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STABILITY OF SPACELIKE HYPERSURFACES IN DE SITTER SPACE

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(Communicated by Lei Ni)

This paper is dedicated to our advisors

ABSTRACT. This paper discusses compact linear Weingarten space-like hypersurfaces in a de Sitter space We prove that a compact linear Weingarten spacelike hypersurface in a de Sitter space is stable if and only if it is totally umbilical.

1. INTRODUCTION

The notion of stability with respect to the hypersurfaces of constant mean curvature in Riemannian space forms was first studied by Barbosa and do Carmo in [2], and Barbosa, do Carmo and Eschenburg in [3], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations. Later on, Alencar, do Carmo and Colares [1] extended such results to the context of closed hypersurfaces with constant scalar curvature. In order to do that, they assumed that the Riemannian ambient space has a positive constant section curvature. After that, Barbosa and Colares [4] studied closed hypersurfaces with constant r-th mean curvature H_r immersed in a Riemannian space form. In this setting, they showed that such hypersurfaces are r-stable if, and only if, they are geodesic spheres. More recently, in [6], Chen and Wang studied the stability of closed hypersurfaces satisfying $(n-1)H_2 + aH = b$, where a and b are real constants, in the Euclidean sphere S^{n+1}. They proved that such a hypersurface M can be characterized as a critical point of an appropriated functional for volume-preserving variations, and that M is stable if, and only if, it is totally umbilical and non-totally geodesic.

In 1993, Barbosa and Oliker [5] considered the analogous problem in the Lorentz context. They proved that constant mean curvature spacelike hypersurfaces in Lorentz manifolds are also critical points of the area functional for volume preserving variations. In this sense, the variational methods for Riemannian and Lorentz manifolds coincide. Then they computed the second variation formula and obtained in the de Sitter space S_1^{n+1} that spheres maximize the area functional for volume preserving variations. This fact determines the definition of stability, which will be

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given below in a more general form. It was proved in [5] that if M is a complete spacelike immersed hypersurface in S_1^{n+1} with constant mean curvature H, then M is stable if M is compact, or $H^2 \geq 1$ or $n^2H^2 < 4(n-1)$. Recently, in [11], Liu and Yang obtained an extension of the result in [5] for spacelike hypersurfaces with constant scalar curvature.

In this paper, we consider Chen-Wang's analogue result for linear Weingarten hypersurfaces in de Sitter space S_1^{n+1}. Following [6, 7, 10], we first give the definition of linear Weingarten hypersurfaces in an $(n+1)$-dimensional de Sitter space S_1^{n+1}:

Definition 1.1. Let M be a hypersurface in an $(n+1)$-dimensional de Sitter space S_1^{n+1}. Then M is called a linear Weingarten hypersurface if $cR+dH+e=0$, where c, d and e are constants such that $c^2+d^2 \neq 0$, R and H are the scalar curvature and the mean curvature of M, respectively.

Remark 1.1. In Definition 1.1, when $d=0$ a linear Weingarten hypersurface M reduces to a hypersurface with constant scalar curvature; when $c=0$, a linear Weingarten hypersurface M reduces to a hypersurface with constant mean curvature. In this case, the linear Weingarten hypersurfaces can be regarded as a natural generalization of hypersurfaces with constant scalar curvature or with constant mean curvature.

Remark 1.2. By the Gauss equation, we can rewrite the condition $cR+dH+e=0$ as $(n-1)\widetilde{e}H_2+aH=b$, where H_2 is the second mean curvature, a, b and $\widetilde{e}$ are constants such that $a^2+\widetilde{e}^2>0$. When $\widetilde{e}=0$, it reduces to the constant mean curvature case; when $\widetilde{e}\neq 0$, without loss of generality, we can assume $\widetilde{e}=1$, that is $(n-1)H_2+aH=b$.

