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ATTRACTING SETS ON SURFACES

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Abstract. Let f be a continuous endomorphism of a surface M, and A an attracting set such that the restriction $f|_A : A \to A$ is a $d:1$ covering map. We show that if f is a local homeomorphism, then f is also a $d:1$ covering of the immediate basin of A. Moreover, the techniques provide a characterization of invariant $d:1$ continua on surfaces. These results are no longer true on manifolds of dimensions at least three.

1. Introduction

Let f be a continuous endomorphism of a topological space X and U a nonempty open proper subset of X. The pair (f, U) is an *attracting pair* of X, if the closure of $f(U)$ is contained in U. By discarding the possibilities $U = \emptyset$ and $U = X$ we avoid trivial cases. The *attracting set* associated to the attracting pair (f, U) is defined as

$$A = A(f, U) = \bigcap_{n \geq 0} f^n(U).$$

Note that we use the terminology "attracting set" instead of "attractor" as we do not require this set to be transitive which is often the case in the literature.

Examples of attracting sets frequently appear. By a theorem of C. Conley ([Con]), a homeomorphism which is not chain recurrent defined in a compact space X, always has an attracting set. Moreover, hyperbolic attractors are always attracting sets.

The basin B of the attracting set A is defined as the set of points in X such that $(f^n(x))_{n\in\mathbb{N}}$ converges to A. The immediate basin of A is the union of the connected components of B that intersect A, and is denoted by B^0_A.

A point p is a critical point of a continuous map f if f is not local homeomorphism at p, meaning that there does not exist a neighborhood V of p such that the restriction of f to V is a homeomorphism onto $f(V)$. The set of critical points of f is denoted by S_f.

It is easy to prove that whenever U is relatively compact, the attracting set A is compact, and if, in addition, X is locally connected, then A has finitely many components. This is shown in item 4 of Section 2. Therefore, as every connected component of A is periodic, we will assume from now on that A is connected.

We are interested in a kind of global problem. Does there exist a preimage of A contained in the immediate basin of A? More precisely, we want to know if $f^{-1}(A) \setminus A$ may intersect B^0_A.

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We will prove the following:

Theorem 1. *If M is a compact oriented surface and f is injective in U, then either f is injective in B_A^0 or there are critical points of f in B_A^0.*

This result has important consequences regarding C^1 structural stability of endomorphisms. Indeed, it is well known that a C^1 stable map must be Axiom A (see [AMS]), thus there exist transitive attracting sets. Moreover, as proved by Przytycki [Prz], the stability implies also that the map is injective in every attractor A. In addition, as critical points are forbidden for C^1 stability, one has the following consequence.

Corollary 1. *If f is a C^1 stable map of an oriented compact surface M, then its restriction to the immediate basin of an attractor is injective.*

Notice that Theorem 1 generalizes the corresponding results in low dimensional manifolds. Indeed, the result has a trivial proof if X is equal to the circle S^1. Moreover, it is well known that for a rational map in the Riemann sphere every attractor is a periodic orbit and the local inverse of f defined in a neighborhood of the attractor can be extended until its domain hits a critical point of f.

However, as will be shown in the final section, the result cannot be extended to manifolds of dimensions three or more.

It is natural to ask if similar conclusions can be obtained when the map f is not one-to-one, but a covering map from U to $f(U)$. It may happen that different points in A have a different number of preimages in A, so to extend the result above we need the following restriction.

Definition 1. An attracting pair (f, U) is called normal if $f^{-1}(A) \setminus A$ is a closed set and $S_f \cap U = \emptyset$.

In particular, this condition holds whenever f is injective in U. We will prove in the next section that if A is a connected attracting set, then (f, U) is normal if and only if $f : A \to A$ is a covering map. The degree d of this covering will be called the degree of A.

Theorem 2. *Let M be an oriented, compact surface, and (f, U) a normal attracting pair with associated attracting set A.*

If f restricted to A is a covering of degree $d > 1$ and $S_f \cap B_A^0 = \emptyset$, then the restriction of f to the immediate basin of A is also a $d : 1$ covering. In addition, there exists a neighborhood U of A which is a topological open annulus such that $U \backslash A$ has exactly two annular connected components.

Normal attractors have another interesting property: the complement of A also has finitely many components. Moreover, if A is a hyperbolic attractor of a C^1 map f on a two dimensional manifold, and the restriction of f to A is $d : 1$ with d greater than one, then the restriction of f to the immediate basin of A is conjugated to

$$(z, y) \in S^1 \times \mathbb{R} \to (z^d, y/2).$$

Many properties of this map are well known (see [BKRU] and [Tsu]) and will be explained in the final section. It will follow that for a generic perturbation f' of f it holds that (f', U) is an attracting pair that fails to be normal, and the complement of the attracting set may have infinitely many components.

We finally show that the assertions of both theorems and the corollary are false in manifolds of dimension at least three.

2. Finiteness of components

This section is devoted to the statement of some general facts concerning normal attracting pairs. The main goal is to prove the following:

Proposition 1. *If M is a compact oriented manifold without boundary, and (f, U) is a normal attracting pair in M, then $M \setminus A$ has at most finitely many components.*

Definition 2. Let (f, U) be an attracting pair and assume that U' is an open subset of X. We say that (f, U') is equivalent to (f, U) if it is an attracting pair and defines the same attracting set.

Let (f, U) be an attracting pair with associated attracting set A. The set of connected components of a set Y will be denoted by $\Pi_0(Y)$.

We begin by stating some general facts:

(1) The attracting set A is closed, since $A = \bigcap f^n(U) = \bigcap \overline{f^n(U)}$. Besides, $f(A) = A$.
(2) If X is compact, and U' is a neighborhood of A such that $f^{n+1}(U) \subset U' \subset f^n(U)$ for some positive integer n, then the pair (f, U') is an attracting pair equivalent to (f, U). In particular, $(f, f(U))$ is equivalent to (f, U).
(3) Without loss of generality, one may suppose that every connected component of U intersects A. Indeed, one can remove unnecessary components and obtain an equivalent attracting pair.
(4) If X is locally connected and the closure of U is compact, then A has finitely many connected components.

Proof. As X is locally connected (that is, every point has a basis of connected neighborhoods), then the connected components of U are open; moreover, they intersect A and are pairwise disjoint. Therefore, as A is compact, U has finitely many connected components. Besides, if $u \in \Pi_0(U)$, then $f(u)$ intersects exactly one element of $\Pi_0(U)$. Then, $A = \bigcap_{n>0} f^n(U)$ has finitely many components (exactly as many as U). □

Finiteness of connected components of attracting sets was proved with a different set of hypothesis in [Bue, Theorem 1. 4. 6].

