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MODULES OF HIGHER ORDER INVARIANTS

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Abstract. Let k be an algebraically closed field of characteristic $p \geq 0$. Let A be a commutative k-algebra with multiplicative identity and let M be an A-module. Let G be a linear algebraic group acting rationally on both A and M. In this paper we study A^G-modules of nth order invariants, $I_n(M,G)$. The $I_n(M,G)$ are defined inductively by $I_0(M,G) = \{0\}$ and $I_n(M,G) = \{m \in M : g \cdot m - m \in I_{n-1}(M,G) \text{ for all } g \in G\}$. We show that some fundamental problems concerning these modules can be reduced to the case $I_n(k[G],G)$ where G acts on itself by right translation. We study the questions as to when $I_n(M,G)$ is a finitely generated A^G-module and how the $I_n(M,G)$ are related to equivariant mappings. For the classical case of $\mathbb{G}_a$ acting on binary forms, we describe the $I_n(M,G)$ and determine when they are Cohen-Macaulay.

1. Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$. Let A be a commutative k-algebra with multiplicative identity and let M be an A-module. Let G be a linear algebraic group having identity e and suppose that G acts rationally on both A and M. In this paper we study A^G-modules of higher order invariants, $I_n(M,G)$. These modules first arose in our work on the reconstruction problem in ordinary differential equations [4]. Here, we study their algebraic properties. The $I_n(M,G)$ are defined in Section 2 where basic properties are proved and examples given. In Section 3, we show that some fundamental problems concerning these modules can be reduced to the case $I_n(k[G],G)$ where G acts on itself by right translation. For example, we show that when G is connected and reductive, $I_n(M,G) = M^G$. For arbitrary G, when is $I_n(M,G)$ a finitely generated A^G-module? This question is studied in Section 4 where it is shown that finite generation follows from the finite generation of A^G when $char k = 0$. For general G, we study the relation of this question to the finite generation ideal. The $I_n(M,G)$ are closely related to equivariant mappings. This relationship is studied in Section 5. In Section 6, we study the classical case of $\mathbb{G}_a$ acting on binary forms. We describe the $I_n(M,G)$ and determine when they are Cohen-Macaulay. Section 7 gives an overview of the applications to ordinary differential equations.

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2. Definition and basic properties

Let k be an algebraically closed field of characteristic $p \geq 0$. Let A be a commutative k-algebra with multiplicative identity and let M be an A-module. Let G be a linear algebraic group having identity e and let G^o be the irreducible component of e in G. The group G acts on $k[G]$ by left and right translation, denoted respectively by ℓ_g and r_g. Suppose that G acts rationally on both A and M. These actions will be denoted by $(g, a) \to g \cdot a$ and $(g, m) \to g \cdot m$, respectively, for $a \in A, m \in M$, and $g \in G$. We shall also assume that the actions are compatible, that is, that $g \cdot am = (g \cdot a)(g \cdot m)$. (When $A = k$, we shall assume that the action of G is trivial.) We denote the identity mapping on M by I_M. If $f \in A$ is not nilpotent, we denote the localization of A at f by A_f. Let X be an irreducible affine G-variety and let $k[X]$ be the algebra of polynomial functions on X. If $f \in k[X]$, then $X_f = \{x \in X : f(x) \neq 0]$. It is known that $k[X_f] = k[X]_f$.

Definition 1. Let $I_0(M, G) = \{0\} \subset M$. For $n \geq 1$, let

$$I_n(M, G) = \{m \in M : g \cdot m - m \in I_{n-1}(M, G) \text{ for all } g \in G\}.$$

We shall call $I_n(M, G)$ the *module of* n*th order invariants.*

Lemma 1. (a) $I_1(M, G) = M^G = \{m \in M : g \cdot m = m \textit{ for all } g \in G\}$.

(b) $I_n(M, G)$ *is an* A^G*-module and a* G*-module.*

(c) $I_0(M, G) \subset I_1(M, G) \subset \ldots \subset I_n(M, G) \subset I_{n+1}(M, G) \subset \ldots.$

(d) $I_n(M, G) = \{m \in M : (g_1 - I_M) \ldots (g_n - I_M) \cdot m = 0 \textit{ for all } g_1, \ldots, g_n \in G\}$.

(e) *If* $char k = 0$ *and* G *is connected with Lie algebra* $\mathcal{L}$, *then* $I_n(M, G) = \{m \in M : L_{X_1} \ldots L_{X_n} \cdot m = 0 \textit{ for all } L_{X_1}, \ldots, L_{X_n} \in \mathcal{L}\}$.

Proof. An element $m \in M$ is in $I_1(M, G)$ if and only if $g \cdot m - m \in I_0(M, G) = \{0\}$ for all $g \in G$, that is, if and only if $g \cdot m = m$. To prove the remaining properties, we proceed by induction, the cases $n = 0, 1$ always being immediate. So, suppose the properties hold for $I_n(M, G)$, $n \geq 1$. For (b), let $a \in A^G$, $m \in I_{n+1}(M, G)$, and $g, g' \in G$. We first show that $am \in I_{n+1}(M, G)$. Indeed, $g \cdot am - am = (g \cdot a)(g \cdot m) - am = a(g \cdot m - m)$. By the definition of $I_{n+1}(M, G)$, $(g \cdot m - m) \in I_n(M, G)$ which is an A^G-module by the induction hypothesis so $a(g \cdot m - m) \in I_n(M, G)$. Next, we show that $g' \cdot m \in I_n(M, G)$. Indeed, $g \cdot (g' \cdot m) - g' \cdot m = ((gg') \cdot m - m) - (g' \cdot m - m)$ and both terms are in $I_n(M, G)$ by definition. For (c), suppose that $I_{n-1}(M, G) \subset I_n(M, G)$ and let $m \in I_n(M, G)$. Then, $g \cdot m - m \in I_{n-1}(M, G) \subset I_n(M, G)$ so $m \in I_{n+1}(M, G)$. Property (d) follows immediately from induction and the definition of $I_n(M, G)$. For (e), we recall that when $char k = 0$ and G is connected, then for any rational G-module V and $v \in V$, $g \cdot v = v$ for all $g \in G$ is equivalent to $L_X \cdot v = 0$ for all $L_X \in \mathcal{L}$. Therefore, $I_{n+1}(M, G) = \{m \in M : L_X \cdot m \in I_n(M, G)$ for all $L_X \in \mathcal{L}\}$. ☐

Example 1 (Unipotent groups)**.** Let U be a unipotent group. Let V be a finite-dimensional U-module.

