a 2–handle to \( K \) giving a surgery cobordism \( W \) from \( Y \) to \( Y_\lambda(K) \). There is a unique Spin structure on \( D^2 \times D^2 \), which restricts to the bounding Spin structure on each framed circle \( \partial D^2 \times \{\text{point}\} \) in \( \partial D^2 \times D^2 \). Thus the Spin structure on \( Y \) extends over \( W \) if and only if its restriction to \( K \), viewed as a framed submanifold via the framing \( \lambda \), is the bounding Spin structure. Note that this is equivalent, symmetrically, to the restriction of \( t \) to the submanifold \( \lambda \)-framed by \( K \) being the bounding Spin structure. Moreover, the extension over \( W \) is unique if it exists.

Let \( L, L_0, L_1 \) be three links in \( S^3 \) differing only in a 3–ball \( B \) as in Figure 1. The double cover of \( B \) branched along the pair of arcs \( B \cap L \) is a solid torus \( \tilde{B} \) with core \( C \). The boundary of a properly embedded disk in \( B \) which separates the two branching arcs lifts to a disjoint pair of meridians of \( \tilde{B} \). The preimage in \( \Sigma(L) \) of the curve \( \lambda_0 \) shown in Figure 2 is a pair of parallel framings for \( C \); denote one of these by \( \tilde{\lambda}_0 \). Similarly, let \( \tilde{\lambda}_1 \) denote one of the components of the preimage in \( \Sigma(L) \) of \( \lambda_1 \). Since \( \lambda_0 \) is homotopic in \( B - L \) to the boundary of a disk separating the two components of \( L_0 \cap B \), we see that \( \Sigma(L_0) \) is obtained from \( \Sigma(L) \) by \( \tilde{\lambda}_0 \)-framed surgery on \( C \). Similarly, \( \lambda_1 \) is homotopic in \( B - L \) to the boundary of a disk separating the two components of \( L_1 \cap B \), and \( \Sigma(L_1) \) is obtained from \( \Sigma(L) \) by \( \tilde{\lambda}_1 \)-framed surgery on \( C \).

The two framings \( \tilde{\lambda}_0 \) and \( \tilde{\lambda}_1 \) differ by a meridian of \( C \). In the terminology from \[14\], the manifolds \( \Sigma(L), \Sigma(L_0) \) and \( \Sigma(L_1) \) form a triad and there are surgery cobordisms

\[
V : \Sigma(L_1) \to \Sigma(L) \quad \text{and} \quad W : \Sigma(L) \to \Sigma(L_0).
\]

The surgery cobordism \( W \) is built by attaching a 2–handle to \( \Sigma(L) \) along the knot \( C \) with framing \( \tilde{\lambda}_0 \). The cobordism \( V \) is built by attaching a 2–handle to \( \Sigma(L_1) \). Dualising this handle structure, \( V \) is obtained by attaching a 2–handle to \( \Sigma(L) \) along the knot \( C \) with framing \( \tilde{\lambda}_1 \) (and reversing orientation).

\[\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{loops.png}
\caption{The loops \( \lambda_0 \) and \( \lambda_1 \).}
\end{figure}\]

**Proposition 1.** For any orientation \( o \) on \( L \) such that the crossing shown in Figure 1 is positive, the Spin structure \( t_{(L,o)} \) extends to a unique Spin structure \( s_o \) on the cobordism \( V \) and does not admit an extension over \( W \). The restriction of \( s_o \) to \( \Sigma(L_1) \) is the Spin structure \( t_{(L_1,o_1)} \), where \( o_1 \) is the orientation on \( L_1 \) induced by \( o \).
Proof. Let \( \pi : \Sigma(L) \to S^3 \) be the branched covering map. The Spin structure \( t_{(L,o)} \) is the lift \( \tilde{s} \) of the Spin structure restricted from \( S^3 \) to \( S^3 - L \), twisted by \( h \in H^1(\Sigma(L) - \pi^{-1}(L); \mathbb{Z}/2\mathbb{Z}) \), where the value of \( h \) on a curve \( \gamma \) is the parity of half the sum of the linking numbers of \( \pi \circ \gamma \) about the components of \( L \) (following Turaev [21 \( \S \) 2.2]). Suppose that the crossing in Figure 1 is positive as, for example, illustrated in Figure 3, so that the orientation \( o \) induces an orientation \( o_1 \) on \( L_1 \).