Lemma 1. *Let X be a compact metrizable space and (f, U') a normal attracting pair in X defining an attracting set A. If A is connected, then the restriction of f to A is a $d:1$ covering map onto A. Moreover, there exists a neighborhood U of A such that*

(1) *The restriction of f to U is a $d:1$ covering map onto $f(U)$.*
(2) *The pair (f, U) is equivalent to (f, U').*

This assertion is trivial when f is injective in U'.

Proof. Note that the restriction of f to A is a local homeomorphism. As A is compact and connected, the first assertion follows immediately.

By normality, there exists an open set $V \subset U'$ such that $f^{-1}(A) \cap \overline{V} = A$. Moreover, we claim that there exists V_0, a neighborhood of A contained in V, such that every point in V_0 has exactly d preimages in V. It is clear that such a neighborhood exists where every point has at least d preimages. Assume by contradiction that there exists a nested sequence $(V_n)_{n\in\mathbb{N}} \subset V$ of neighborhoods of A such that $\bigcap_{n\in\mathbb{N}} V_n = A$ and for each $n \in \mathbb{N}$ there exists a point $y_n \in V_n$ such that

$f^{-1}(y_n)$ has $d+1$ different points $\{x_n^1, \dots, x_n^{d+1}\} \subset V$. By taking subsequences if necessary, we may assume that $y_n \to y \in A$, and $x_n^i \to x^i \in f^{-1}(A) \cap \overline{V} = A$. As there are no critical points in $\overline{V}$, the x^i's are all different. This is a contradiction because $f(x^i) = y \in A$, and f is $d:1$ in A.

Let n be such that $f^n(U')$ is contained in V_0 and let $U = f^{-1}(f^{n+1}(U')) \cap V_0$. Let us show that U satisfies assertions (1) and (2). To prove (2) we first prove that $\overline{f(U)} \subset U$. This is because $\overline{f(U)} \subset \overline{f^{n+1}(U')} \subset f^n(U') \subset V_0$, and $\overline{f(U)} \subset \overline{f^{n+1}(U')} \subset f^{-1}(f^{n+1}(U'))$. To show that $(f, U) \sim (f, U')$ we will show that there exist $m, k \in \mathbb{N}$ such that $f^k(U) \subset U'$ and $f^m(U') \subset U$. First, $f(U) \subset f^{n+1}(U') \subset U'$ follows directly from the definition of U. Besides, $f^n(U') \subset U$ because $f^n(U') \subset f^{-1}f^{n+1}(U')$, and $f^n(U') \subset V_0$ by the choice of n. □

We obtain the following lemma:

Lemma 2. *If X is a compact metric space, A is connected and $S_f \cap U = \emptyset$, then the attracting pair (f, U) is normal if and only if $f|_A$ is a covering onto A.*

Proof. To prove the remaining part, take a sequence $(x_n)_{n\in\mathbb{N}} \subset f^{-1}(A) \backslash A$ and assume by contradiction that $x_n \to x \in A$. For each $n \geq 0$, $f(x_n) \in A$ has d preimages in A, denoted $y_n^1, \dots, y_n^d$. By passing to subsequences, assume that each $\{y_n^i\}_{n\in\mathbb{N}}$ is convergent to a point $y^i \in A$. Note that there exists $\epsilon > 0$ such that $d(x, y) \geq \epsilon$ whenever x, y are different preimages of the same $z \in A$. Moreover, $d(y^i, y^j) \geq \epsilon$ for $i \neq j$ and $d(x, y^i) \geq \epsilon$ for all $i = 1, \dots, d$. It follows that $f(x)$ has at least $d+1$ preimages in A: this contradiction proves the assertion. □

Let M be a compact manifold. We begin the proof of Proposition 1 with a simple topological fact.

Lemma 3. *Let A_1 and A_2 be closed disjoint subsets of M. Assume that $\{d_n\}_{n>0}$ is a sequence of connected, pairwise disjoint sets in M such that the boundary of each d_n is contained in $A_1 \cup A_2$ and intersects A_1. Then there exists N such that for all $n > N$ the boundary of d_n is disjoint from A_2.*

Proof. Note that one can choose a metric d in M such that for some $\epsilon_0 > 0$ the ball $B(x, \epsilon)$ of center x and radius ϵ is connected for every $\epsilon \leq \epsilon_0$. Let $\epsilon < \epsilon_0$ be such that the distance between A_1 and A_2 is greater than 2ϵ. As $\{d_n\}$ is a disjoint sequence of open sets and M is compact, there exists $N > 0$ such that d_n does not contain a ball of radius ϵ for every $n > N$. Take any $n > N$ and for $i = 1, 2$ define $a_n^i = \{x \in d_n \; : \; B(x; \epsilon) \cap A_i \neq \emptyset\}$. Note that $d_n = a_n^1 \cup a_n^2$ (because $B(x, \epsilon)$ is connected), that a_n^1 and a_n^2 are open and disjoint in d_n and that a_n^1 is not empty. It follows that a_n^2 is empty. As $n > N$ was arbitrary, the lemma is proved. □

From now on it is assumed that (f, U) is a normal attracting pair defined in a compact manifold M, where U satisfies the conditions of Lemma 1.

Note that f acts on $\Pi_0(U \setminus A)$. More precisely,

(1) If $d \in \Pi_0(U \setminus A)$, then $f(d)$ is contained in some $d' \in \Pi_0(U \setminus A)$ because $f^{-1}(A) \cap U = A$. Define $F : \Pi_0(U \setminus A) \to \Pi_0(U \setminus A)$ by $F(d) = d'$.
(2) F is surjective: if $d' \in \Pi_0(U \setminus A)$, then $d' \cap f(U) \neq \emptyset$, because $f(U)$ is a neighborhood of A and the boundary of d intersects A. It follows that there exists $x \in U \setminus A$ such that $f(x) \in d'$, but every $x \in U \setminus A$ belongs to some $d \in \Pi_0(U \setminus A)$, showing that $F(d) = d'$.