(a) If dim $V = n$, then $V \subset I_n(V, U)$;

(b) $M = \bigcup_{n \geq 1} I_n(M, U)$.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis for V relative to which each element in U is upper triangular, i.e., if $u \in U$, then $uv_1 = v_1$ and $uv_i - v_i \in \langle v_1, \ldots, v_{i-1}\rangle$, the vector space spanned by $\{v_1, \ldots, v_{i-1}\}$ for all $i = 2, \ldots, n$. We show by induction that $v_i \in I_i(V, U)$ for $i = 1, \ldots, n$. Indeed, $v_1 \in V^U = I_1(V, U)$. Now, suppose that

$v_j \in I_j(V,U)$ for $j = 1,\ldots,i$. We show that $v_{i+1} \in I_{i+1}(V,U)$. Indeed, if $u \in U$, then $uv_{i+1} - v_{i+1} \in \langle v_1,\ldots,v_i\rangle$. By Lemma 1(c) and induction, $\langle v_1,\ldots,v_i\rangle$ is contained in $I_i(V,U)$. Then, by definition, $v_{i+1} \in I_{i+1}(V,U)$. Thus, for $i = 1,\ldots,n$, $v_i \in I_i(V,U) \subset I_n(V,U)$. This proves statement (a). Statement (b) follows from the fact that any $m \in M$ is contained in a finite-dimensional vector space invariant under the action of U. □

Example 2 ([4, Theorem 4, p. 1834] Maximal unipotent subgroups of reductive groups)**.** Suppose that $char k = 0$. Let G be a connected reductive algebraic group. Let $B = TU$ be a Borel subgroup of G where U is the unipotent radical and T is a maximal torus in G which normalizes U. Let Φ (resp.Φ_+) be the set of roots of G (resp. U) with respect to T and let $\{\alpha_1,\ldots,\alpha_m\}$ be the corresponding fundamental root system in Φ_+. Let e_α be the T-weight vector in $\mathcal{L}$ corresponding to the root $\alpha \in \Phi$. Let V be a finite-dimensional irreducible representation of G having highest weight ω with respect to B and highest weight vector v_o. We denote the endomorphism of V corresponding to the action of e_α by D_α. Then $I_{n+1}(V,U)$ is spanned over k by all elements of the form $D_{-\alpha_{i_1}}\ldots D_{-\alpha_{i_r}} v_o$ where $0 \le r \le n$.

Example 3 (Homomorphisms from G to $\mathbb{G}_a$)**.** Let G act on itself by right translation. Then $I_2(k[G],G) = \{f \in k[G] : f - f(e)$ is a homomorphism from G to $\mathbb{G}_a\}$.

Proof. According to Lemma 1(d), $f \in I_2(k[G],G)$ if and only if $(g_1 - I_{k[G]})(g_2 - I_{k[G]})f = 0$ for any $g_1, g_2 \in G$. This is true if and only if for any $x \in G$, we have $f(xg_1g_2) - f(xg_1) - f(xg_2) + f(x) = 0$. If $f \in I_2(k[G],G)$, we put $x = e$ and $f_1 = f - f(e)$; then, f_1 is a homomorphism from G to $\mathbb{G}_a$. Conversely if $f_1 = f - f(e)$ is a homomorphism from G to $\mathbb{G}_a$, then $(g_1 - I_{k[G]})(g_2 - I_{k[G]})f_1 = 0$ by the equation above so $f_1 \in I_2(k[G],G)$. Then $f = f_1 + f(e)$ is also in $I_2(k[G],G)$ since $f(e) \in k = I_1(k[G],G) \subset I_2(k[G],G)$. □

Example 4 (p-polynomials)**.** Suppose that $char k = p > 0$. We identify $\mathbb{G}_a$ with $\{u_b = \left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right) : b \in k\}$ and $k[\mathbb{G}_a]$ with $k[x]$ where $x(u_b) = b$. The group $\mathbb{G}_a$ acts on itself by right translation and $r_{u_b}(x) = x + b$. For $n = 1, 2, \ldots$, a basis over k of $I_n(k[\mathbb{G}_a],\mathbb{G}_a)$ is

$$\{x^{c_0+c_1p+c_2p^2+\ldots+c_Np^N} : 0 \le c_i < p, \sum_{i=0}^{N} c_i \le n-1\}.$$

Proof. We show first that $I_n(k[\mathbb{G}_a],\mathbb{G}_a)$ is spanned by certain x^d. Indeed, let $T = \{t_a = \left(\begin{smallmatrix}a & 0\\ 0 & 1/a\end{smallmatrix}\right) : a \in k^*\}$. Since T normalizes $\mathbb{G}_a$, T maps $k[\mathbb{G}_a]$ to $k[\mathbb{G}_a]$ and $I_n(k[\mathbb{G}_a],\mathbb{G}_a)$ to itself. Since $t_a x = x/a^2$, the weight vectors of T acting on $k[\mathbb{G}_a]$ are the $x^m, m \ge 0$. The x^m have distinct T-weights so we need to determine which x^m are in $I_n(k[\mathbb{G}_a],\mathbb{G}_a)$. We proceed by induction the case $n = 1$ being immediate since $I_1(k[\mathbb{G}_a],\mathbb{G}_a) = k$. In general, suppose that the desired result holds for $I_{n-1}(k[\mathbb{G}_a],\mathbb{G}_a)$ and let $f = x^m \in I_n(k[\mathbb{G}_a],\mathbb{G}_a)$. We write m as

