MODULES OF HIGHER ORDER INVARIANTS

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(Communicated by Harm Derksen)

ABSTRACT. Let $k$ be an algebraically closed field of characteristic $p \geq 0$. Let $A$ be a commutative $k$-algebra with multiplicative identity and let $M$ be an $A$-module. Let $G$ be a linear algebraic group acting rationally on both $A$ and $M$. In this paper we study $A^G$-modules of $n$th order invariants, $I_n(M, G)$. The $I_n(M, G)$ are defined inductively by $I_0(M, G) = \{0\}$ and $I_n(M, G) = \{m \in M : g \cdot m - m \in I_{n-1}(M, G) \text{ for all } g \in G\}$. We show that some fundamental problems concerning these modules can be reduced to the case $I_n(k[G], G)$ where $G$ acts on itself by right translation. We study the questions as to when $I_n(M, G)$ is a finitely generated $A^G$-module and how the $I_n(M, G)$ are related to equivariant mappings. For the classical case of $G_a$ acting on binary forms, we describe the $I_n(M, G)$ and determine when they are Cohen-Macaulay.

1. INTRODUCTION

Let $k$ be an algebraically closed field of characteristic $p \geq 0$. Let $A$ be a commutative $k$-algebra with multiplicative identity and let $M$ be an $A$-module. Let $G$ be a linear algebraic group having identity $e$ and suppose that $G$ acts rationally on both $A$ and $M$. In this paper we study $A^G$-modules of higher order invariants, $I_n(M, G)$. These modules first arose in our work on the reconstruction problem in ordinary differential equations [4]. Here, we study their algebraic properties. The $I_n(M, G)$ are defined in Section 2 where basic properties are proved and examples given. In Section 3, we show that some fundamental problems concerning these modules can be reduced to the case $I_n(k[G], G)$ where $G$ acts on itself by right translation. For example, we show that when $G$ is connected and reductive, $I_n(M, G) = M^G$. For arbitrary $G$, when is $I_n(M, G)$ a finitely generated $A^G$-module? This question is studied in Section 4 where it is shown that finite generation follows from the finite generation of $A^G$ when $\text{char} k = 0$. For general $G$, we study the relation of this question to the finite generation ideal. The $I_n(M, G)$ are closely related to equivariant mappings. This relationship is studied in Section 5. In Section 6, we study the classical case of $G_a$ acting on binary forms. We describe the $I_n(M, G)$ and determine when they are Cohen-Macaulay. Section 7 gives an overview of the applications to ordinary differential equations.

Received by the editors September 26, 2012 and, in revised form, May 22, 2013.
2010 Mathematics Subject Classification. Primary 13A50; Secondary 37C80.
Key words and phrases. Invariants, modules.
The authors thank the referee for a very careful reading of the manuscript and many helpful suggestions.

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2. Definition and basic properties

Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \). Let \( A \) be a commutative \( k \)-algebra with multiplicative identity and let \( M \) be an \( A \)-module. Let \( G \) be a linear algebraic group having identity \( e \) and let \( G^o \) be the irreducible component of \( e \) in \( G \). The group \( G \) acts on \( k[G] \) by left and right translation, denoted respectively by \( \ell_g \) and \( r_g \). Suppose that \( G \) acts rationally on both \( A \) and \( M \). These actions will be denoted by \( (g,a) \rightarrow g \cdot a \) and \( (g,m) \rightarrow g \cdot m \), respectively, for \( a \in A, m \in M \), and \( g \in G \). We shall also assume that the actions are compatible, that is, that \( g \cdot am = (g \cdot a)(g \cdot m) \). (When \( A = k \), we shall assume that the action of \( G \) is trivial.) We denote the identity mapping on \( M \) by \( I_M \). If \( f \in A \) is not nilpotent, we denote the localization of \( A \) at \( f \) by \( A_f \). Let \( X \) be an irreducible affine \( G \)-variety and let \( k[X] \) be the algebra of polynomial functions on \( X \). If \( f \in k[X] \), then \( X_f = \{ x \in X : f(x) \neq 0 \} \). It is known that \( k[X_f] = k[X]_f \).

**Definition 1.** Let \( I_0(M,G) = \{0\} \subset M \). For \( n \geq 1 \), let

\[ I_n(M,G) = \{ m \in M : g \cdot m - m = I_{n-1}(M,G) \text{ for all } g \in G \}. \]

We shall call \( I_n(M,G) \) the module of \( n \)th order invariants.

**Lemma 1.** (a) \( I_1(M,G) = M^G = \{ m \in M : g \cdot m = m \text{ for all } g \in G \} \).

(b) \( I_n(M,G) \) is an \( A^G \)-module and a \( G \)-module.

(c) \( I_0(M,G) \subset I_1(M,G) \subset \ldots \subset I_n(M,G) \subset I_{n+1}(M,G) \subset \ldots \).

(d) \( I_n(M,G) = \{ m \in M : (g_1 - I_M) \ldots (g_n - I_M) \cdot m = 0 \text{ for all } g_1, \ldots, g_n \in G \} \).

(e) If \( \text{char} k = 0 \) and \( G \) is connected with Lie algebra \( \mathcal{L} \), then \( I_n(M,G) = \{ m \in M : L_{X_1} \cdots L_{X_n} \cdot m = 0 \text{ for all } L_{X_1}, \ldots, L_{X_n} \in \mathcal{L} \} \).