Lemma 4. *No component of $M \setminus A$ is contained in U.*

Proof. Assume by contradiction that $c \in \Pi_0(M \setminus A)$ is contained in U. It follows that $f^n(c)$ is a component of $M \setminus A$ for every $n > 0$ because f is an open mapping in U and $f^{-1}(A) \cap U = A$. This implies that $f^n(c) \in \Pi_0(U \setminus A)$ for every $n \geq 0$. Note that c is not F-periodic, otherwise c would be contained in $\bigcap_{n>0} f^n(U)$ which is a contradiction. Let $\{d_n\}_{n\in\mathbb{Z}}$ be a whole orbit of F such that $d_0 = c$ (there exists such an orbit because F is surjective). Note that the sets $\{d_n\}_{n\in\mathbb{Z}}$ are pairwise disjoint since c is not periodic. Moreover, as $c \cap A = \emptyset$, then there exists $x \in c$ and $N \in \mathbb{N}$ such that $f^{-N}(x) \cap U = \emptyset$; it follows that there exists a minimum $N_0 > 0$ such that d_{-N_0} is strictly contained in the component c_{-N_0} of $M \setminus A$ that contains d_{-N_0}. Therefore, d_{-n} is strictly contained in the component c_{-n} of $M \setminus A$ that contains d_{-n}, for every $n \geq N_0$. So, the sequence $\{d_n\}_{-n>N_0}$ satisfies the following properties:

It is a sequence of connected pairwise disjoint subsets of M, the boundary of each d_n is contained in $A \cup \partial U$, the boundary of each d_n intersects A, and the boundary of each d_n intersects ∂U (because the contrary assumption implies that the boundary of d_n is contained in A and so d_n is equal to a component of the complement of A in M).

As A and ∂U are closed disjoint sets, Lemma 3 implies the assertion. □

Proof of Proposition 1. The boundary of each $d \in \Pi_0(U \setminus A)$ intersects ∂U by the previous lemma, and it certainly intersects A.

By Lemma 3 it follows that $\Pi_0(U \setminus A)$ is finite. In particular, as each connected component of $M\setminus A$ intersects U, $M\setminus A$ has finitely many connected components. □

3. Proof of the theorems

We devote this section to proving Theorem 1. We motivate the preliminary work with the following example. Suppose that A is an attracting fixed point for a differentiable function f. We may take $U = B(A, r)$ for a suitable r and obtain $\overline{f(U)} \subset U$. In this case, U retracts onto $f(U)$. However, one may take a point $y \in U\setminus\overline{f(U)}$ and consider $U' = U\setminus\{y\}$. In this case $\overline{f(U')} \subset U'$, but U' does not retract onto $f(U')$. A fundamental part of the proof of Theorem 1 is finding a neighborhood U of the attracting set such that U retracts onto $f(U)$ (see the comments preceding Lemma 9).

3.1. **Genus and nexus of a continuum.** Throughout this section, M will stand for a compact oriented connected surface without boundary. A two dimensional manifold with boundary S is always diffeomorphic to the space obtained by attaching a finite number ($g(S)$) of handles to a two sphere and then removing a finite number ($\kappa(S)$) of open discs with disjoint closures and smooth boundaries (see, for example, [Tho]). The Euler characteristic of S, $\chi(S) = 2 - 2g(S) - \kappa(S)$ is an invariant under homeomorphisms. If $f : S_1 \to S_2$ is a covering map of degree d, then $\chi(S_1) = d\chi(S_2)$.

The *genus* of S is defined as the number of handles attached, and is denoted by $g(S)$. Equivalently, the genus of S is equal to the maximal number of simple closed disjoint curves one can delete from S without disconnecting it. In this form, the definition can be extended to the class of all connected open subsets of M. Furthermore, the genus of a connected compact subset K of M is now defined as the minimal genus of an open connected set that contains K. This definition makes sense because whenever W and V are open sets such that $V \subset W$, then $g(V) \leq g(W)$. We denote by $\mathcal{G}(K)$ the class of all open neighborhoods of K having minimal genus and whose closure is a subsurface of M.

The set $\mathcal{G}(K)$ can be characterized as follows:

Lemma 5. *If S is an open neighborhood of K whose closure is a submanifold of M with boundary, then $S \in \mathcal{G}(K)$ if and only if each simple closed curve in S disconnects S or intersects K.*

Proof. Let γ be a closed simple curve that does not disconnect S nor intersects K. Then $S' = S \setminus \{\gamma\}$ is connected, is a neighborhood of K and has $g(S') \leq g(S) - 1$; thus $S \notin \mathcal{G}(K)$.

If $S \notin \mathcal{G}(K)$, there exists a compact subset of S whose interior S' belongs to $\mathcal{G}(K)$. As the genus of S is greater than the genus of S', there exists a simple closed curve γ in $S \setminus S'$ that does not disconnect S, and does not intersect K. □

The *nexus* of a compact, connected, orientable surface with nonempty boundary S is defined as the number of connected components of its boundary. It can also be defined in terms of crosscuts. A crosscut is an arc in S with extreme points in the boundary of S. The nexus of S is κ if $\kappa - 1$ is the minimal number of disjoint crosscuts needed to connect the boundary of S. Note that $\kappa > 0$ because we assume that the surface has nonempty boundary. We are interested in extending this definition to a continuum $K \subset M$.

Lemma 6. *The number of connected components of $S \setminus K$ is the same for every $S \in \mathcal{G}(K)$.*

Proof. Take S_1 and S_2 in $\mathcal{G}(K)$. The intersection of S_1 and S_2 is a neighborhood of K, but not necessarily a submanifold, not necessarily connected. There exists $S' \in \mathcal{G}(K)$ such that the closure of S' is contained in $S_1 \cap S_2$. If the number of components of $S_1 \setminus K$ is different from the number of components of $S_2 \setminus K$, then the number of one of them is different from the number of components of $S' \setminus K$. Therefore one can assume that $S_2 \subset S_1$, and the number of connected components of $S_1 \setminus K$ is strictly smaller than the number of connected components of $S_2 \setminus K$. So each connected component of $S_2 \setminus K$ is contained in a connected component of $S_1 \setminus K$. By assumption, there must exist a connected component c of $S_1 \setminus K$ that contains at least two connected components d_1, d_2 of $S_2 \setminus K$. The boundary of d_1 contains a simple closed curve γ contained in ∂S_1 that does not intersect K and $S_0 = S_1 \setminus \gamma$ is connected. This contradicts Lemma 5. □

Definition 3. Let K be a compact and connected subset of M. Define the nexus of K as the number of components of $S \setminus K$ where $S \in \mathcal{G}(K)$. Denote by $\kappa(K)$ the nexus of K.