We now consider the stability of closed linear Weingarten spacelike hypersurfaces in a de Sitter space S_1^{n+1}. Such a concept arises from considering the variational problem of minimizing a suitable linear combination of the 2nd area for volume-preserving variations. We will show that a closed linear Weingarten spacelike hypersurface in the de Sitter space S_1^{n+1} is stable if, and only if, it is totally umbilical. More precisely, we have:

Theorem 1.1. *Let M be a compact orientable hypersurface in a de Sitter space S_1^{n+1} satisfying $(n-1)H_2+aH=b$ for some constants $a>0$ and b. By choosing the orientation if possible, we assume that $H>0$. Then M is stable if, and only if, M is totally umbilical.*

Remark 1.3. Comparing with the Main Theorem in [5, 11], we withdraw the constant mean curvature or constant scalar curvature and obtain the same result.

2. Preliminaries

Let M be a hypersurface in a de Sitter space S_1^{n+1}, and $\{e_1, \cdots, e_{n+1}\}$ be a local frame of semi-Riemannian orthonormal vector fields in S_1^{n+1} such that, restricted to M, the vectors $\{e_1, \cdots, e_n\}$ are tangent to M and the vector $e_{n+1}=N$ is the time-like unit normal field. We use the following convention on the range of indices:

$$1 \leq i, j, k, \cdots, \leq n.$$

We denote the principal curvatures of M by $k_1, \cdots, k_n$. Let H, H_2 and H_3 denote the mean curvature, the 2nd mean curvature and the 3rd mean curvature of M, respectively; namely,

$$H=\frac{1}{n}S_1=\frac{1}{n}\sum_{i=1}^{n}k_i,$$

$$H_2=\frac{2}{n(n-1)}S_2=\frac{2}{n(n-1)}\sum_{1\le i_1<i_2\le n}k_{i_1}k_{i_2},$$

$$H_3=\frac{6}{n(n-1)(n-2)}S_3=\frac{6}{n(n-1)(n-2)}\sum_{1\le i_1<i_2<i_3\le n}k_{i_1}k_{i_2}k_{i_3}.$$

The Riemannian curvature tensor has the following components:

$$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})-(h_{il}h_{jk}-h_{ik}h_{jl}).$$

The normalized scalar curvature R is given by

$$n(n-1)R=n(n-1)-n^2H^2+S,$$

where $S=\sum_{\alpha,i,j}(h_{ij}^{\alpha})^2$ is the squared norm of the second fundamental form of M.

Let f be a smooth function on M, its gradient and Hessian is defined by

$$df=\sum_{i=1}^{n}f_i\omega_i,$$

$$\sum_{j=1}^{n}f_{ij}\omega_j=df_i+\sum_{j=1}^{n}f_j\omega_{ji},$$

where $\{\omega_1,\cdots,\omega_{n+1}\}$ are dual coframe of $\{e_1,\cdots,e_n\}$. The Laplacian of f is defined by the trace of Hessian:

$$\Delta f=\mathrm{tr}(\mathrm{Hess}(f))=\sum_i f_{ii}.$$

Following Cheng-Yau [9], we introduce the operator $\square$ [8] associated to ϕ acting on any C^2-function f by

$$\square f=\sum_{i,j}\phi_{ij}f_{ij}=\sum_{i,j}(nH\delta_{ij}-h_{ij})f_{ij}, \tag{2.1}$$

where f_{ij} are the components of the Hessian of f.

3. The variation problem for linear Weingarten hypersurfaces

Let X be a variation of $x:M\to S_1^{n+1}$, which is a differentiable map $X:(-\varepsilon,\varepsilon)\times M\to S_1^{n+1}$, $\varepsilon>0$, such that $X_0=x$ and for each $t\in(-\varepsilon,\varepsilon)$, $X_t(\cdot)=X(t,\cdot)$ is an immersion from M to S_1^{n+1}, and $X_t|_{\partial M}=x|_{\partial M}$. Denote by $W=\frac{\partial X}{\partial t}|_{t=0}$ the variation vector field of the variation X. We define the volume function $V:(-\varepsilon,\varepsilon)\to R$ of X by

$$V(t)=\int_{[0,t]\times M}X^*dS_1^{n+1}.$$

The variation is normal if W is parallel to the unit normal vector field N of x and volume-preserving if $V(t)=V(0)$ for all t.