**Proof.** An element \( m \in M \) is in \( I_1(M,G) \) if and only if \( g \cdot m - m = I_0(M,G) = \{0\} \) for all \( g \in G \), that is, if and only if \( g \cdot m = m \). To prove the remaining properties, we proceed by induction, the cases \( n = 0, 1 \) always being immediate. So, suppose the properties hold for \( I_n(M,G) \), \( n \geq 1 \). For (b), let \( a \in A^G, m \in I_{n+1}(M,G) \), and \( g, a \in G \). We first show that \( am \in I_{n+1}(M,G) \). Indeed, \( g \cdot am - am = (g \cdot a)(g \cdot m) - am = a(g \cdot m - m) \). By the definition of \( I_{n+1}(M,G) \), \( g \cdot m - m \in I_n(M,G) \) which is an \( A^G \)-module by the induction hypothesis so \( a(g \cdot m - m) \in I_n(M,G) \). Next, we show that \( g \cdot m \in I_n(M,G) \). Indeed, \( g \cdot (g \cdot m) = (gg) \cdot (m - m) = (g \cdot m - m) \) and both terms are in \( I_n(M,G) \) by definition. For (c), suppose that \( I_{n-1}(M,G) \subset I_n(M,G) \) and let \( m \in I_n(M,G) \). Then, \( g \cdot m - m = I_{n-1}(M,G) \subset I_n(M,G) \) so \( m \in I_{n+1}(M,G) \). Property (d) follows immediately from induction and the definition of \( I_n(M,G) \). For (e), we recall that when \( \text{char} k = 0 \) and \( G \) is connected, then for any rational \( G \)-module \( V \) and \( v \in V, g \cdot v = v \) for all \( g \in G \) is equivalent to \( L_X \cdot v = 0 \) for all \( L_X \in \mathcal{L} \). Therefore, \( I_{n+1}(M,G) = \{ m \in M : L_X \cdot m \in I_n(M,G) \text{ for all } L_X \in \mathcal{L} \} \). \( \square \)

**Example 1** (Unipotent groups). Let \( U \) be a unipotent group. Let \( V \) be a finite-dimensional \( U \)-module.

(a) If \( \dim V = n \), then \( V \subset I_n(V,U) \).

(b) \( M = \bigcup_{n \geq 1} I_n(M,U) \).

**Proof.** Let \( \{v_1, \ldots, v_n\} \) be a basis for \( V \) relative to which each element in \( U \) is upper triangular, i.e., if \( u \in U \), then \( uv_1 = v_1 \) and \( uv_i - v_i \in \langle v_1, \ldots, v_{i-1} \rangle \), the vector space spanned by \( \{v_1, \ldots, v_{i-1}\} \) for all \( i = 2, \ldots, n \). We show by induction that \( v_i \in I_i(V,U) \) for \( i = 1, \ldots, n \). Indeed, \( v_1 \in V^U = I_1(V,U) \). Now, suppose that
to the root $v_j \in I_j(V,U)$ for $j = 1, \ldots, i$. We show that $v_{i+1} \in I_{i+1}(V,U)$. Indeed, if $u \in U$, then $u v_{i+1} - v_{i+1} \in \langle v_1, \ldots, v_i \rangle$. By Lemma 1(c) and induction, $\langle v_1, \ldots, v_i \rangle$ is contained in $I_i(V,U)$. Then, by definition, $v_{i+1} \in I_{i+1}(V,U)$. Thus, for $i = 1, \ldots, n$, $v_i \in I_i(V,U) \subset I_n(V,U)$. This proves statement (a). Statement (b) follows from the fact that any $m \in M$ is contained in a finite-dimensional vector space invariant under the action of $U$.

\[\square\]

Example 2 ([4, Theorem 4, p. 1834] Maximal unipotent subgroups of reductive groups). Suppose that $char k = 0$. Let $G$ be a connected reductive algebraic group. Let $B = TU$ be a Borel subgroup of $G$ where $U$ is the unipotent radical and $T$ is a maximal torus in $G$ which normalizes $U$. Let $\Phi$ (resp. $\Phi_+$) be the set of roots of $G$ (resp. $U$) with respect to $T$ and let $\{\alpha_1, \ldots, \alpha_m\}$ be the corresponding fundamental root system in $\Phi_+$. Let $e_\alpha$ be the $T$-weight vector in $L$ corresponding to the root $\alpha \in \Phi$. Let $V$ be a finite-dimensional irreducible representation of $G$ having highest weight $\omega$ with respect to $B$ and highest weight vector $v_\omega$. We denote the endomorphism of $V$ corresponding to the action of $e_\alpha$ by $D_\alpha$. Then $I_{n+1}(V,U)$ is spanned over $k$ by all elements of the form $D_{-\alpha_1} \cdots D_{-\alpha_r} v_\omega$ where $0 \leq r \leq n$.

Example 3 (Homomorphisms from $G$ to $G_a$). Let $G$ act on itself by right translation. Then $I_2(k[G],G) = \{f \in k[G] : f - f(e)\}$ is a homomorphism from $G$ to $G_a$.

\[\text{Proof.}\] According to Lemma 1(d), $f \in I_2(k[G],G)$ if and only if $(g_1 - I_k[G])(g_2 - I_k[G])f = 0$ for any $g_1, g_2 \in G$. This is true if and only if for any $x \in G$, we have $f(x g_1 g_2) - f(x g_1) - f(x g_2) + f(x) = 0$. If $f \in I_2(k[G],G)$, we put $x = e$ and $f_1 = f - f(e)$; then, $f_1$ is a homomorphism from $G$ to $G_a$. Conversely if $f_1 = f - f(e)$ is a homomorphism from $G$ to $G_a$, then $(g_1 - I_k[G])(g_2 - I_k[G])f_1 = 0$ by the equation above so $f_1 \in I_2(k[G],G)$. Then $f = f_1 + f(e)$ is also in $I_2(k[G],G)$ since $f(e) \in k = I_1(k[G],G) \subset I_2(k[G],G)$.

\[\square\]

Example 4 ($p$-polynomials). Suppose that $char k = p > 0$. We identify $G_a$ with \{ $u_b = (\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) : b \in k$ \} and $k[G_a]$ with $k[x]$ where $x(u_b) = b$. The group $G_a$ acts on itself by right translation and $u_b(x) = x + b$. For $n = 1, 2, \ldots$, a basis over $k$ of $I_n(k[G_a],G_a)$ is

\[\{x^{c_0 + c_1 p + c_2 p^2 + \cdots + c_N p^N} : 0 \leq c_i < p, \sum_{i=0}^{N} c_i \leq n - 1 \}.