A priori, the nexus of K may be infinite. However, in our context, we will only work with subsets K such that $M \setminus K$ has finitely many connected components. From now on, we assume that K is a continuum such that $M \setminus K$ has finitely many connected components. Now we will relate $\kappa(K)$ with $\kappa(S)$ for $S \in \mathcal{G}(K)$.

Definition 4. $\mathcal{G}_\kappa(K)$ is the class of surfaces $S \in \mathcal{G}(K)$ having exactly one connected component of ∂S in each connected component of $\overline{S}\backslash K$.

It is clear that if $S \in \mathcal{G}_\kappa(K)$, then $\kappa(S) = \kappa(K)$.

We will often make use of the following

Remark 1. If $S \in \mathcal{G}_\kappa(K)$, then $S\backslash K$ is a disjoint union of open annuli.

Indeed, $S\backslash K$ has genus zero as $S \in \mathcal{G}(K)$. Furthermore, $\kappa(S) = \kappa(K)$ gives us that each connected component of $S\backslash K$ has only two ends, therefore it is an annulus.

Lemma 7. *If K is a continuum such that $M \setminus K$ has at most finitely many components, then $\mathcal{G}_\kappa(K)$ is a basis of neighborhoods of K.*

Proof. It is enough to show that for any $S \in \mathcal{G}(K)$ there exists $S' \in \mathcal{G}_\kappa(K)$ such that $S' \subset S$. Note that as $M \setminus K$ has finitely many components, then $S \setminus K$ has finitely many components. In each connected component of $S\backslash K$ whose closure does not contain a connected component of ∂S, we choose a disc d_i. Then define $S'_0 = S\backslash \bigcup_i d_i$. For each connected component C of $S'_0\backslash K$ having $k > 1$ boundary components of S (denoted $b_1, \ldots, b_k$) take pairwise disjoint crosscuts γ_j $0 < j < k$ in C connecting b_j with b_{j+1}. Enlarge carefully the curves γ_j to a small strip c_j and define $\tilde{C} = C\backslash(\bigcup_j c_j)$. Note that each $\tilde{C}$ is connected (each b_i is a circle), contained in S'_0 and has exactly one boundary component not intersecting K. Finally define S' as the union of K with the union of the $\tilde{C}$ for $C \in \Pi_0(S'_0 \setminus K)$. □

We still need another topological fact.

Lemma 8. *Let $W \in \mathcal{G}_\kappa(K)$ and let $V \in \mathcal{G}(K)$, $V \subset W$. Then there exists $S' \in \mathcal{G}_\kappa(K)$ satisfying the following properties:*

(1) $V \subset S' \subset W$,
(2) $\partial S' \subset \partial V$,
(3) $\kappa(S') = \kappa(K)$.

Proof. Clearly, $\kappa(V) \geq \kappa(K)$. If $\kappa(V) = \kappa(K)$, we let $S' = V$. Otherwise, $\kappa(V) > \kappa(K)$. This means that there exists $u \in \Pi_0(W\backslash K)$ such that V has at least two boundary components in u. As u is an annulus (see Remark 1), only one of these boundary components is nontrivial in $\pi_1(u, x_0)$. Let Γ_u be the set of all $\gamma \in \Pi_0(\partial V)$ that are trivial in $\pi_1(u, x_0)$. Then, any $\gamma \in \Gamma_u$ is the boundary of a disc $D_\gamma \subset u$. The surface $S^{'}$, obtained as the union of V with D_γ , $\gamma \in \Gamma_u$ and $u \in \Pi_0(W\backslash K)$, satisfies all the conclusions of the lemma. □

3.2. **An invariant $d : 1$ continuum $K \neq M$, $d > 1$, has nexus two.** We turn now to dynamics. This subsection is devoted to the proof of the last assertion of Theorem 2. In fact we prove a more general result that applies also to invariant continua, not necessarily attracting.

The argument below rests on a simple fact: if $f : S_1 \to S_2$ is a $d : 1$ covering between connected surfaces S_1 and S_2, $d > 1$, and S_2 is an annulus, then S_1 is also an annulus.

Theorem 3. *Let $K \subsetneq M$ be an f-invariant continuum such that:*

- $S_f \cap K = \emptyset$,
- $f|_K : K \to K$ *is a* $d:1$ *covering map,* $d > 1$,
- $f^{-1}(K)\backslash K$ *is closed,*
- $M\backslash K$ *has finitely many connected components.*

Then, K *has genus zero and nexus two.*

Proof. As $M\backslash K$ has finitely many connected components, by Lemma 7, there exists $U \in \mathcal{G}_\kappa(K)$ such that $U \cap S_f = \emptyset$ and $U \cap (f^{-1}(K)\backslash K) = \emptyset$. Also there exists $V \in \mathcal{G}_\kappa(K)$ such that the connected component V' of $f^{-1}(V)$ containing K is contained in U. Our choice of U implies that $f|_{V'} : V' \to V$ is a $d:1$ covering map. Moreover, as $K \subset V' \subset U$ one has $g(V') = g(V) = g$. We begin by proving that $\kappa(V') = \kappa(V)$. As $V \in \mathcal{G}_\kappa(K)$, the connected components of $V\backslash K$ are topological open annuli (see Remark 1). As $V'\backslash K$ covers $V\backslash K$, it follows that the connected components of $V'\backslash K$ are also annuli. This implies that there is exactly one connected component of $\partial V'$ in each connected component of $\overline{U}\backslash K$, that is, $\kappa(V') = \kappa(V) = \kappa(U)$.

The fact that $f|_{V'} : V' \to V$ is a $d:1$ covering map gives $\chi(V') = d\chi(V)$, that is, $2-2g(V')-\kappa(V') = d(2-2g(V)-\kappa(V))$. But as $g(V) = g(V')$ and $\kappa(V') = \kappa(V)$, it comes that $2-2g-k = d(2-2g-k)$. Moreover, $d \neq 1$ implies $\chi(V) = \chi(V') = 0$, showing that $V \in \mathcal{G}_\kappa(K)$ is an annulus. □

Remark 2.
1. One deduces that, for instance, a figure 8 does not admit a $2:1$ normal covering (compare with the nonnormal examples in Section 4).
2. For $z \to z^2-1$, the Julia set K satisfies all properties of the theorem above except for the fact that $M\backslash K$ has infinitely many connected. Of course, this set K does not satisfy the thesis of Theorem 3.