The area functional $A : (-\varepsilon, \varepsilon) \to R$ is given by

$$A(t) = \int_M (-S_1) dM_t,$$

where dM_t denotes the volume element of the metric induced in M by X_t. Let $f = \langle W, N\rangle$. We have the following classical result

Lemma 3.1 ([1–3, 11])**.**

$$\left.\frac{dV}{dt}\right|_{t=0} = \int_M f dM.$$

Lemma 3.2 ([4, 11])**.** *Let M be an n-dimensional compact hypersurface in S_1^{n+1}. We have*

$$\frac{d(S_1)}{dt} = \Delta f - (S_1^2 - 2S_2)f - nf + \sum_i W_i S_{1,i},$$

$$\frac{dS_2}{dt} = \Box f - (S_1 S_2 - 3S_3)f - (n-1)S_1 f + \sum_i W_i S_{2,i},$$

$$\frac{d(dM_t)}{dt} = (S_1 f + div(\sum_i W_i e_i))dM_t,$$

where $W = \sum_i W_i e_i + uN$ is the variational vector field of $x : M \to S_1^{n+1}$, $S_{r,i}$ are the first covariant derivatives of S_r in the e_i direction, $r = 1, 2$.

For the constant a, the Jacobi functional associated to the variation X is given by $J : (-\varepsilon, \varepsilon) \to R$

$$J(t) = A(t) - a.$$

We can obtain the following proposition:

Proposition 3.1 (First Variation Formula)**.** *Let M be an n-dimensional spacelike hypersurface in the de Sitter space S_1^{n+1}. For any variation of $x : M \to S_1^{n+1}$, we have*

$$\frac{dJ(t)}{dt} = \int_M (-2S_2 + n - aS_1) f dM_t. \tag{3.1}$$

Proof. From Lemma 3.2, we have

$$\begin{aligned} \frac{dJ(t)}{dt} &= \int_M [\frac{d(-S_1)}{dt} + (-S_1 - a)(S_1 f + div(\sum_i W_i e_i))]dM_t \\ &= \int_M (-2S_2 + n - aS_1) f dM_t - \int_M \Delta f dM_t + \int_M div(\sum_i (S_1 - a) W_i e_i))dM_t. \end{aligned}$$

Since M is compact and S_1^{n+1} has constant sectional curvature, then

$$\int_M \Delta f dM_t = 0,$$

$$\int_M div(\sum_i (S_1 - a) W_i e_i))dM_t = 0.$$

Then, we get

$$\frac{dJ(t)}{dt} = \int_M (-2S_2 + n - aS_1) f dM_t. \qquad \square$$

As a direct application of Lemma 3.2 and Proposition 3.1, after a long but direct computation, we obtain

Proposition 3.2 (Second Variation Formula). *Let $x : M \to S_1^{n+1}$ be a linear Weingarten hypersurface satisfying $(n-1)H_2 + aH = b$, where a and b are constants. X is a variation of x, then the second derivative of J at $t = 0$ is given by*

$$J''(0)(f) = \int_M \left\{ -(2\square + a\Delta)f + 2[S_1S_2 - 3S_3 + (n-1)S_1]f + a(S_1^2 - 2S_2 + n)f \right\} f dM. \tag{3.2}$$

Definition 3.1. Let $x : M \to S_1^{n+1}$ be a linear Weingarten hypersurface satisfying $(n-1)H_2 + aH = b$, where a and b are constants. The immersion x is said to be stable if $J''(0)(f) \leq 0$ for all volume-preserving variations of x.