\[\text{Proof.}\] We show first that $I_n(k[G_a],G_a)$ is spanned by certain $x^d$. Indeed, let $T = \{t_a = (\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}) : a \in k^* \}$. Since $T$ normalizes $G_a$, $T$ maps $k[G_a]$ to $k[G_a]$ and $I_n(k[G_a],G_a)$ to itself. Since $t_a x = x/a^2$, the weight vectors of $T$ acting on $k[G_a]$ are the $x^m$, $m \geq 0$. The $x^m$ have distinct $T$-weights so we need to determine which $x^m$ are in $I_n(k[G_a],G_a)$. We proceed by induction the case $n = 1$ being immediate since $I_1(k[G_a],G_a) = k$. In general, suppose that the desired result holds for $I_{n-1}(k[G_a],G_a)$ and let $f = x^m \in I_n(k[G_a],G_a)$. We write $m$ as
Example 5. When $\text{char} k = 0$, a basis over $k$ of $I_n(k[G_a], G_a)$ consists of all the $x^m$, $m \leq n - 1$. This is proved by modifying the argument given in Example 4.

Example 6 (Cyclic groups of order $p$). Suppose that $G$ is a cyclic group and let $\sigma$ be a generator for $G$. We have (by induction and Lemma 1(b)) that for each $n = 1, 2, \ldots, I_n(M, G) = \{m \in M : (\sigma - I_M)^p m = 0\}$. In particular, suppose that $\text{char} k = p > 0$ and that $G = {c}$ is the cyclic group of order $p$. Then $I_p(M, G) = \{m \in M : (\sigma - I_M)^p m = 0\}$. But $(\sigma - I_M)^p = (\sigma^p - I_M) = (I_M - I_M) = 0$ so $I_p(M, G) = M$. In a similar way, it can be shown that $I_{p-1}(M, G) = \{m \in M : Tr_G m = 0\}$ using the combinatorial formula $(p^r - 1) \equiv (-1)^r \pmod{p}$. (Here, $Tr_G$ is the transfer map.) For the special case when $M = k[G]$, we follow the notation introduced in Example 4 and identify $G$ with the subgroup of $G_a$ generated by $u_1$. Let $x = x|G$. Then $k[G]$ may be identified with polynomials in $x$ of degree $\leq p - 1$. Reasoning as in Example 4, we may then prove that for $n \leq p$, a basis over $k$ of $I_n(k[G], G)$ consists of all the $x^m$, $m \leq n - 1$. A direct calculation shows that the $I_n(k[G], G)$, $1 \leq n \leq p$, are the $p$-indecomposable representations of $G$ [2] Lemma 7.1.1, p.105].

3. $I_n(M, G)$ AND $I_n(k[G], G)$

The action of $G$ on $M$ and on $k[G]$ by left translation gives an action on $k[G] \otimes M$. Let $\Phi : (k[G] \otimes M)^G \rightarrow M$ be defined by $\Phi(\sum_i f_i \otimes m_i) = \sum_i f_i(\sigma)m_i$. The Transfer Principle [3] Theorem 9.1, p.49] says that (a) $\Phi$ is a vector space isomorphism which is also an algebra isomorphism if $M$ is a $k$-algebra; (b) if $H$ is a subgroup of $G$ acting on $k[G]$ by right translation, then $\Phi$ is an $H$-equivariant mapping; (c) in particular, $\Phi$ gives an isomorphism from $(k[G]^H \otimes M)^G$ to $M^H$. When $H$ is a subgroup of $G$, we may consider the modules, $I_n(k[G], H)$ obtained by taking the action of $H$ on $k[G]$ by right translation; $G$ acts by left translation on $I_n(k[G], H)$.

Lemma 2. Let $H$ be an algebraic subgroup of $G$ and let $M$ be a $G$-module. Let $v \in M$ and suppose that for each $g \in G$, we have $g \cdot v = \sum_{j=1}^N f_j(g)v_j$ where $\{v_1, \ldots, v_N\}$ is linearly independent over $k$. Then $v \in I_n(M, H)$ if and only if $f_j \in I_n(k[G], H)$ for each $j = 1, \ldots, N$. 

\[ c_0 + c_1p + c_2p^2 + \ldots + c_Np^N, \quad 0 \leq c_i < p \text{ for some non-negative integer } N. \]

Then, the highest power of $x$ appearing in this expansion is $(c_i - 1)p^i + c_i p^j + c_i p^\ell + \ldots$. Since $r_{u_1}(x^m) = I_{n-1}(k[G_a], G_a)$, we see by induction that $(c_i - 1) + c_j + c_\ell + \ldots \leq n - 2$ so $c_i + c_j + c_\ell + \ldots \leq n - 1$ and $m$ has the desired form. Conversely, if $m$ has the desired form, the equation above shows that $r_{u_1}(x^m) = I_{n-1}(k[G_a], G_a)$. \[ \square \]
Proof. We first note that for any \( g \in G, h \in H \), we have
\[
g \cdot (h \cdot v - v) = \sum_{j=1}^{N} (f_j(gh) - f_j(g))v_j = \sum_{j=1}^{N} ((h \cdot f_j)(g) - f_j(g))v_j.
\]
The proof now proceeds by induction, the case \( n = 0 \) being immediate. So, suppose the statement is true for \( n - 1 \); we shall show it holds for \( n \). Now, \( v \in I_n(M, H) \)
if and only if for each \( h \in H \), we have \( h \cdot v - v \in I_{n-1}(M, H) \). By the induction hypothesis and equation above, this is true if and only if the function \( (h \cdot f_j) - f_j \) is in \( I_{n-1}(k[G], H) \). By definition, this means that \( f_j \in I_n(k[G], H) \).

\[ \square \]

**Theorem 1.** (a) For each \( n \geq 0 \), \( \Phi : (I_n(k[G], H) \otimes M)^G \to I_n(M, H) \) is a vector space isomorphism.

(b) If \( r \in (k[G]^H \otimes A)^G \) and \( m \in (I_n(k[G], H) \otimes M)^G \), then \( \Phi(r) \in A^H \) and \( \Phi(rm) = \Phi(r)\Phi(m) \).