$c_0 + c_1p + c_2p^2 + \ldots + c_Np^N$, $0 \le c_i < p$ for some non-negative integer N. Then

$$\begin{aligned} r_{u_b}(x^m) - x^m &= (x+b)^{c_0}((x+b)^{c_1})^p \ldots ((x+b)^{c_N})^{p^N} - x^m \\ &= (x^{c_0} + \binom{c_0}{1} x^{c_0-1}b + \ldots + b^{c_0}) \times \ldots \\ &\times (x^{c_N} + \binom{c_N}{1} x^{c_N-1}b + \ldots + b^{c_N})^{p^N} - x^m. \end{aligned}$$

Suppose that the non-zero $c's$ are $c_i, c_j, c_\ell, \ldots$ where $0 \le i < j < \ell < \ldots \le N$. Then, the highest power of x appearing in this expansion is $(c_i - 1)p^i + c_jp^j + c_\ell p^\ell + \ldots$. Since $r_{u_b}(x^m) - x^m$ is in $I_{n-1}(k[\mathbb{G}_a], \mathbb{G}_a)$, we see by induction that $(c_i - 1) + c_j + c_\ell + \ldots \le n-2$ so $c_i + c_j + c_\ell + \ldots \le n-1$ and m has the desired form. Conversely, if m has the desired form, the equation above shows that $r_{u_b}(x^m) - x^m$ is in $I_{n-1}(k[\mathbb{G}_a], \mathbb{G}_a)$. □

Example 5. When $char k = 0$, a basis over k of $I_n(k[\mathbb{G}_a], \mathbb{G}_a)$ consists of all the x^m, $m \le n-1$. This is proved by modifying the argument given in Example 4.

Example 6 (Cyclic groups of order p)**.** Suppose that G is a cyclic group and let σ be a generator for G. We have (by induction and Lemma 1(b)) that for each $n = 1, 2, \ldots$, $I_n(M, G) = \{m \in M : (\sigma - I_M)^n m = 0\}$. In particular, suppose that $char k = p > 0$ and that $G = C_p$ is the cyclic group of order p. Then $I_p(M, G) = \{m \in M : (\sigma - I_M)^p m = 0\}$. But $(\sigma - I_M)^p = (\sigma^p - I_M) = (I_M - I_M) = 0$ so $I_p(M, G) = M$. In a similar way, it can be shown that $I_{p-1}(M, G) = \{m \in M : Tr_G m = 0\}$ using the combinatorial formula $\binom{p-1}{i} \equiv (-1)^i (\operatorname{mod} p)$. (Here, Tr_G is the transfer map.)

For the special case when $M = k[G]$, we follow the notation introduced in Example 4 and identify G with the subgroup of $\mathbb{G}_a$ generated by u_1. Let $\overline{x} = x|G$. Then $k[G]$ may be identified with polynomials in $\overline{x}$ of degree $\le p-1$. Reasoning as in Example 4, we may then prove that for $n \le p$, a basis over k of $I_n(k[G], G)$ consists of all the $\overline{x}^m$, $m \le n-1$. A direct calculation shows that the $I_n(k[G], G), 1 \le n \le p$, are the p-indecomposable representations of G [2, Lemma 7.1.1, p.105].

3. $I_n(M, G)$ AND $I_n(k[G], G)$

The action of G on M and on $k[G]$ by left translation gives an action on $k[G] \otimes M$. Let $\Phi : (k[G] \otimes M)^G \to M$ be defined by $\Phi(\sum_i f_i \otimes m_i) = \sum_i f_i(e)m_i$. The Transfer Principle [5, Theorem 9.1, p.49] says that (a) Φ is a vector space isomorphism which is also an algebra isomorphism if M is a k-algebra; (b) if H is a subgroup of G acting on $k[G]$ by right translation, then Φ is an H-equivariant mapping; (c) in particular,Φ gives an isomorphism from $(k[G]^H \otimes M)^G$ to M^H. When H is a subgroup of G, we may consider the modules, $I_n(k[G], H)$ obtained by taking the action of H on $k[G]$ by right translation; G acts by left translation on $I_n(k[G], H)$.

Lemma 2. *Let H be an algebraic subgroup of G and let M be a G-module. Let $v \in M$ and suppose that for each $g \in G$, we have $g \cdot v = \sum_{j=1}^N f_j(g)v_j$ where $\{v_1, \ldots, v_N\}$ is linearly independent over k. Then $v \in I_n(M, H)$ if and only if $f_j \in I_n(k[G], H)$ for each $j = 1, \ldots, N$.*

Proof. We first note that for any $g \in G, h \in H$, we have

$$g \cdot (h \cdot v - v) = \sum_{j=1}^{N} (f_j(gh) - f_j(g))v_j = \sum_{j=1}^{N} ((h \cdot f_j)(g) - f_j(g))v_j.$$

The proof now proceeds by induction, the case $n = 0$ being immediate. So, suppose the statement is true for $n - 1$; we shall show it holds for n. Now, $v \in I_n(M, H)$ if and only if for each $h \in H$, we have $h \cdot v - v \in I_{n-1}(M, H)$. By the induction hypothesis and equation above, this is true if and only if the function $(h \cdot f_j) - f_j$ is in $I_{n-1}(k[G], H)$. By definition, this means that $f_j \in I_n(k[G], H)$. □

Theorem 1. *(a) For each $n \geq 0$, $\Phi : (I_n(k[G], H) \otimes M)^G \to I_n(M, H)$ is a vector space isomorphism.*

(b) If $r \in (k[G]^H \otimes A)^G$ and $m \in (I_n(k[G], H) \otimes M)^G$, then $\Phi(r) \in A^H$ and $\Phi(rm) = \Phi(r)\Phi(m)$.