![Figure 3. The oriented link \((L, o)\) together with the oriented resolution \((L_1, o_1)\) and the unoriented resolution \(L_0\).](image)

Then, we can compute from Figure 2 that \( h(\tilde{\lambda}_1) = 0 \) and \( h(\tilde{\lambda}_0) = 1 \). The Spin structure on \( S^3 \) restricts to the bounding structure on each of \( \lambda_0 \) and \( \lambda_1 \) using the 0-framing. The map \( \pi \) restricts to a diffeomorphism on neighbourhoods of \( \tilde{\lambda}_0 \) and \( \tilde{\lambda}_1 \). Therefore, the restriction of \( \tilde{s} \) to each of \( \tilde{\lambda}_0 \) and \( \tilde{\lambda}_1 \) using the pullback of the 0-framing is also the bounding structure. Also note that the preimage under \( \pi \) of a disk bounded by \( \lambda_i \) is an annulus with core \( C \), so the framing of \( \tilde{\lambda}_i \) given by \( C \) is the same as the pullback of the 0-framing.

The spin structure \( t_{(L,o)} \) is equal to \( \tilde{s} \) twisted by \( h \). Since \( \tilde{s} \) restricts to the bounding spin structure on \( \tilde{\lambda}_1 \), and \( h(\tilde{\lambda}_1) = 0 \), we see that \( t_{(L,o)} \) restricts to the bounding Spin structure on \( \tilde{\lambda}_1 \) using the framing given by \( C \). On the other hand since \( h(\tilde{\lambda}_0) = 1 \), \( t_{(L,o)} \) restricts to the Lie Spin structure on \( \tilde{\lambda}_0 \), again using the framing given by \( C \). It follows that \( t_{(L,o)} \) admits a unique extension \( s_o \) over the 2-handle giving the cobordism \( V \), and does not extend over the cobordism \( W \).

The restriction of \( s_o \) to \( \Sigma(L_1) \) coincides with \( t_{(L_1,o_1)} \) outside of the solid torus \( \tilde{B} \), and therefore also on the closed manifold \( \Sigma(L_1) \). \( \square \)

3. Relations between correction terms

By [14 Proposition 2.1] we have the following exact triangle:

\[
\begin{array}{ccc}
\widehat{HF}(\Sigma(L_1)) & \xrightarrow{F_V} & \widehat{HF}(\Sigma(L)) \\
| \ & | & |
\end{array}
\]

where the maps \( F_V \) and \( F_W \) are induced by the surgery cobordisms of \( \Sigma \). (All the Heegaard Floer groups are taken with \( \mathbb{Z}/2\mathbb{Z} \) coefficients.)

By [14 Proposition 3.3] (and notation as in that paper), if \( L \subset S^3 \) is a quasi-alternating link and \( L_0 \) and \( L_1 \) are resolutions of \( L \) as in Definition 1 then \( \Sigma(L), \Sigma(L_0) \) and \( \Sigma(L_1) \) are \( L \)-spaces. Moreover, by assumption we have

\[
|H^2(\Sigma(L); \mathbb{Z})| = |H^2(\Sigma(L_0); \mathbb{Z})| + |H^2(\Sigma(L_1); \mathbb{Z})|.
\]
Since for every $L$–space $Y$ we have $|H^2(Y;\mathbb{Z})| = \dim \widehat{HF}(Y)$, the Heegaard Floer surgery exact triangle reduces to a short exact sequence:

\[(4)\quad 0 \to \widehat{HF}(\Sigma(L_1)) \xrightarrow{F_v} \widehat{HF}(\Sigma(L)) \xrightarrow{F_w} \widehat{HF}(\Sigma(L_0)) \to 0.\]

The type of argument employed in the proof of the following proposition goes back to [9] and was also used in [20].