Proof. We prove statement (a) by induction on \( n \), the case \( n = 0 \) being immediate. So, suppose that (a) is true for \( n-1 \); we shall show it holds for \( n \). First, suppose that \( v = \Phi(\sum_i f_i \otimes m_i) \) where \( (\sum_i f_i \otimes m_i) \in (I_n(k[G], H) \otimes M)^G \) and \( \{m_i\} \) is linearly independent over \( k \). Then, for any \( g \in G \), we have \( v = \Phi(\sum_i (\ell_i g f_i) \otimes g \cdot m_i) = \sum_i (\ell_i g f_i(e)g \cdot m_i) \), that is, \( g^{-1} \cdot v = \sum_i f_i(g^{-1})m_i \). Applying Lemma 2, we may conclude that \( v \in I_n(M, H) \). Conversely, suppose that \( v \in I_n(M, H) \) and that \( v = \Phi(\sum_i f_i \otimes m_i) \) where \( (\sum_i f_i \otimes m_i) \in (k[G] \otimes M)^G \) and \( \{m_i\} \) is linearly independent over \( k \). As above, this means that \( g^{-1} \cdot v = \sum_i f_i(g^{-1})m_i \) for any \( g \in G \). Again applying Lemma 2, we see that each \( f_i \in I_n(k[G], H) \).

To show statement (b), let \( r = \sum_j f_j \otimes a_j \in (k[G]^H \otimes A)^G \) and \( m = \sum_i f_i \otimes m_i \in (I_n(k[G], H) \otimes M)^G \). Then,
\[
\Phi(rm) = \Phi(\sum_j f_j f_i \otimes a_j m_i) = \sum_j f_j(e) f_i(e) a_j m_i = \sum_j f_j(e) a_j \sum_i f_i(e) m_i = \Phi(r)\Phi(m).
\]

\[ \square \]

**Corollary 1.** If \( G \) is a finite group, then there is an \( N, 1 \leq N \leq |G| \) so that \( I_n(M, G) = I_N(M, G) \) for all \( n \geq N \).

Proof. First, suppose that \( M = k[G] \). In the sequence of subspaces \( I_0(k[G], G) \subset I_1(k[G], G) \subset \ldots \subset I_m(k[G], G) \subset \ldots \), there must be an \( m \) so that \( I_m(k[G], G) = I_{m+1}(k[G], G) \) since \( \dim k[G] = |G| \). Then \( I_m(k[G], G) = I_n(k[G], G) \) for all \( n \geq m \).

The general result now follows from Theorem 1 since \((I_n(k[G], G) \otimes M)^G \) is isomorphic to \( I_n(M, G) \).

The corollary above is of interest only in the modular case (i.e., when \( p \) divides \( |G| \)) and then it is the best possible according to Example 6. In the non-modular case, all the \( I_n(M, G) = M^G \) (Theorem 3).

If \( G \) is reductive and \( R \) is a commutative, finitely generated \( k \)-algebra on which \( G \) acts rationally, then \( R^G \) is a finitely generated \( k \)-algebra. This celebrated result is due mainly to H. Weyl, C. Chevalley, D. Mumford, M. Nagata, and W. Haboush. For our purposes, we take \( A \) to be a finitely generated \( k \)-algebra and \( R = A \oplus M \) with the multiplication \((a, m)(a', m') = (aa', am' + a'm)\). If \( M \) is a finitely generated \( A \)-module, then \( A \oplus M \) is a finitely generated \( k \)-algebra. Then, by the result just cited, \( R^G = A^G \oplus M^G \) is a finitely generated \( k \)-algebra. Consequently, \( M^G \) is a finitely generated \( A^G \)-module. So that we can cite it later, we state this next.
Theorem 2. Let $A$ be a finitely generated, commutative $k$-algebra. Let $M$ be an $A$-module. Let $G$ be a reductive group which acts rationally on $A$ and $M$. Suppose that the actions of $G$ are compatible with the structure of $M$ as an $A$-module. If $M$ is a finitely generated $A$-module, then $M^G$ is a finitely generated $A^G$-module.

Corollary 2. Suppose that $G$ is a reductive group, $A$ is finitely generated over $k$, and $M$ is a finitely generated $A$-module. If $I_n(k[G],H)$ is a finitely generated $k[G]^H$-module, then $I_n(M,H)$ is a finitely generated $A^H$-module.

Proof. Since $I_n(k[G],H)$ is a finitely generated $k[G]^H$-module, $I_n(k[G],H) \otimes M$ is a finitely generated $k[G]^H \otimes A$-module. Then, by Theorem 2, we see that $(I_n(k[G],H) \otimes M)^G$ is a finitely generated $(k[G]^H \otimes A)^G$-module. We now apply Theorem 1. □

Theorem 3. Suppose that $G$ is a reductive group.

(a) If $p$ does not divide $[G : G^0]$, then $I_n(M,G) = M^G$ for all $n \geq 1$.

(b) If $p$ divides $[G : G^0]$, then $I_n(M,G) = I_n(M^{G^0},G/G^0)$ and there is an $N$, $1 \leq N \leq [G : G^0]$ so that $I_n(M,G) = I_N(M,G)$ for all $n \geq N$.

Proof. To prove statement (a), we first show that it is true when $G$ acts on $A = k[G]$ by right translation, i.e., we shall show that $I_n(k[G],G) = k$ for all $n \geq 1$. This is true for $n = 1$ so we assume it is true for $n$ and show it for $n + 1$. Let $f \in I_{n+1}(k[G],G)$. By the induction hypothesis, $(g_1 - I_G)(g_2 - I_G)f = 0$ for any $g_1, g_2 \in G$. Proceeding as in Example 3, we see $f_1 = f - f(e)$ is a homomorphism from $G$ to $G^0$. Since the semi-simple elements are dense in $G^0$, $f_1|G^0$ is identically 0. Then, $f_1 = 0$ since $p \nmid [G : G^0]$. For the action of $G$ on $M$, we apply Theorem 1 to see that $I_n(M,G) = \Phi(I_n(k[G],G) \otimes M^G) = \Phi(k \otimes M)^G = M^G$.