Proof. We prove statement (a) by induction on n, the case $n = 0$ being immediate. So, suppose that (a) is true for $n-1$; we shall show it holds for n. First, suppose that $v = \Phi(\sum_i f_i \otimes m_i)$ where $(\sum_i f_i \otimes m_i) \in (I_n(k[G], H) \otimes M)^G$ and $\{m_i\}$ is linearly independent over k. Then, for any $g \in G$, we have $v = \Phi(\sum_i \ell_g f_i \otimes g \cdot m_i) = \sum_i (\ell_g f_i)(e) g \cdot m_i$, that is, $g^{-1} \cdot v = \sum_i f_i(g^{-1}) m_i$. Applying Lemma 2, we may conclude that $v \in I_n(M, H)$. Conversely, suppose that $v \in I_n(M, H)$ and that $v = \Phi(\sum_i f_i \otimes m_i)$ where $(\sum_i f_i \otimes m_i) \in (k[G] \otimes M)^G$ and $\{m_i\}$ is linearly independent over k. As above, this means that $g^{-1} \cdot v = \sum_i f_i(g^{-1}) m_i$ for any $g \in G$. Again applying Lemma 2, we see that each $f_i \in I_n(k[G], H)$.

To show statement (b), let $r = \sum_j f_j' \otimes a_j' \in (k[G]^H \otimes A)^G$ and $m = \sum_i f_i \otimes m_i \in (I_n(k[G], H) \otimes M)^G$. Then,

$$\Phi(rm) = \Phi(\sum f_j' f_i \otimes a_j' m_i) = \sum f_j'(e) f_i(e) a_j' m_i$$
$$= \sum f_j'(e) a_j' \sum f_i(e) m_i = \Phi(r)\Phi(m).$$

□

Corollary 1. *If G is a finite group, then there is an N, $1 \leq N \leq |G|$ so that $I_n(M, G) = I_N(M, G)$ for all $n \geq N$.*

Proof. First, suppose that $M = k[G]$. In the sequence of subspaces $I_0(k[G], G) \subset I_1(k[G], G) \subset \ldots \subset I_m(k[G], G) \subset \ldots$, there must be an m so that $I_m(k[G], G) = I_{m+1}(k[G], G)$ since $\dim k[G] = |G|$. Then $I_m(k[G], G) = I_n(k[G], G)$ for all $n \geq m$. The general result now follows from Theorem 1 since $(I_n(k[G], G) \otimes M)^G$ is isomorphic to $I_n(M, G)$. □

The corollary above is of interest only in the modular case (i.e., when p divides $|G|$) and then it is the best possible according to Example 6. In the non-modular case, all the $I_n(M, G) = M^G$ (Theorem 3).

If G is reductive and R is a commutative, finitely generated k-algebra on which G acts rationally, then R^G is a finitely generated k-algebra. This celebrated result is due mainly to H. Weyl, C. Chevalley, D. Mumford, M. Nagata, and W. Haboush [6]. For our purposes, we take A to be a finitely generated k-algebra and $R = A \oplus M$ with the multiplication $(a, m)(a', m') = (aa', am' + a'm)$. If M is a finitely generated A-module, then $A \oplus M$ is a finitely generated k-algebra. Then, by the result just cited, $R^G = A^G \oplus M^G$ is a finitely generated k-algebra. Consequently, M^G is a finitely generated A^G-module. So that we can cite it later, we state this next.

Theorem 2. *Let A be a finitely generated, commutative k-algebra. Let M be an A-module. Let G be a reductive group which acts rationally on A and M. Suppose that the actions of G are compatible with the structure of M as an A-module. If M is a finitely generated A-module, then M^G is a finitely generated A^G-module.*

Corollary 2. *Suppose that G is a reductive group, A is finitely generated over k, and M is a finitely generated A-module. If $I_n(k[G],H)$ is a finitely generated $k[G]^H$-module, then $I_n(M,H)$ is a finitely generated A^H-module.*

Proof. Since $I_n(k[G],H)$ is a finitely generated $k[G]^H$-module, $I_n(k[G],H)\otimes M$ is a finitely generated $k[G]^H\otimes A$-module. Then, by Theorem 2, we see that $(I_n(k[G],H)\otimes M)^G$ is a finitely generated $(k[G]^H\otimes A)^G$-module. We now apply Theorem 1. □

Theorem 3. *Suppose that G is a reductive group.*

(a) If p does not divide $[G:G^o]$, then $I_n(M,G)=M^G$ for all $n\geq 1$.

(b) If p divides $[G:G^o]$, then $I_n(M,G)=I_n(M^{G^o},G/G^o)$ and there is an N, $1\leq N\leq [G:G^o]$ so that $I_n(M,G)=I_N(M,G)$ for all $n\geq N$.

Proof. To prove statement (a), we first show that it is true when G acts on $A=k[G]$ by right translation, i.e., we shall show that $I_n(k[G],G)=k$ for all $n\geq 1$. This is true for $n=1$ so we assume it is true for n and show it for $n+1$. Let $f\in I_{n+1}(k[G],G)$. By the induction hypothesis, $(g_1-I_G)(g_2-I_G)f=0$ for any $g_1,g_2\in G$. Proceeding as in Example 3, we see $f_1=f-f(e)$ is a homomorphism from G to $\mathbb{G}_a$. Since the semi-simple elements are dense in G^0, $f_1|G^0$ is identically 0. Then, $f_1=0$ since $p\nmid [G:G^o]$. For the action of G on M, we apply Theorem 1 to see that $I_n(M,G)=\Phi(I_n(k[G],G)\otimes M)^G=\Phi(k\otimes M)^G=M^G$.