**Proposition 2.** Let $L$ be a quasi–alternating link and let $L_0$, $L_1$ be resolutions of $L$ as in Definition [1]. Let $o$ be an orientation on $L$ for which the crossing of Figure [1] is positive, and let $o_1$ be the induced orientation on $L_1$. Then, the following holds:

\[-4d(\Sigma(L), t_{(L,o)}) = -4d(\Sigma(L_1), t_{(L_1,o_1)}) - 1.\]

**Proof.** Since $\Sigma(L)$, $\Sigma(L_1)$, and $\Sigma(L_0)$ are $L$–spaces, we may think of the Spin<sup>c</sup> structures on these spaces as generators of their $\widehat{HF}$–groups, and we shall abuse our notation accordingly. Let $V : \Sigma(L_1) \to \Sigma(L)$ be the surgery cobordism of (2), and let $s_o$ be the unique Spin structure on $V$ which extends $t_{(L,o)}$ as in Proposition [1].

Recall that, by definition, the map $F_U$ associated to a cobordism $U : Y_1 \to Y_2$ is given by

\[F_U = \sum_{s \in \text{Spin}^c(U)} F_{U,s},\]

where $F_{U,s} : \widehat{HF}(Y_1, t_1) \to \widehat{HF}(Y_2, t_2)$ and $t_i = s|_{Y_i}$ for $i = 1, 2$. We claim that

\[(5)\quad F_{V,s_o}(t_{(L_1,o_1)}) = t_{(L,o)}.\]

The Heegaard Floer $\widehat{HF}$–groups admit a natural involution, usually denoted by $\mathcal{J}$. The maps induced by cobordisms are equivariant with respect to the $\mathbb{Z}/2\mathbb{Z}$–actions associated to conjugation on Spin<sup>c</sup> structures and the $\mathcal{J}$–map on the Heegaard Floer groups, in the sense that, if $\overline{x} := \mathcal{J}(x)$ for an element $x$, we have

\[(6)\quad F_{W,s}(\overline{x}) = F_{W,s}(x)\]

for each $s \in \text{Spin}^c(W)$. Since by Proposition [1] there are no Spin structures on the surgery cobordism $W : \Sigma(L) \to \Sigma(L_0)$ of (2) which restrict to $t_{(L,o)}$, the element $F_W(t_{(L,o)}) \in \widehat{HF}(\Sigma(L_0))$ has no Spin component. In fact, since $t_{(L,o)}$ is fixed under conjugation and we are working over $\mathbb{Z}/2\mathbb{Z}$, (6) implies that the contribution of each non–Spin $s \in \text{Spin}^c(W)$ to a Spin component of $F_W(t_{(L,o)})$ is cancelled by the contribution of $\overline{s}$ to the same component. Therefore we may write

\[F_W(t_{(L,o)}) = x + \overline{x}\]

for some $x \in \widehat{HF}(\Sigma(L_0))$. By the surjectivity of $F_W$ there is some $y \in \widehat{HF}(\Sigma(L))$ with $F_W(y) = x$, therefore $F_W(t_{(L,o)} + y + \overline{y}) = 0$, and by the exactness of (4) we have $t_{(L,o)} + y + \overline{y} = F_V(z)$ for some $z \in \widehat{HF}(\Sigma(L_0))$. Since $F_V(\overline{z}) = F_V(z) = F_V(z)$, the injectivity of $F_V$ implies $z = \overline{z}$. Moreover, $z$ must have some nonzero Spin component, otherwise we could write $z = u + \overline{u}$ and

\[F_V(u + \overline{u}) = F_V(u) + F_V(\overline{u}) = F_V(u) + F_V(u)\]

could not have the Spin component $t_{(L,o)}$. This shows that there is a Spin structure $t \in \widehat{HF}(\Sigma(L_1))$ such that $F_V(t) = t_{(L,o)}$. But, as we argued before for $F_W(t_{(L,o)})$, in order for $F_V(t)$ to have a Spin component it must be the case that there is some Spin structure $s$ on $V$ such that $F_V(s) = t_{(L,o)}$. Applying Proposition [1] we conclude $s = s_o$ and therefore $t = t_{(L_1,o_1)}$. This establishes claim (5).
Using equation (3) and the fact that $\det(L_1) > 0$ it is easy to check that $V$ is negative definite. The statement follows immediately from equation (5) and the degree–shift formula in Heegaard Floer theory [15, Theorem 7.1] using the fact that $c_1(s_\sigma) = 0$, $\sigma(V) = -1$ and $\chi(V) = 1$. □