To prove (b), we first note that $I_n(M,G) \subset I_n(M,G^0) = M^{G^0}$ by Lemma 1(d) and what was just proved. Then, an induction argument shows that $I_n(M,G) = I_n(M^{G^0},G/G^0)$. Since $G/G^0$ is finite, we may apply Corollary 1. □

4. Finite generation of $I_n(M,G)$, $\text{chark} = 0$

For applications to ordinary differential equations, we need $A^G$ to be a finitely generated algebra over $\mathbb{C}$ and $I_n(M,G)$ to be a finitely generated $A^G$-module. Example 4 shows that even in the simplest cases finite generation may not hold when $\text{chark} > 0$. In this section, we study the finite generation property in the case $\text{chark} = 0$.

Theorem 4. Suppose that $\text{chark} = 0$. Suppose also that $G$ is connected, $A^G$ is a finitely generated $k$-algebra, and $M^G$ is a finitely generated $A^G$-module. Then $I_n(M,G)$ is a finitely generated $A^G$-module for all $n = 0, \ldots$.}

Proof. Let $\mathcal{L}$ be the Lie algebra of $G$. According to Lemma 1(e), each $L_X \in \mathcal{L}$ gives an $A^G$-module homomorphism from $I_n(M,G)$ to $I_{n-1}(M,G)$. We now show by induction on $n$ that $I_n(M,G)$ is a finitely generated $A^G$-module for all $n = 0, \ldots$. The case $n = 0$ is immediate. So, assume that $n \geq 1$ and that $I_{n-1}(M,G)$ is a finitely generated $A^G$-module. Let $\{L_{X_1}, \ldots, L_{X_s}\}$ be a basis for $\mathcal{L}$. We show by induction on $i$ that

$$\ker L_{X_1} \cap \ldots \cap \ker L_{X_i} \cap I_n(M,G)$$

is a finitely generated $A^G$-module. When $i = 1$, we have $m \in \ker L_{X_1} \cap \ldots \cap \ker L_{X_i}$ if and only if $L_X \cdot m = 0$ for all $L_X \in \mathcal{L}$, i.e., if and only if $m \in M^G$. Hence,
\[ \ker L_{X_1} \cap \ldots \cap \ker L_{X_n} \cap I_n(M, G) = M^G \] which is a finitely generated \( A^G \)-module by assumption. Now suppose that \( \ker L_{X_1} \cap \ldots \cap \ker L_{X_n} \cap I_n(M, G) \) is a finitely generated \( A^G \)-module for \( i \geq 2 \). We show that \( M^* = \ker L_{X_1} \cap \ldots \cap \ker L_{X_n} \cap I_n(M, G) \) is also a finitely generated \( A^G \)-module. Let \( L^*_{X_{n-1}} = L_{X_{n-1}} | M^*. \) Then \( L^*_{X_{n-1}}(M^*) \subset L_{X_{n-1}}(I_n(M, G)) \subset I_{n-1}(M, G) \) so \( L^*_{X_{n-1}}(M^*) \) is an \( A^G \)-submodule of the finitely generated \( A^G \)-module \( I_{n-1}(M, G) \). Also, \[ \ker L^*_{X_{n-1}} = \ker L_{X_{n-1}} \cap M^* = \ker L_{X_{n-1}} \cap \ldots \cap \ker L_{X_n} \cap I_n(M, G) \] which is a finitely generated \( A^G \)-module by induction. Hence, \( M^* \) is a finitely generated \( A^G \)-module.

In particular, \( \ker L_{X_n} \cap I_n(M, G) \) is a finitely generated \( A^G \)-module. Consider \( L_{X_n} : I_n(M, G) \rightarrow I_{n-1}(M, G) \). Now, \( L_{X_n}(I_n(M, G)) \) is an \( A^G \)-submodule of the finitely generated \( A^G \)-module \( I_{n-1}(M, G) \) and, so, is finitely generated. We have just shown that the kernel of \( L_{X_n} \) is a finitely generated \( A^G \)-module. It follows that \( I_n(M, G) \) is a finitely generated \( A^G \)-module. \( \qed \)

**Theorem 5.** Suppose that \( \text{char} k = 0 \), \( A^G \) is a finitely generated \( k \)-algebra, and \( M^G \) is a finitely generated \( A^G \)-module. Then \( I_n(M, G) \) is a finitely generated \( A^G \)-module for all \( n = 0, \ldots, \).

**Proof.** Since \( A^G \) is finitely generated, so is \( A \) since \( G/G^o \) is a finite group. Furthermore, \( A^G \) is an integral extension of \( A \). By Theorem 4, \( I_n(M, G^o) \) is a finitely generated \( A^G \)-module for all \( n = 0, \ldots, \). Thus, each \( I_n(M, G^o) \) is a finitely generated \( A^G \)-module. Since \( I_n(M, G) \) is an \( A^G \)-submodule of \( I_n(M, G^o) \), it is also finitely generated. \( \Box \)

**Corollary 3.** With respect to the action of \( G \) on itself by right translation, \( I_n(k[G], G) \) is a finite-dimensional vector space for all \( n = 0, \ldots, \).

In general, the algebra \( k[X]^G \) is not finitely generated over \( k \). However, there are non-zero elements \( f \in k[X]^G \) so that the localization \( k[X]^G \) is a finitely generated \( k \)-algebra. The set of all such \( f \) together with 0 forms an ideal in \( k[X]^G \). Most recently, this ideal has been studied by Derksen and Kemper who call it the finite generation ideal \( G \). Section 2.2.

**Corollary 4.** Suppose that \( \text{char} k = 0 \) and let \( f \neq 0 \) be in the finite generation ideal for \( k[X] \). Then \( k[X]^G \) is finitely generated over \( k \) and each \( I_n(k[X], G) \) is a finitely generated \( k[X]^G \)-module.