Proof of Theorem 2. We include an easy proof here inspired by the discussions above. A different, unified proof of both Theorems 1 and 2 is given in the next section. Take an annular neighborhood U of A such that $f^{-1}(A) \cap U = A$. The connected component U_1 of $f^{-1}(U)$ containing A is also an annulus because $U \subset B_A^0$ and $B_A^0 \cap S_f = \emptyset$. Moreover, $f : U_1 \to U$ is $d:1$ by the choice of U. Define by induction U_n as to be the connected component of $f^{-1}(U_{n-1})$ containing A. Note that U_n is an annulus contained in B_A^0 for all $n \geq 1$, but we cannot assume that $U_n \cap f^{-1}(A)$ is contained in A (see Example 4.4). Assuming that $f : U_n \to U_{n-1}$ is a $d:1$ covering it will be shown that so is $f : U_{n+1} \to U_n$. Let c_1 and c_2 be the connected component of $U_n\backslash A$. Each one is an annulus, one of whose boundary components is contained in A. Let c_i' be the connected component of $f^{-1}(c_i)$ that is contained in U_{n+1}. Then c_i' is an annulus, and $f : c_i' \to c_i$ a covering map. It is claimed now that the component of the boundary of c_i' intersecting $f^{-1}(A)$ is contained in A: otherwise, both connected components of the boundary of c_i' are mapped by f in A, which is impossible. So $f^{-1}(A) \cap (\overline{c_1' \cup c_2'}) = A$, and as $f : U_{n+1} \to U_n$ is a covering map, then such covering is $d:1$. As $\bigcup_n U_n = B_A^0$ the result follows.

3.3. **Unified proof.** Let M be a two dimensional manifold and (f, U) be a normal attracting pair.

Proposition 2. *There exists a neighborhood U' of A such that (f, U') is equivalent to (f, U), $f|_{U'}$ is a $d:1$ covering onto $f(U')$ and $U' \in \mathcal{G}_\kappa(A)$.*

Proof. By Lemma 7 there exists $W \in \mathcal{G}_\kappa(A)$, such that $\overline{W} \subset f(U)$ and $\overline{W} \cap f(\partial U) = \emptyset$. Note that every connected component of $U \setminus A$ contains exactly one connected component of $W \setminus A$. It follows that each connected component of $M \setminus W$ contains at least one connected component of $M \setminus U$ (otherwise U would contain a connected component of $M \setminus A$; that is not possible as stated in Lemma 4 of the previous section).

Then let $n > 0$ be such that the closure of $f^n(U)$ is contained in W. Choose some open set V whose closure is a manifold with boundary and such that the closure of $f^{n+1}(U)$ is contained in V and V is contained in $f^n(U)$. By Lemma 8, there exists an open set S', between V and W with $\kappa(S') = \kappa(A)$.

It will be proved that $\overline{f(S')} \subset S'$. Note that S' is equal to the union of V with those components of the complement of V that are contained in W. Note first that $\overline{f(V)} \subset V$ because $f^{n+1}(U) \subset V \subset f^n(U)$.

It remains to prove that if a component d of the complement of V is contained in W, then $f(d) \subset W$. Denote by b the boundary of d. As f has no critical points in U, the boundary of $f(d)$ is contained in $f(b)$, that is, contained in V, and hence in W. Assume by contradiction that $f(d)$ is not contained in W. Then there exists a component of the boundary of W contained in $f(d)$. As each component of the complement of W contains at least one component of the complement of U, it follows that U does not contain $f(d)$. But this is a contradiction since $f(d) \subset f(U) \subset U$, hence $f(d) \subset W$, and therefore $\overline{f(S')} \subset S'$.

Now define $U' = f^{-1}(S') \cap U$. Then, $f|_{U'} : U' \to f(U')$ is a $d : 1$ covering: first note that it is proper, as $f(\partial U') \subset \partial f(U')$. Indeed, suppose that $x \in \partial U'$ and $f(x)$ lies in the interior of U', and take a neighborhood W_1 of $f(x)$, $W_1 \subset f(U') \subset S' \cap f(U)$. Note that $x \in U$ because $x \in \partial U^{'}$ implies that $x \in \overline{f^{-1}(S')} \subset \overline{f^{-1}(W)} = f^{-1}(\overline{W})$ and $f^{-1}(\overline{W}) \cap \partial U = \emptyset$, by the choice of W at the beginning of this proof.

It follows that $x \in f^{-1}(W_1) \cap U \subset f^{-1}(S^{'}) \cap U = U^{'}$. But $f^{-1}(W_1) \cap U$ is open contradicting $x \in \partial U^{'}$. Besides, $f|_{U'} : U' \to f(U')$ is a covering as there are no critical points in U, and it is $d : 1$ because its restriction to A is $d : 1$ and $f^{-1}(A) \cap U = A$.

Moreover $U' \in \mathcal{G}_\kappa(A)$ because S' does, and $\overline{f(U')} \subset U'$ by the claim. Clearly the pair (f, U') is equivalent to (f, U). □

Corollary 2. *Let $U \in \mathcal{G}_\kappa(A)$ where A is the attracting set defined by the attracting pair (f, U) and $f : U \to f(U)$ a covering map. Then $U \setminus f(U)$ is the union of a finite number of annuli.*

Proof. Note that $\overline{f(U)}$ is a submanifold and that the genus of U and $f(U)$ coincide as both contain A. As $U = (U \setminus f(U)) \cup f(U)$, it follows that the genus of $U \setminus f(U)$ is equal to zero. It is claimed now that $f(U)$ also belongs to $\mathcal{G}_\kappa(A)$. As $g(U) = g(f(U))$, one just has to prove $\kappa(U) = \kappa(f(U))$. Using that the restriction of f to U is a covering, it follows that the number of components of the boundary of $f(U)$ is less than or equal to $\kappa(U)$. Now, $f(U) \in \mathcal{G}(A)$ implies that $\kappa(U) \leq \kappa(f(U))$. Then, $\kappa(U) = \kappa(f(U))$, proving the claim.