To prove (b), we first note that $I_n(M,G)\subset I_n(M,G^o)=M^{G^o}$ by Lemma 1(d) and what was just proved. Then, an induction argument shows that $I_n(M,G)=I_n(M^{G^o},G/G^o)$. Since G/G^o is finite, we may apply Corollary 1. □

4. Finite generation of $I_n(M,G), char k=0$

For applications to ordinary differential equations, we need A^G to be a finitely generated algebra over $\mathbb{C}$ and $I_n(M,G)$ to be a finitely generated A^G-module. Example 4 shows that even in the simplest cases finite generation may not hold when $char k>0$. In this section, we study the finite generation property in the case $char k=0$.

Theorem 4. *Suppose that $char k=0$. Suppose also that G is connected, A^G is a finitely generated k-algebra, and M^G is a finitely generated A^G-module. Then $I_n(M,G)$ is a finitely generated A^G-module for all $n=0,\ldots$.*

Proof. Let $\mathcal{L}$ be the Lie algebra of G. According to Lemma 1(e), each $L_X\in\mathcal{L}$ gives an A^G-module homomorphism from $I_n(M,G)$ to $I_{n-1}(M,G)$. We now show by induction on n that $I_n(M,G)$ is a finitely generated A^G-module for all $n=0,\ldots$. The case $n=0$ is immediate. So, assume that $n\geq 1$ and that $I_{n-1}(M,G)$ is a finitely generated A^G-module. Let $\{L_{X_1},\ldots,L_{X_s}\}$ be a basis for $\mathcal{L}$. We show by induction on i that

$$\ker L_{X_i}\cap\ldots\cap\ \ker L_{X_s}\cap I_n(M,G)$$

is a finitely generated A^G-module. When $i=1$, we have $m\in\ker L_{X_1}\cap\ldots\cap\ker L_{X_s}$ if and only if $L_X\cdot m=0$ for all $L_X\in\mathcal{L}$, i.e., if and only if $m\in M^G$. Hence, ker $L_{X_1} \cap \ldots \cap$ ker $L_{X_s} \cap I_n(M,G) = M^G$ which is a finitely generated A^G-module by assumption. Now suppose that ker $L_{X_{i-1}} \cap \ldots \cap$ ker $L_{X_s} \cap I_n(M,G)$ is a finitely generated A^G-module for $i \geq 2$. We show that $M^* =$ ker $L_{X_i} \cap \ldots \cap$ ker $L_{X_s} \cap I_n(M,G)$ is also a finitely generated A^G-module. Let $L^*_{X_{i-1}} = L_{X_{i-1}}|M^*$. Then $L^*_{X_{i-1}}(M^*) \subset L_{X_{i-1}}(I_n(M,G)) \subset I_{n-1}(M,G)$ so $L^*_{X_{i-1}}(M^*)$ is an A^G-submodule of the finitely generated A^G-module $I_{n-1}(M,G)$. Also,

$$\text{ker } L^*_{X_{i-1}} = \text{ker } L_{X_{i-1}} \cap M^* = \text{ker } L_{X_{i-1}} \cap \ldots \cap \text{ ker} L_{X_s} \cap I_n(M,G)$$

which is a finitely generated A^G-module by induction. Hence, M^* is a finitely generated A^G-module.

In particular, ker $L_{X_s} \cap I_n(M,G)$ is a finitely generated A^G-module. Consider $L_{X_s} : I_n(M,G) \to I_{n-1}(M,G)$. Now, $L_{X_s}(I_n(M,G))$ is an A^G-submodule of the finitely generated A^G-module $I_{n-1}(M,G)$ and, so, is finitely generated. We have just shown that the kernel of L_{X_s} is a finitely generated A^G-module. It follows that $I_n(M,G)$ is a finitely generated A^G-module. □

Theorem 5. *Suppose that char$k = 0$, A^{G^o} is a finitely generated k-algebra, and M^{G^o} is a finitely generated A^{G^o}-module. Then $I_n(M,G)$ is a finitely generated A^G-module for all $n = 0, \ldots$.*

Proof. Since A^{G^o} is finitely generated, so is A^G since G/G^o is a finite group. Furthermore, A^{G^o} is an integral extension of A^G. By Theorem 4, $I_n(M,G^o)$ is a finitely generated A^{G^o}-module for all $n = 0, \ldots$. Thus, each $I_n(M,G^o)$ is a finitely generated A^G-module. Since $I_n(M,G)$ is an A^G-submodule of $I_n(M,G^o)$, it is also finitely generated. □

Corollary 3. *With respect to the action of G on itself by right translation, $I_n(k[G],G)$ is a finite-dimensional vector space for all $n = 0, \ldots$.*

In general, the algebra $k[X]^G$ is not finitely generated over k. However, there are non-zero elements $f \in k[X]^G$ so that the localization $k[X]^G_f$ is a finitely generated k-algebra. The set of all such f together with 0 forms an ideal in $k[X]^G$. Most recently, this ideal has been studied by Derksen and Kemper who call it the *finite generation ideal* [3, Section 2.2].

Corollary 4. *Suppose that char$k = 0$ and let $f \neq 0$ be in the finite generation ideal for $k[X]$. Then $k[X_f]^G$ is finitely generated over k and each $I_n(k[X_f],G)$ is a finitely generated $k[X_f]^G$-module.*

Proof. Since f is in the finite generation ideal for $k[X]$, $k[X_f]^G = k[X]^G_f$ is finitely generated over k. Applying Theorem 5, we see that each $I_n(k[X_f],G)$ is a finitely generated $k[X_f]^G$-module. □

5. G-EQUIVARIANT POLYNOMIAL MAPPINGS

Let $\mathcal{R}_uG$ denote the unipotent radical of G. As we have seen in Examples 4 and 5, the modules $I_n(k[X],G)$ may or may not be finitely generated over $k[X]^G$. In this section, we show that for any finite-dimensional G-module W, the $k[X]^G$-module of W-relative invariants, $(I_n(k[X],G) \otimes W)^G$ is finitely generated whenever $k[X]^{\mathcal{R}_uG}$ is a finitely generated k-algebra. The key idea is to relate the $I_n(k[X],G)$ to equivariant mappings.