**Proof.** Since \( f \) is in the finite generation ideal for \( k[X] \), \( k[X]^G = k[X]^G \) is finitely generated over \( k \). Applying Theorem 5, we see that each \( I_n(k[X], G) \) is a finitely generated \( k[X]^G \)-module. \( \Box \)

5. G-EQUIVARIANT POLYNOMIAL MAPPINGS

Let \( R_n G \) denote the unipotent radical of \( G \). As we have seen in Examples 4 and 5, the modules \( I_n(k[X], G) \) may or may not be finitely generated over \( k[X]^G \). In this section, we show that for any finite-dimensional \( G \)-module \( W \), the \( k[X]^G \)-module of \( W \)-relative invariants, \( (I_n(k[X], G) \otimes W)^G \) is finitely generated whenever \( k[X]^G \) is a finitely generated \( k \)-algebra. The key idea is to relate the \( I_n(k[X], G) \) to equivariant mappings.
Let $W$ be a finite-dimensional $G$-module. The vector space of all polynomial mappings from $X$ to $W$ may be identified with $k[X] \otimes W$ and is naturally a $k[X]$-module. A polynomial mapping $F : X \to W$ is said to be $G$-equivariant if $F(g \cdot x) = g \cdot F(x)$ for all $g \in G, x \in X$. The vector space of all $G$-equivariant polynomial mappings from $X$ to $W$ may be identified with $(k[X] \otimes W)^G$. It is a $k[X]^G$-module.

**Theorem 6.** Let $G$ act regularly on an irreducible affine variety $X$; let $W$ be a finite-dimensional $G$-module. If $k[X]^{R_aG}$ is a finitely generated $k$-algebra, then $k[X]^G$ is a finitely generated $k$-algebra and $(k[X] \otimes W)^G$ is a finitely generated $k[X]^G$-module.

**Proof.** First, we note that $k[X]^G$ is a finitely generated $k$-algebra since $k[X]^G = (k[X]^{R_aG})^{G/R_aG}$ and $G/R_aG$ is reductive. If $W^{R_aG} = W$, then, $(k[X] \otimes W)^G = ((k[X] \otimes W^{R_aG})^{G/R_aG} = (k[X]^{R_aG} \otimes W)^{G/R_aG}$. Since $(k[X]^{R_aG} \otimes W)$ is a finitely generated $k[X]^{R_aG}$-module, $(k[X]^{R_aG} \otimes W)^{G/R_aG}$ is a finitely generated $(k[X]^{R_aG})^{G/R_aG}$-module by Theorem 2.

We now proceed by induction on $\dim W$. If $\dim W = 1$, $W^{R_aG} = W$. Otherwise, suppose that $\dim W^{R_aG} < \dim W$ and let $\pi : W \to W/W^{R_aG}$. The subspace $W^{R_aG}$ is $G$-invariant since $R_aG$ is a normal subgroup of $G$. Thus, a $G$-equivariant mapping $F : X \to W$ gives a $G$-equivariant mapping $\pi \circ F : X \to W/W^{R_aG}$. Let $\overline{F} = \pi \circ F$. The mapping $F \to \overline{F}$ is a $k[X]^G$-homomorphism from the $k[X]^G$-module $(k[X] \otimes W)^G$ to the $k[X]^G$-module $(k[X] \otimes W/W^{R_aG})^G$. The image is a finitely generated $k[X]^G$-module by the induction assumption since $1 \leq \dim W^{R_aG} < \dim W$. The kernel consists of $G$-equivariant maps from $X$ to $W^{R_aG}$ and is a finitely generated $k[X]^G$-module by what was proved above. Hence, $(k[X] \otimes W)^G$ is a finitely generated $k[X]^G$-module.

For finite group actions, Theorem 6 is proved in [1, Proposition 3.3], for example.

**Corollary 5.** If $k[X]^{R_aG}$ is a finitely generated $k$-algebra, then $(I_n(k[X], G) \otimes W)^G$ is a finitely generated $k[X]^G$-module for all $n \geq 0$.

**Proof.** This follows from Lemma 1(b) and Theorem 6 since $k[X]^G$ is finitely generated and $(I_n(k[X], G) \otimes W)^G$ is a $k[X]^G$-submodule of $(k[X] \otimes W)^G$.

**Corollary 6.** Let $X$ be an irreducible, affine $G$-variety. Let $J \subset k[X]^{R_aG}$ be the finite generation ideal for $R_aG$. Suppose that there is a non-zero $f \in J \cap k[X]^G$. Then, $(k[X^f] \otimes W)^G$ and $(I_n(k[X^f], G) \otimes W)^G$ are finitely generated $k[X^f]^G$-modules.

**Proof.** Since $f \in J \cap k[X]^G$, $k[X^f]^{R_aG}$ is finitely generated over $k$. The corollary now follows from Theorem 6 and Corollary 5.

**Lemma 3.** Let $\dim W = n_0$. If $U$ is a unipotent group, then $(k[X] \otimes W)^U = (I_{n_0}(k[X], U) \otimes W)^U$.

**Proof.** It is enough to show that $(k[X] \otimes W)^U \subset I_{n_0}(k[X], U) \otimes W$. Let $\{w_1, \ldots, w_{n_0}\}$ be a basis for $W$ and $\{\lambda_1, \ldots, \lambda_{n_0}\}$ the dual basis. Let $F = \sum_i (f_i \otimes w_i) \in (k[X] \otimes W)^U$ be a $U$-equivariant polynomial mapping from $X$ to $W$. Let $F^* : k[W] \to k[X]$ be its dual map. Then, $F^*(\lambda_i) = f_i$. Since $F$ is $G$-equivariant so is $F^*$. By Example 1, $W^* \subset I_{n_0}(k[W], U)$ so we see that $F^*(W^*) \subset I_{n_0}(k[X], U)$, i.e., $f_i \in I_{n_0}(k[X], U)$.
The condition required in Corollary 6 that there is a non-zero \( f \in J \cap k[X]^G \) is always satisfied if \( k[X]^{R_u,G} \) is finitely generated or if \( G \) is unipotent. For arbitrary linear algebraic groups \( G \), Renner and Rittatore have considered somewhat related questions \[\text{[7]}\]. They define the action of \( G \) on \( X \) to be observable if for any \( G \)-invariant proper, closed subset \( Y \) of \( X \), there is a non-zero \( f \in k[X]^G \) so that \( f|_Y = 0 \). Now, suppose that \( G \) is solvable and let \( E_G(X) = \{ \chi \in X(G) : \text{there is a non-zero} \ f \in k[X] \text{such that} \ gf = \chi(g)f \text{for all} \ g \in G \} \). Then, the action of \( G \) on \( X \) is observable if and only if \( E_G(X) \) is a group \[\text{[7]}\] Corollary 3.16.