It follows that both $U \setminus A$ and $f(U) \setminus A$ are disjoint union of annuli. Moreover, as $f : U \to f(U)$ is a covering map, ∂U is mapped onto $\partial f(U)$ and no component of $f(\partial U)$ can be homotopically trivial in U. So, the connected components of $f(\partial U)$

are necessarily homotopic to connected components of ∂U, proving that $U \setminus f(U)$ is a finite union of annuli. $\square$

It follows that $f(U)$ is a deformation retract of U. In particular the attracting set is *simple*:

Definition 5. Let (U, f) be an attracting pair with $f : U \to f(U)$ a $d : 1$ covering map, $d \geq 1$, and consider $w \in f(U)$. The pair (U, f) is *simple* if $i_* : \pi_1(f(U), w) \to \pi_1(U, w)$ is onto, where i_* is the map induced by the inclusion $i : (f(U), w) \to (U, w)$.

The following lemma, valid in any dimension, proves that simple attractors have the property we search for: being a $d : 1$ covering in a neighborhood implies the same property in the whole immediate basin.

Lemma 9. *If (U, f) is a simple attracting pair with $f : U \to f(U)$ a $d : 1$ covering map and $S_f \cap B_A^0 = \emptyset$, then f is $d : 1$ in B_A^0.*

Proof. Let $w \in f(U)$ and $\{w_1, \dots, w_d\}$ be the d preimages of w in U. We will show that for all $i = 1, \dots, d$, w_i has exactly d preimages in $f^{-1}(U) \cap B_A^0$.

Fix $i \in \{1, \dots, d\}$. If β is an arc joining w and w_i, then the lift β_j of β starting at w_j defines a preimage x_j of w_i, $j \in 1, \dots, d$. Besides, as we assume that there are no critical points in B_A^0, the points $x_j, j = 1, \dots, d$ are all different. We will show that for any other arc γ joining w and w_i the lifts γ_j starting at w_j have their other extremity in $\{x_1, \dots, x_d\}$, $j \in 1, \dots, d$. Note that this implies the result. Indeed, if x is a preimage of w_i in $f^{-1}(U) \cap B_A^0$, and δ is an arc joining w_i and x, then $f(\delta)$ is an arc joining w and w_i, and therefore the lift of $f(\delta)$ starting at w_i has its other extremity in $\{x_1, \dots, x_d\}$, showing that $x \in \{x_1, \dots, x_d\}$.

By hypothesis there exists a loop $\alpha \subset f(U)$ with basepoint at w such that $[\beta\gamma^{-1}]_{\pi_1(U,w)} = [\alpha]_{\pi_1(U,w)}$. As $f : U \to f(U)$ is $d : 1$, the lifts α_j of α starting at w_j have their other extremities at $\{w_1, \dots, w_d\}$, $j = 1, \dots, d$. Let $h_j : S^1 \times [0, 1] \to U$ be the lift of the homotopy joining α and $\beta\gamma^{-1}$ starting at α_j, $j = 1, \dots, d$. Then, $h_j((t, 1))$ is the lift of $\beta\gamma^{-1}$ starting at w_j. So, $h_j((t, 1)) = \beta_j\rho$, where ρ is an arc starting at x_j and having its other extremity $z \in \{w_1, \dots, w_d\}$. This implies that ρ^{-1} is the lift of γ starting at z, showing that the lifts of γ starting at $\{w_1, \dots, w_d\}$ have their other extremities at $\{x_1, \dots, x_d\}$. $\square$

4. Examples and applications

4.1. **Homotopy of the immediate basin.**

It is not known if there exists a C^1-structurally stable map in the homotopy class of a linear hyperbolic nonexpanding automorphism of the 2 - torus $\mathbb{T}^2$ (see [IPR1], where this problem was posed, C^1-stability was characterized in dimension 2, and some examples of C^1-stable maps on $\mathbb{T}^2$ were explained). Note that whenever f_* (the induced map in homology) is a hyperbolic linear automorphism of $\mathbb{T}^2$ with determinant greater than 1, then

$$\bigcap_{k \in \mathbb{Z}} f_*^k(\mathbb{Z}^2) = \{0\}.$$

The following result shows that the immediate basin of an injective attracting set (as those of C^1- structurally stable maps) is contained in a disc of $\mathbb{T}^2$.

Proposition 3. *Let $A \subset \mathbb{T}^2$ be the attracting set associated to the attracting pair (f, U), and assume that $f|_A$ is injective. If $\bigcap_{k \in \mathbb{Z}} f_*^k(\mathbb{Z}^2) = \{0\}$, then every closed curve contained in B_A^0 is null homotopic in $\mathbb{T}^2$.*

Note that this does not say that B_A^0 is simply connected; for example, when A is a Plykin attractor immersed in the torus.

Proof. Assume by contradiction that there exists a curve α in B_A^0 that is not null homotopic. Now Theorem 1 implies that $f' := f|_{B_A^0}$ is injective, so for every $k \in \mathbb{Z}$, it holds that $\alpha_k = f'^{-k}(\alpha)$ is a closed curve which is not null homotopic. Then $[\alpha] \in \cap_{k\in\mathbb{Z}} f_*^k(\mathbb{Z}^2)$, a contradiction. □

4.2. **The hyperbolic normal attractor.** Let f be a map of the cylinder $S^1 \times \mathbb{R}$ given by

$$f(z, y) = (z^d, \lambda y + \tau(z)),$$

where $\lambda \in (0, 1)$ and τ is a continuous function.

If $\tau = 0$, then there exists a normal attractor of degree d, $A = S^1 \times \{0\}$. F. Przytycki (see [Prz]) used this map (with $\tau = 0$) to show that C^1 Ω-inverse stable does not imply C^1 Ω-stable. The inverse limit is conjugated to a solenoid. M. Tsujii gave a proof of the fact that for λ close to 1, and generic τ of class C^2, there exists a unique physical measure that is absolutely continuous (see [Tsu]). In a related paper ([AGT]), Avila, Gouëzel and Tsujii studied the smoothness of the density of the SBR measure in more detail, and the mixing properties of f for smooth τ. In [BKRU] some topological properties of this map were obtained: it has a global attractor that coincides with the nonwandering set, and if τ is Lipschitz, then for λ close to 1 the attractor is an annulus. As f is hyperbolic, this provides an example of a hyperbolic attractor with nonempty interior. As will be seen in Lemma 10, this is not possible if the attractor is normal.

These properties show the complexity of the dynamics of the family. We show here (Proposition 4) that this example is frequent.