Let W be a finite-dimensional G-module. The vector space of all polynomial mappings from X to W may be identified with $k[X] \otimes W$ and is naturally a $k[X]$-module. A polynomial mapping $F : X \to W$ is said to be G-equivariant if $F(g \cdot x) = g \cdot F(x)$ for all $g \in G, x \in X$. The vector space of all G-equivariant polynomial mappings from X to W may be identified with $(k[X] \otimes W)^G$. It is a $k[X]^G$-module.

Theorem 6. *Let G act regularly on an irreducible affine variety X; let W be a finite-dimensional G-module. If $k[X]^{\mathcal{R}_u G}$ is a finitely generated k-algebra, then $k[X]^G$ is a finitely generated k-algebra and $(k[X] \otimes W)^G$ is a finitely generated $k[X]^G$-module.*

Proof. First, we note that $k[X]^G$ is a finitely generated k-algebra since $k[X]^G = (k[X]^{\mathcal{R}_u G})^{G/\mathcal{R}_u G}$ and $G/\mathcal{R}_u G$ is reductive. If $W^{\mathcal{R}_u G} = W$, then, $(k[X] \otimes W)^G = ((k[X] \otimes W)^{\mathcal{R}_u G})^{G/\mathcal{R}_u G} = (k[X]^{\mathcal{R}_u G} \otimes W)^{G/\mathcal{R}_u G}$. Since $(k[X]^{\mathcal{R}_u G} \otimes W)$ is a finitely generated $k[X]^{\mathcal{R}_u G}$-module, $(k[X]^{\mathcal{R}_u G} \otimes W)^{G/\mathcal{R}_u G}$ is a finitely generated $(k[X]^{\mathcal{R}_u G})^{G/\mathcal{R}_u G} = k[X]^G$-module by Theorem 2.

We now proceed by induction on $\dim W$. If $\dim W = 1$, $W^{\mathcal{R}_u G} = W$. Otherwise, suppose that $\dim W^{\mathcal{R}_u G} < \dim W$ and let $\pi : W \to W/W^{\mathcal{R}_u G}$. The subspace $W^{\mathcal{R}_u G}$ is G-invariant since $\mathcal{R}_u G$ is a normal subgroup of G. Thus, a G-equivariant mapping $F : X \to W$ gives a G-equivariant mapping $\pi \circ F : X \to W/W^{\mathcal{R}_u G}$. Let $\overline{F} = \pi \circ F$. The mapping $F \to \overline{F}$ is a $k[X]^G$-homomorphism from the $k[X]^G$-module $(k[X] \otimes W)^G$ to the $k[X]^G$-module $(k[X] \otimes W/W^{\mathcal{R}_u G})^G$. The image is a finitely generated $k[X]^G$-module by the induction assumption since $1 \leq \dim W^{\mathcal{R}_u G} < \dim W$. The kernel consists of G-equivariant maps from X to $W^{\mathcal{R}_u G}$ and is a finitely generated $k[X]^G$-module by what was proved above. Hence, $(k[X] \otimes W)^G$ is a finitely generated $k[X]^G$-module. □

For finite group actions, Theorem 6 is proved in [1, Proposition 3.3], for example.

Corollary 5. *If $k[X]^{\mathcal{R}_u G}$ is a finitely generated k-algebra, then $(I_n(k[X], G) \otimes W)^G$ is a finitely generated $k[X]^G$-module for all $n \geq 0$.*

Proof. This follows from Lemma 1(b) and Theorem 6 since $k[X]^G$ is finitely generated and $(I_n(k[X], G) \otimes W)^G$ is a $k[X]^G$-submodule of $(k[X] \otimes W)^G$. □

Corollary 6. *Let X be an irreducible, affine G-variety. Let $J \subset k[X]^{\mathcal{R}_u G}$ be the finite generation ideal for $\mathcal{R}_u G$. Suppose that there is a non-zero $f \in J \cap k[X]^G$. Then, $(k[X_f] \otimes W)^G$ and $(I_n(k[X_f], G) \otimes W)^G$ are finitely generated $k[X_f]^G$-modules.*

Proof. Since $f \in J \cap k[X]^G$, $k[X_f]^{\mathcal{R}_u G}$ is finitely generated over k. The corollary now follows from Theorem 6 and Corollary 5. □

Lemma 3. *Let $\dim W = n_0$. If U is a unipotent group, then $(k[X] \otimes W)^U = (I_{n_0}(k[X], U) \otimes W)^U$.*

Proof. It is enough to show that $(k[X] \otimes W)^U \subset I_{n_0}(k[X], U) \otimes W$. Let $\{w_1, \ldots, w_{n_0}\}$ be a basis for W and $\{\lambda_1, \ldots, \lambda_{n_0}\}$ the dual basis. Let $F = \sum_i (f_i \otimes w_i) \in (k[X] \otimes W)^U$ be a U-equivariant polynomial mapping from X to W. Let $F^* : k[W] \to k[X]$ be its dual map. Then, $F^*(\lambda_i) = f_i$. Since F is G-equivariant so is F^*. By Example 1, $W^* \subset I_{n_0}(k[W], U)$ so we see that $F^*(W^*) \subset I_{n_0}(k[X], U)$, i.e., $f_i \in I_{n_0}(k[X], U)$. □

The condition required in Corollary 6 that there is a non-zero $f \in J \cap k[X]^G$ is always satisfied if $k[X]^{\mathcal{R}_u G}$ is finitely generated or if G is unipotent. For arbitrary linear algebraic groups G, Renner and Rittatore have considered somewhat related questions [7]. They define the action of G on X to be *observable* if for any G-invariant proper, closed subset Y of X, there is a non-zero $f \in k[X]^G$ so that $f|_Y = 0$. Now, suppose that G is solvable and let $E_G(X) = \{\chi \in X(G)$: there is a non-zero $f \in k[X]$ such that $gf = \chi(g)f$ for all $g \in G\}$. Then, the action of G on X is observable if and only if $E_G(X)$ is a group [7, Corollary 3.16].