**Lemma 4.** Suppose that \( G \) is solvable, say \( G = TU \), where \( T \) is a maximal torus and \( U \) is the unipotent radical. If the action of \( G \) on \( X \) is observable, then there is a non-zero \( f \in k[X]^G \) so that \( k[X]^G \) is a finitely generated \( k \)-algebra.

**Proof.** Let \( J \) be the finite generation ideal for \( U \) acting on \( X \). Since \( U \) is a normal subgroup of \( G \), the ideal \( J \) is invariant under \( G \). Thus, there is a non-zero \( \varphi \) in the ideal \( J \) which is a \( T \)-weight vector corresponding, say, to the character \( \chi \in X(T) \). Since \( E_G(X) \) is a group, there is a \( \psi \in k[X]^U \) corresponding to the character \(-\chi\). Let \( f = \varphi \psi \). Then, \( f \in J \cap k[X]^G \).

\[\square\]

**6. Representations of the additive group**

Throughout this section, we assume that \( \text{char} k = 0 \). Let \( G = SL_2(k) \) be the group consisting of \( 2 \times 2 \) matrices \((a_{ij})\) whose determinant is 1. Let \( k[G] = k[x_{11}, x_{12}, x_{21}, x_{22}] \) where \( x_{ij}(g) = a_{ij} \). Let \( U \) (resp. \( U^- \)) be the subgroup of \( G \) consisting of all upper triangular (resp. lower triangular) matrices with 1’s on the diagonal. We may identify \( U \) with the additive group \( G_a \). The group \( G \) acts by left multiplication on the vector space \( V \) consisting of \( 2 \times 1 \) column matrices. The actions of \( G \) on \( V \) and \( V^* \) are equivariant. Let \( S(V) \) be the symmetric algebra on \( V \) and let \( S^d(V) \) be elements in \( S(V) \) homogeneous of degree \( d \). We consider the natural action of \( G \) on \( S^d(V) \) and the action of \( G \) by left translation on \( k[G]^U \) and \( I_{n+1}(k[G], U) \)

**Theorem 7.** With respect to the action of \( U \) on \( k[G] \) by right translation, for each \( n \geq 0, I_{n+1}(k[G], U) \) is a free \( k[G]^U \)-module with basis \( \{x_{12}^{d}x_{22}^{-d} : d = 0, \ldots, n\} \). In particular, \( k[G]^U \otimes S^n(V) \) and \( I_{n+1}(k[G], U) \) are isomorphic as \( G \)-modules.

**Proof.** Let \( D_\alpha \) (resp. \( D_{-\alpha} \)) be a basis of the Lie algebra of \( U \) (resp. \( U^- \)). Then, \( D_\alpha : x_{11} = D_\alpha : x_{21} = 0, D_\alpha : x_{12} = x_{11}, D_\alpha : x_{22} = x_{21} \) and \( D_{-\alpha} : x_{11} = x_{12}, D_{-\alpha} : x_{21} = x_{22}, D_{-\alpha} : x_{12} = D_{-\alpha} : x_{22} = 0 \). It is known that \( k[G]^U = k[x_{11}, x_{21}] \). Now, \( k[G] \) is a direct sum of irreducible representations of \( G \) acting by right translation on \( k[G] \), say \( k[G] = \bigoplus V_i \). Let \( v_i \) be a non-zero element in \( V_i^U \); any \( a \in k[G]^U \) is a linear combination of the \( v_i \). By Lemma 1(d), \( I_{n+1}(k[G], U) = \bigoplus I_{n+1}(V_i, U) \). Thus, applying Example 2, we see that \( I_{n+1}(k[G], U) \) is the vector space over \( k \) spanned by all the \((D_{-\rho})^r \cdot \alpha, \) where \( 0 \leq r \leq n \) and \( \alpha \in k[G]^U \). Since \( k[G]^U = k[x_{11}, x_{21}] \), from the usual rules for differentiating a product, we see that any element in \( I_{n+1}(k[G], U) \) is a \( k[G]^U \)-linear combination of elements of the form \( x_{12}^{d}x_{22}^{-d} \) for \( 0 \leq d \leq r \leq n \). However, \( 1 = x_{11}x_{22} - x_{12}x_{21} \). Therefore, for \( r < n, x_{12}^{d}x_{22}^{-d} \) can be written as a \( k[G]^U \)-linear combination of the \( x_{12}^{d}x_{22}^{-d} \) since \( x_{12}^{d}x_{22}^{-d} = x_{12}^{d}x_{22}^{-d} \times (x_{11}x_{22} - x_{12}x_{21})^{\alpha-r} \).

Finally, we show that the \( x_{12}^{d}x_{22}^{-d} \) are linearly independent over \( k[G]^U \). Indeed, suppose that \( \sum a_d x_{12}^{d}x_{22}^{-d} = 0 \) where each \( a_d \in k[G]^U \). Let \( r \) be the smallest
Theorem 8 ([8]). If $W = V \oplus S^2(V)$, then $(k[W] \otimes S^n(V))^G$ is a Cohen-Macaulay $k[W]^G$-module for all $n$. If $W = V \oplus S^d(V)$ with $d > 2$, then $(k[W] \otimes S^n(V))^G$ is a Cohen-Macaulay $k[W]^G$-module if and only if $n + 1 < \frac{(d+1)^2}{4}$ for $d$ odd or $n + 1 < \frac{d(d+2)}{4}$ for $d$ even.