4. The main result and a corollary

Theorem 1. Let $(L, o)$ be an oriented link. If $L$ is quasi–alternating, then

$$\sigma(L, o) = -4d(\Sigma(L), t_{(L,o)}) - 4d(\Sigma(L), t_{(L,0)}).$$

Proof. The statement trivially holds for the unknot, because the unknot has zero signature and the two–fold cover of $S^3$ branched along the unknot is $S^3$, whose only correction term vanishes. If $L$ is not the unknot and $L$ is quasi–alternating, there are quasi–alternating links $L_0$ and $L_1$ such that $\det(L) = \det(L_0) + \det(L_1)$ and $L$, $L_0$ and $L_1$ are related as in Figure 1. To prove the theorem it suffices to show that if the statement holds for $L_0$ and $L_1$, then it holds for $L$ as well.

Denote by $L^m$ the mirror image of $L$, and by $o^m$ the orientation on $L^m$ naturally induced by an orientation $o$ on $L$. The orientation–reversing diffeomorphism from $S^3$ to itself taking $L$ to $L^m$ lifts to one from $\Sigma(L)$ to $\Sigma(L^m)$ sending $t_{(L,o)}$ to $t_{(L^m,o^m)}$. Thus by [8, Theorem 8.10] and [13, Proposition 4.2] we have

$$\sigma(L^m, o^m) = -\sigma(L, o)$$

and

$$4d(\Sigma(L^m), t_{(L^m,o^m)}) = 4d(-\Sigma(L), t_{(L,o)}) = -4d(\Sigma(L), t_{(L,o)}),$$

therefore equation (1) holds for $(L, o)$ if and only if it holds for $(L^m, o^m)$. Hence, without loss of generality we may now fix an orientation $o$ on $L$ so that the crossing appearing in Figure 1 is positive.

Denote by $o_1$ the orientation on $L_1$ naturally induced by $o$. By [11, Lemma 2.1]

$$\sigma(L, o) = \sigma(L_1, o_1) - 1.$$

Since we are assuming that the statement holds for $L_1$, we have

$$\sigma(L_1, o_1) = -4d(\Sigma(L_1), t_{(L_1,o_1)}).$$

Equations (7) and (8) together with Proposition 2 immediately imply equation (1). □

Corollary 3. Let $(L, o)$ be an oriented, quasi–alternating link. Then,

$$\tau(\Sigma(L), t_{(L,o)}) = -\frac{1}{12} \frac{V'_L(-1)}{V_L(-1)},$$

where $\tau$ is Turaev’s torsion function and $V_{(L,o)}(t)$ is the Jones polynomial of $(L, o)$.

Proof. By [18, Theorem 3.4] we have

$$d(\Sigma(L), t_{(L,o)}) = 2\lambda(\text{HF}_\text{red}(\Sigma(L))) + 2\tau(\Sigma(L), t_{(L,o)}) - \lambda(\Sigma(L)),$$

where $\lambda$ denotes the Casson–Walker invariant, normalized so that it takes value $-2$ on the Poincaré sphere oriented as the boundary of the negative $E_8$ plumbing. Moreover, since $L$ is quasi–alternating $\Sigma(L)$ is an $L$–space; therefore the first
summand on the right–hand side of (9) vanishes. By [12, Theorem 5.1], when \( \det(L) > 0 \) we have

\[
\lambda(\Sigma(L)) = -\frac{1}{6} \frac{V'(L,o)(-1)}{V(L,o)(-1)} + \frac{1}{4} \sigma(L,o).
\]

Therefore, when \((L,o)\) is an oriented quasi–alternating link, Theorem 4 together with equations (9) and (10) yield the statement. \( \square \)

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