Lemma 4. *Suppose that G is solvable, say $G = TU$, where T is a maximal torus and U is the unipotent radical. If the action of G on X is observable, then there is a non-zero $f \in k[X]^G$ so that $k[X]^U_f$ is a finitely generated k-algebra.*

Proof. Let J be the finite generation ideal for U acting on X. Since U is a normal subgroup of G, the ideal J is invariant under G. Thus, there is a non-zero φ in the ideal J which is a T-weight vector corresponding, say, to the character $\chi \in X(T)$. Since $E_G(X)$ is a group, there is a $\psi \in k[X]^U$ corresponding to the character $-\chi$. Let $f = \varphi\psi$. Then, $f \in J \cap k[X]^G$. □

6. Representations of the additive group

Throughout this section, we assume that *chark* $= 0$. Let $G = SL_2(k)$ be the group consisting of 2×2 matrices (a_{ij}) whose determinant is 1. Let $k[G] = k[x_{11}, x_{12}, x_{21}, x_{22}]$ where $x_{ij}(g) = a_{ij}$. Let U (resp. U^-) be the subgroup of G consisting of all upper triangular (resp. lower triangular) matrices with 1's on the diagonal. We may identify U with the additive group $\mathbb{G}_a$. The group G acts by left multiplication on the vector space V consisting of 2×1 column matrices. The actions of G on V and V^* are equivariant. Let $S(V)$ be the symmetric algebra on V and let $S^d(V)$ be elements in $S(V)$ homogeneous of degree d. We consider the natural action of G on $S^d(V)$ and the action of G by left translation on $k[G]^U$ and $I_{n+1}(k[G], U)$

Theorem 7. *With respect to the action of U on $k[G]$ by right translation, for each $n \geq 0$, $I_{n+1}(k[G], U)$ is a free $k[G]^U$-module with basis $\{x_{12}^d x_{22}^{n-d} : d = 0, \dots n\}$. In particular, $k[G]^U \otimes S^n(V)$ and $I_{n+1}(k[G], U)$ are isomorphic as G-modules.*

Proof. Let D_α (resp. $D_{-\alpha}$) be a basis of the Lie algebra of U (resp. U^-). Then, $D_\alpha \cdot x_{11} = D_\alpha \cdot x_{21} = 0$, $D_\alpha \cdot x_{12} = x_{11}$, $D_\alpha \cdot x_{22} = x_{21}$ and $D_{-\alpha} \cdot x_{11} = x_{12}$, $D_{-\alpha} \cdot x_{21} = x_{22}$, $D_{-\alpha} \cdot x_{12} = D_{-\alpha} \cdot x_{22} = 0$. It is known that $k[G]^U = k[x_{11}, x_{21}]$. Now, $k[G]$ is a direct sum of irreducible representations of G acting by right translation on $k[G]$, say $k[G] = \bigoplus V_i$. Let v_i be a non-zero element in V_i^U; any $a \in k[G]^U$ is a linear combination of the v_i. By Lemma 1(d), $I_{n+1}(k[G], U) = \bigoplus I_{n+1}(V_i, U)$. Thus, applying Example 2, we see that $I_{n+1}(k[G], U)$ is the vector space over k spanned by all the $(D_{-\alpha})^r \cdot a$, where $0 \leq r \leq n$ and $a \in k[G]^U$. Since $k[G]^U = k[x_{11}, x_{21}]$, from the usual rules for differentiating a product, we see that any element in $I_{n+1}(k[G], U)$ is a $k[G]^U$-linear combination of elements of the form $x_{12}^d x_{22}^{r-d}$ for $0 \leq d \leq r \leq n$. However, $1 = x_{11}x_{22} - x_{12}x_{21}$. Therefore, for $r < n$, $x_{12}^d x_{22}^{r-d}$ can be written as a $k[G]^U$-linear combination of the $x_{12}^{d'} x_{22}^{n-d'}$ since $x_{12}^d x_{22}^{r-d} = x_{12}^d x_{22}^{r-d} \times (x_{11}x_{22} - x_{12}x_{21})^{n-r}$.

Finally, we show that the $x_{12}^d x_{22}^{n-d}$ are linearly independent over $k[G]^U$. Indeed, suppose that $\sum_d a_d x_{12}^d x_{22}^{n-d} = 0$ where each $a_d \in k[G]^U$. Let r be the smallest

d for which $a_d \neq 0$. We divide the equation by x_{12}^r and see that $(*)$ $a_r x_{22}^{n-r} = -\sum_{d>r} a_d x_{12}^{d-r} x_{22}^{n-d}$ so x_{12} divides the right-hand side of this equation. This is impossible. Indeed, we define an element $g \in G$ as follows. First, since a_r is a polynomial in x_{11} and x_{21}, we may choose $a_{11} \neq 0$ and a_{21} so that $a_r(a_{11}, a_{21}) \neq 0$. Then, let $a_{12} = 0$ and $a_{22} = 1/a_{11}$. The left-hand side of $(*)$ is non-zero whereas the right-hand side is 0.