Theorem 9. $I_{n+1}(k[S^d(V)], U)$ is a Cohen-Macaulay module over $k[S^d(V)]^U$ for all $n$ when $d = 2$. When $d > 2$, $I_{n+1}(k[S^d(V)], U)$ is a Cohen-Macaulay module over $k[S^d(V)]^U$ if and only if $n + 1 < \frac{(d+1)^2}{4}$ for $d$ odd or $n + 1 < \frac{d(d+2)}{4}$ for $d$ even.

Proof. We first apply Theorem 1 to see that $I_{n+1}(k[S^d(V)], U)$ is isomorphic to $(I_{n+1}(k[G], U) \otimes k[S^d(V)])^G$ which, by Theorem 7, is $(k[G]^U \otimes S^n(V) \otimes k[S^d(V)])^G$. Now, $k[G]^U = k[x_{11}, x_{21}]$ which may be identified with $k[V]$ as a $G$-module. Therefore, $I_{n+1}(k[S^d(V)], U)$ is isomorphic to

$$(k[V] \otimes k[S^d(V)] \otimes S^n(V))^G = (k[V \times S^d(V)] \otimes S^n(V))^G.$$

Let $W = V \times S^d(V)$. Then, we have $k[W]^G = k[V \times S^d(V)]^G = (k[G]^U \otimes k[S^d(V)])^G = k[S^d(V)]^U$ according to Theorem 1. We now apply Theorem 8 with $W = V \times S^d(V)$ to prove the theorem. □

7. Differential equations

Our interest in modules of higher invariants comes from our study of $G$-symmetric ordinary differential equations [4]. Thus, it seems appropriate to sketch some of the connections of this paper with ordinary differential equations. Let $\text{char} k = 0$ (in particular, we may take $k = \mathbb{C}$) and let $W := k^n$. Consider the ordinary differential equation

$$\dot{x} = \frac{dx}{dt} = F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

on $W$ with each $f_i \in k[W] = k[x_1, \ldots, x_n]$. (One may think of solutions as formal power series, in general.) To $F$ we assign the derivation $L_F = \sum_i f_i \frac{\partial}{\partial x_i}$ of $k[x_1, \ldots, x_n]$. Then, for any polynomial $\phi(x_1, \ldots, x_n)$ and any solution $v(t)$ to the differential equation above, we have $\frac{d}{dt} \phi(v(t)) = (L_F \phi)(v(t))$: we write $\phi$ instead of $L_F \phi$. We call $L_F$ a $G$-equivariant derivation if $L_F$ commutes with the natural action of $G$ on $k[W]$. One verifies that $L_F$ is $G$-equivariant if and only if the corresponding map $F$ is $G$-equivariant.

As always, let $X$ be an irreducible, $G$-invariant affine subvariety of $W$ and assume that $L_F$ sends the vanishing ideal of $X$ to itself. We may then consider the $f_i$ as elements of $k[X]$ and $L_F$ as a derivation of $k[X]$. In this case, it can be shown that
if $x \in X$, then $F(x)$ is tangent to $X$. Thus, the differential equation above can be considered as a differential equation on $X$.

It follows immediately from Lemma 1(d) that if $L_F$ is $G$-equivariant then it maps each $I_m(k[X], G)$, $m = 1, 2, \ldots$ to itself. For $m = 1$ this fact has been utilized for a long time to obtain a reduced system corresponding to the differential equation above as follows. Assume that $I_1(k[X], G) = k[X]^G$ admits a finite set of generators $\phi_1, \ldots, \phi_r$. Then there exist polynomials $\psi_j$ such that $\phi_j = L_F(\phi_j) = \psi_j(\phi_1, \ldots, \phi_r)$ for $1 \leq j \leq r$. Letting

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_r \end{pmatrix},$$

the previous equation shows that $\Phi$ sends solutions of $\dot{x} = F(x)$ to solutions $(\phi_1(x(t)), \ldots, \phi_r(x(t)))$ of the “reduced system” $\dot{y} = \Psi(y)$ on $Y := \Phi(X)$.

We can extend this procedure as follows. If the module $I_m(k[X], G)$ has a finite set $\theta_{m,1}, \ldots, \theta_{m,\ell_m}$ of generators, there exist polynomials $h_{ij}$ such that

$$\theta_{m,i} = L_F(\theta_{m,i}) = \sum_j h_{ij}(\phi_1, \ldots, \phi_r)\theta_{m,j}, \quad 1 \leq i \leq \ell_m. \quad (*)$$

Thus $\Theta_m := (\theta_{m,1}, \ldots, \theta_{m,\ell_m})$ maps solutions of $\dot{x} = F(x)$ to solutions of the non-autonomous linear differential equation

$$\dot{z} = H(y) \cdot z, \quad H(y) := (h_{ij}(y)),$$

where $y$ stands for a solution of the reduced equation above.

This observation may not provide useful information for arbitrary groups, but it is quite valuable if $G = U$ is unipotent and $k[X]^U$ is finitely generated. Indeed, by Example 1, each $x_i \in I_n(k[X], U)$. It then follows from Theorem 4 and equation $(*)$ above, read with $m := n$, that once solutions $(\phi_1(x(t)), \ldots, \phi_r(x(t)))$ to the reduced system are found, solutions to the original system can be found by solving a system of non-autonomous linear differential equations. In other words, given a $U$-symmetric differential equation $\dot{x} = F(x)$ on the variety $X$, there remains only a non-autonomous linear differential equation modulo the reduced system $\dot{y} = \Psi(y)$. Using the inclusions $I_2(k[X], U) \subseteq \cdots \subseteq I_n(k[X], U)$, one may refine this to obtain a kind of echelon form for the linear system. In the special case $X = k^n$, this result was proved under stronger hypotheses in [4 Proposition 1 and Corollary 1]. The comments above give an analogous statement for an arbitrary irreducible variety $X$. The extension to varieties of the rest of the program carried out in [4] will be taken up elsewhere.

References


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