We will need to recall some facts about hyperbolic attractors of endomorphisms. Assume that f is a C^1 map without critical points having a hyperbolic attractor A which does not reduce to a periodic orbit. The local stable set of size ϵ of a point $x \in A$, denoted $W_\epsilon^s(x)$, is defined as the set of points $y \in M$ such that $d(f^n(x), f^n(y)) < \epsilon$ for all $n \geq 0$. There exists $\epsilon > 0$ such that the collection of local stable sets of the points in A constitute a C^1 foliation of a neighborhood of A. This will be referred to in the sequel as the local stable foliation. Unstable manifolds are not well defined for points in A but for orbits in A. Indeed, if $\bar{x} = \{x_n\}_{n\in\mathbb{Z}}$ is an orbit of f contained in A, then the unstable set of $\bar{x}$ is defined by $\bar{W}^u(\bar{x}) = \{\bar{y} : \lim_{n\to-\infty} d(x_n, y_n) = 0\}$. If $\overleftarrow{A}$ denotes the inverse limit set of A, that is, the set of f-orbits contained in A, then the unstable set of a point $\bar{x} \in \overleftarrow{A}$ is contained in $\overleftarrow{A}$. There is a projection $\pi_0 : \overleftarrow{A} \to A$ given by $\pi_0(\bar{x}) = x_0$. It turns out that $\pi_0(\bar{W}^u(\bar{x}))$ is a C^1 immersion of $\mathbb{R}$ into A, not necessarily injective. When x is a periodic point, we will denote by $W^u(x)$ the projection of $\bar{W}^u(\bar{x})$ in M, where $\bar{x}$ denotes the periodic sequence in $\overleftarrow{A}$ whose 0 coordinate is x. It turns out that $W^u(x)$ is a one dimensional immersed submanifold transverse to each stable leaf; moreover, each intersection of $W^u(x)$ with a stable leaf of a point in A belongs to A (local product structure).

The local stable foliation covers a neighborhood W of A. Let s be any leaf, and let $\phi : \mathbb{R} \to W$ be a C^1 injective immersion transverse to the stable foliation. Let $a < b$ be such that $\phi(a)$ and $\phi(b)$ belong to s but $\phi(t) \notin s$ for every $a < t < b$. Then the closed curve γ formed with the segment of s from $\phi(a)$ to $\phi(b)$ followed by the curve ϕ restricted to the interval $[a, b]$ is a simple closed curve. It is claimed now that γ is essential in W. On the contrary, there exists a disc D contained in W whose boundary is equal to γ (because γ is simple). Since the local stable foliation has neither singularities nor closed leaves, the Poincaré-Bendixson Theorem implies that any leaf entering D has to leave D. This contradicts the transversality between ϕ and the local stable foliation. Indeed, to each $x \in \phi([a, b])$ one can assign the point $y \in \phi([a, b])$ where the leaf of the foliation entering to D at x leaves D for the first time. This function is well defined because the leaf through x is transverse to ϕ, and is continuous by the continuity of the foliation. So it has a fixed point x_0, and this contradicts the transversality at x_0. This proves the claim.

Consequently, using again a Poincaré-Bendixson argument, it comes that if W is an annulus, then the set of intersections of ϕ with s form a monotone sequence.

We will make use of the following fact, which can be seen as a mild version of the λ-lemma:

Let p be a hyperbolic fixed point with both stable and unstable manifolds of dimension one, and let U be a neighborhood of p. Let u be a subset of $W^u(p)$ homeomorphic to an open interval. Let $\gamma : [0, 1] \to U$ be a continuous function such that $\gamma(0) \in u$ and $\gamma(t) \notin u$ for every $t \neq 0$. Let $(x_n)_{n \in \mathbb{N}}$ be a preorbit of $\gamma(0)$ contained in the unstable set of p. In other words, it is asked that $x_n \to p$, $f(x_{n+1}) = x_n$ and $x_0 = \gamma(0)$. For each $n > 0$, let γ_n be the lift of γ under f^n with initial point x_n. Then there exists $y \in W_s(p)$, $y \neq p$ such that every $x \in [p, y]_{W^s(p)}$ is the limit of a sequence $\{z_n\}$, where $z_n \in \gamma_n$ for every $n \in \mathbb{N}$.

A difference with the hypothesis of the λ-lemma is that the curve γ is not asked to be C^1, and so the intersection with u need not be transverse. On the other hand, its conclusion just implies that a segment contained in the stable manifold of p can be obtained as a C^0 limit of segments contained in the lifts of γ. The proof is very easy, just take two curves α_1 and α_2, transverse to u, and such that $\alpha_1(0)$ and $\alpha_2(0)$ lie in u at different sides of $\gamma(0)$. Now apply the usual λ-lemma of Palis.

We are now in a position to give an intelligible proof of the following result.

Proposition 4. *Let A be a normal connected hyperbolic attracting set in a two dimensional manifold M. If the degree of $f|_A$ is d, and $d > 1$, and f has no critical point in A, then either A is homeomorphic to a circle and the restriction of f to A is conjugated to z^d, or f is Anosov.*

Proof. If f is not Anosov, then A is an essential continuum in an annulus U that separates U in two annular connected components. It suffices to show that every (or one) unstable manifold is a circle. This implies that there is a semiconjugacy from $f|_A$ to m_d in S^1, but this must be a homeomorphism since f was hyperbolic in A. Let $p \in A$ be a periodic point and assume it is fixed with positive eigenvalues. One can also assume, replacing U with a smaller annulus if needed, that there is a foliation of U whose leaves are local stable manifolds. Assume as well that the leaf through p joins the two boundary components of U.

Let $\phi : [0, \infty) \to U$ parametrize a dense branch of $W^u(p)$ with $\phi(0) = p$. We prove first that $W^u(p)$ self-intersects; that is, that ϕ is not injective. So, suppose that ϕ is injective.

As $d > 1$, there exists a preimage x of p, other than p, that belongs to A. As the unstable manifold of p is dense in A, it must intersect the stable manifold of x; let y be the first point of intersection between $W^s(x)$ and $\phi([0, +\infty))$. Note that the loop $\gamma = [p, f(y)]_{W^u(p)} \cdot [f(y), p]_{W^s(p)}$ is essential as proved in the claim preceding this proposition. It follows, moreover, that the set of points of intersection of ϕ with $W^s(p)$ is a monotonic sequence, but this is a contradiction since $f^n(y) \to p$.