The action by left translation of G on the vector space spanned by x_{12} and x_{22} is isomorphic to the action of G on V. This proves the last statement in the theorem. □

Theorem 8 ([8]). *If $W = V \oplus S^2(V)$, then $(k[W] \otimes S^n(V))^G$ is a Cohen-Macaulay $k[W]^G$-module for all n. If $W = V \oplus S^d(V)$ with $d > 2$, then $(k[W] \otimes S^n(V))^G$ is a Cohen-Macaulay $k[W]^G$-module if and only if $n + 1 < \frac{(d+1)^2}{4}$ for d odd or $n + 1 < \frac{d(d+2)}{4}$ for d even.*

Theorem 9. *$I_{n+1}(k[S^d(V)], U)$ is a Cohen-Macaulay module over $k[S^d(V)]^U$ for all n when $d = 2$. When $d > 2$, $I_{n+1}(k[S^d(V)], U)$ is a Cohen-Macaulay module over $k[S^d(V)]^U$ if and only if $n+1 < \frac{(d+1)^2}{4}$ for d odd or $n+1 < \frac{d(d+2)}{4}$ for d even.*

Proof. We first apply Theorem 1 to see that $I_{n+1}(k[S^d(V)], U)$ is isomorphic to $(I_{n+1}(k[G], U) \otimes k[S^d(V)])^G$ which, by Theorem 7, is $(k[G]^U \otimes S^n(V) \otimes k[S^d(V)])^G$. Now, $k[G]^U = k[x_{11}, x_{21}]$ which may be identified with $k[V]$ as a G-module. Therefore, $I_{n+1}(k[S^d(V)], U)$ is isomorphic to

$$(k[V] \otimes k[S^d(V)] \otimes S^n(V))^G = (k[V \times S^d(V)] \otimes S^n(V))^G.$$

Let $W = V \times S^d(V)$. Then, we have $k[W]^G = k[V \times S^d(V)]^G = (k[G]^U \otimes k[S^d(V)])^G = k[S^d(V)]^U$ according to Theorem 1. We now apply Theorem 8 with $W = V \times S^d(V)$ to prove the theorem. □

7. Differential equations

Our interest in modules of higher invariants comes from our study of G-symmetric ordinary differential equations [4]. Thus, it seems appropriate to sketch some of the connections of this paper with ordinary differential equations. Let $char k = 0$ (in particular, we may take $k = \mathbb{C}$) and let $W := k^n$. Consider the ordinary differential equation

$$\dot{x} = \frac{dx}{dt} = F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

on W with each $f_i \in k[W] = k[x_1, \ldots, x_n]$. (One may think of solutions as formal power series, in general.) To F we assign the derivation $L_F = \sum_i f_i \frac{\partial}{\partial x_i}$ of $k[x_1, \ldots, x_n]$. Then, for any polynomial $\phi(x_1, \ldots, x_n)$ and any solution $v(t)$ to the differential equation above, we have $\frac{d}{dt}\phi(v(t)) = (L_F\phi)(v(t))$; we write $\dot{\phi}$ instead of $L_F\phi$. We call L_F a G-equivariant derivation if L_F commutes with the natural action of G on $k[W]$. One verifies that L_F is G-equivariant if and only if the corresponding map F is G-equivariant.

As always, let X be an irreducible, G-invariant affine subvariety of W and assume that L_F sends the vanishing ideal of X to itself. We may then consider the f_i as elements of $k[X]$ and L_F as a derivation of $k[X]$. In this case, it can be shown that if $x \in X$, then $F(x)$ is tangent to X. Thus, the differential equation above can be considered as a differential equation on X.

It follows immediately from Lemma 1(d) that if L_F is G-equivariant then it maps each $I_m(k[X],G)$, $m = 1,2,\ldots$ to itself. For $m = 1$ this fact has been utilized for a long time to obtain a reduced system corresponding to the differential equation above as follows. Assume that $I_1(k[X],G) = k[X]^G$ admits a finite set of generators $\phi_1,\ldots,\phi_r$. Then there exist polynomials ψ_j such that $\dot{\phi_j} = L_F(\phi_j) = \psi_j(\phi_1,\ldots,\phi_r)$ for $1 \le j \le r$. Letting

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_r \end{pmatrix},$$

the previous equation shows that Φ sends solutions of $\dot{x} = F(x)$ to solutions $(\phi_1(x(t)),\ldots,\phi_r(x(t)))$ of the "reduced system" $\dot{y} = \Psi(y)$ on $Y := \overline{\Phi(X)}$.

We can extend this procedure as follows. If the module $I_m(k[X],G)$ has a finite set $\theta_{m,1},\ldots,\theta_{m,\ell_m}$ of generators, there exist polynomials h_{ij} such that

$$(*) \qquad \dot{\theta}_{m,i} = L_F(\theta_{m,i}) = \sum_j h_{ij}(\phi_1,\ldots,\phi_r)\theta_{m,j}, \quad 1 \le i \le \ell_m.$$

Thus $\Theta_m := (\theta_{m,1},\ldots,\theta_{m,\ell_m})^t$ maps solutions of $\dot{x} = F(x)$ to solutions of the non-autonomous linear differential equation

$$\dot{z} = H(y)\cdot z, \quad H(y) := (h_{ij}(y)),$$

where y stands for a solution of the reduced equation above.

This observation may not provide useful information for arbitrary groups, but it is quite valuable if $G = U$ is unipotent and $k[X]^U$ is finitely generated. Indeed, by Example 1, each $x_i \in I_n(k[X],U)$. It then follows from Theorem 4 and equation $(*)$ above, read with $m := n$, that once solutions $(\phi_1(x(t)),\ldots,\phi_r(x(t)))$ to the reduced system are found, solutions to the original system can be found by solving a system of non-autonomous linear differential equations. In other words, given a U-symmetric differential equation $\dot{x} = F(x)$ on the variety X, there remains only a non-autonomous linear differential equation modulo the reduced system $\dot{y} = \Psi(y)$. Using the inclusions $I_2(k[X],U) \subseteq \cdots \subseteq I_n(k[X],U)$, one may refine this to obtain a kind of echelon form for the linear system. In the special case $X = k^n$, this result was proved under stronger hypotheses in [4, Proposition 1 and Corollary 1]. The comments above give an analogous statement for an arbitrary irreducible variety X. The extension to varieties of the rest of the program carried out in [4] will be taken up elsewhere.

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