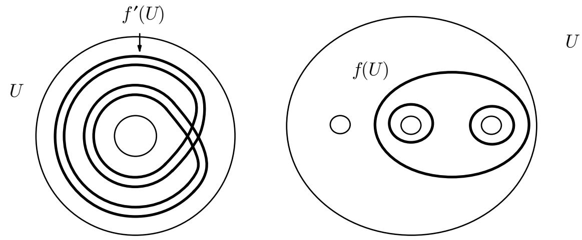
We have just proven that injectivity of ϕ leads to a contradiction; that is, $W^u(p)$ self-intersects. Two possibilities are left: either $W^u(p)$ (and therefore A) is a simple closed curve, or the situation of the C^0- λ-lemma appears, where u is a segment of $W^u(p)$, and γ is also contained in $W^u(p)$, and hence in A. From normality, the preimages γ_n of γ are also contained in A, and by the conclusion of the C^0-λ-lemma, it follows that a segment of $W^s(p)$ is contained in A. Therefore by the local product structure A has interior, contradicting the following lemma. □

Lemma 10. *If a hyperbolic normal attracting set has nonempty interior, then it is open.*

Proof. Let V be an open disc contained in A and let $x \in A$ be a periodic point of f. Let k be the period of x, and note that there exists a sequence $\{y_n\}_{n\geq 0}$ such that $f^k(y_{n+1}) = y_n$ for every $n \geq 0$, $y_n \to x$ and $y_0 \in V$, that is, y_n belongs to the unstable set of x. As $V \subset A$, and A is a normal attractor, the connected component V_j of $f^{-jk}(V)$ that contains y_j is contained in A. But the λ-lemma implies that V_j accumulates at the stable manifold of x as j goes to ∞. In conclusion, the stable manifold of x is contained in A. It follows that every stable and unstable manifold of a point in A is contained in A. This implies that A is open by the local product structure. □

4.3. **Nonnormal examples.** 1) Let $f : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$, $f(z, y) = (z^d, \lambda y)$, $|\lambda| < 1$, as in the previous example with $\tau = 0$. Then, $\overline{f(U)} \subset U$ if $U = S^1 \times (-\epsilon, \epsilon)$ for a suitable $\epsilon > 0$. One can perturb f to a map f' such that $f'(U)$ looks as in Figure 1 (a) (see [Prz]). The resulting attracting set A is not normal.

2) As this example shows, not every nonnormal attractor has nexus two (see Figure 1b). The map $q(z) = z^2$ is a $2 : 1$ covering from $U' = \mathbb{D} \setminus \{0, 1/2, -1/2\}$ onto $\mathbb{D} \setminus \{0, 1/4\}$. Let ϕ be a diffeomorphism defined in $\mathbb{D}$ that fixes 0, sends 1/4 to 1/2 and whose image is a disc contained in $\mathbb{D}$ that avoids the point $-1/2$. Note that $f_1 = \phi q$ fixes 0 and 1/2 and carries U' into itself. Moreover if the derivative of ϕ at 1/4 is adequately chosen, then 1/2 will be a repelling fixed point for the composition ϕq. But (f_1, U') is not an attracting pair because the closure of $f_1(U')$ is not contained in U'. To achieve an attracting pair with the same geometrical features as f_1, we proceed to modify the map f_1 around the origin and the open set U' around the points 0 and 1/2. The new map f will coincide with f_1 for $|z| > 1/3$, and with $q(2z/\epsilon)$ in $|z| < \epsilon$, with $\epsilon < 1/10$. Thus f will have a fixed critical point at the origin and an expanding invariant curve at $|z| = \epsilon^2/4$. Finally, (f, U) is an attractor pair if U is defined as $\mathbb{D} \setminus (D_0 \cup D_{1/2} \cup D_{-1/2})$, where $D_0 = D(0; \epsilon)$, $D_{1/2}$ is a neighborhood of 1/2 such that $f(D_{1/2}) \supset \overline{D}_{1/2}$ and $D_{-1/2}$ is a neighborhood of $-1/2$ such that $f(D_{-1/2}) \supset \overline{D}_{1/2}$. It is claimed that the connected component of $f^{-1}(A)$ containing A is not contained in $f(U)$. This implies that $f^{-1}(A) \setminus A$ is not closed, hence A is not normal. To prove the claim, note that the fundamental group of U has three generators a, b, c, while that of $f(U)$ has only two b, c. Moreover, $f_*(a) = c$, $f_*(b) = 2b$ and $f_*(c) = c$, where f_* is the induced map of fundamental groups. This implies that for all $n \geq 1$ $f^n(U)$ contains a loop of homotopy class bc. So $f^{-1}(f^n(U))$ contains an open connected set containing a loop of the class abc, which gives that the connected component of $f^{-1}(A)$ containing is not contained in $f(U)$ as claimed. Note that the nexus of K is at least 3.



(a) (b)

FIGURE 1

4.4. **The counterexample.** Other interesting conclusions can be derived from Theorem 1; for example, it can be proved that a C^1 stable map which is not a diffeomorphism nor expanding must have a saddle type basic piece and an attracting periodic orbit. These assertions are nontrivial consequences, and we left their proofs to forthcoming works.

The results of Theorems 1 and 2 are not valid in manifolds of dimension greater than two. The following example, introduced in [IPR], is a C^1 stable map that is not a diffeomorphism nor an expanding map. The ambient manifold is $S^1 \times S^2$, the two sphere seen as the compactification of the complex plane. The map f is given by $f(z,w) = (z^2, (z+w)/3)$ for $w \in \mathbb{C}$, and $f(z,\infty) = (z^2,\infty)$. This map has degree two, has no critical points, and its nonwandering set is the union of a solenoid attractor and the expanding basic piece $S^1 \times \infty$. The attractor A is obtained as the intersection of the forward images of the solid torus $S^1 \times \mathbb{D}$, where $\mathbb{D}$ denotes the unit disc. It is easy to see that f is injective in $S^1 \times \mathbb{D}$ and hence in A, and that the immediate basin coincides with the basin and is equal to $S^1 \times \mathbb{C}$. As the map has degree two, it follows that the restriction of the map to the immediate basin is not injective: the set $A' = f^{-1}(A) \setminus A$ is not empty and contained in B_A^0. This example shows that Theorem 1 is not true if the manifold is $S^1 \times S^2$.

For a counterexample to Theorem 2 in dimension greater than 2, consider $f : S^1 \times S^1 \times S^2 \to S^1 \times S^1 \times S^2$ such that $f(z,w,x) = (z^2, w^2, \frac{x+w}{3})$. It has degree 4, and a normal attracting set of degree 2 equal to $S^1 \times A$, A as in the previous example, where the immediate basin is the whole manifold except $S^1 \times S^1 \times \infty$.

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