Then from the above definition, a hypersurface satisfying $(n-1)H_2 + aH = b$ is stable if, and only if, $J''(0)(f) \leq 0$ for all differentiable function f which satisfies $\int_M f dM = 0$. This can be proved after a similar argument as in [3, 11], the details are omitted here.

4. Proof of the theorem

In this section, we will prove our main theorem.

Proof of Theorem 1.1. Firstly, suppose that M is totally an umbilical hypersurface in S_1^{n+1}. Then the principal curvatures and H are constants, and we have

$$S_2 = \frac{n(n-1)}{2}H^2, \; S_3 = \frac{n(n-1)(n-2)}{6}H^3,$$

and

$$\square = (n-1)H\Delta.$$

Choose $f : M \to R$ such that $\int_M f dM = 0$. From the second variation formula (3.2) of J, we have

$$\begin{aligned} J''(0)(f) &= \int_M \left\{ -(2\square + a\Delta)f + 2[S_1S_2 - 3S_3 + (n-1)S_1]f + a(S_1^2 - 2S_2 + n)f \right\} f dM_t \\ &= (a + 2(n-1)H) \int_M (n(H^2+1)f^2 - \|\nabla f\|^2) dM \\ &\leq (a + 2(n-1)H) \int_M (n(H^2+1) - \lambda(M)) f^2 dM, \end{aligned}$$

where $\lambda(M)$ is the first eigenvalue of the Laplacian Δ in M. Since M is totally umbilical, M is sphere. Then we have $\lambda(M) = n(H^2+1)$. By the assumption that $H > 0$ and $a > 0$, we obtain that $J''(0)(f) \leq 0$, for all f with $\int_M f dM = 0$. Therefore we conclude that M is stable.

Now we consider the reversed part. Let $M \subset S_1^{n+1}$ be a stable linear Weingarten hypersurface satisfying $(n-1)H_2 + aH = b$ for some constants $a > 0$ and b. We will show that M is totally umbilical.

Let $x : M \to S_1^{n+1} \subset R_1^{n+2}$. Fix a unit vector $\nu \in R_1^{n+2}$ and define functions f and g on M by

$$f = \langle N, \nu \rangle, \; g = \langle x, \nu \rangle.$$

We will need the following equations for f and g [4, 12]:

$$\begin{aligned}\Box f &= (S_1S_2 - 3S_3)f + 2S_2g + D_{\nu^T}S_2,\\ \Box g &= 2S_2f + (n-1)S_1g,\\ \Delta g &= S_1f + ng,\\ \Delta f &= (S_1^2 - 2S_2)f + S_1g + D_{\nu^T}S_1,\end{aligned}$$

where ν^T is the tangent component of the vector ν used in the definition of f and g.

For the follows, we recall that the $n+2$-dimensional Lorentz-Minkowski space R_1^{n+2} is the real vector space R^{n+2} endowed with the Lorentz metric

$$\langle \nu, \omega\rangle = -\nu_0\omega_0 + \sum_{i=1}^{n+1}\nu_i\omega_i,$$

for all $\nu, \omega \in R^{n+2}$. The $n+1$-dimensional de Sitter space S_1^{n+1} is given by

$$S_1^{n+1} = \{p \in R_1^{n+2}; \langle p, p\rangle = 1\}.$$

Then it is easy to show that the metric induced from $\langle , \rangle$, turns S_1^{n+1} into a Lorentz manifold with constant sectional curvature 1. We choose ν as an element of a canonical basis $a_0, \cdots, a_{n+1}$ of R_1^{n+2} and let f_A and g_A be the above functions for $\nu = a_A$, $A = 0, 1, \cdots, n+1$. Set

$$f_A = \langle N, a_A\rangle,\ g_A = \langle x, a_A\rangle.$$

Now observing that,

$$\sum_{A=1}^{n+1} f_A^2 = 1 + f_0^2,\ \sum_{A=1}^{n+1} g_A^2 = 1 + g_0^2,\ \sum_{A=1}^{n+1} f_Ag_A = f_0g_0.$$

We defined $\overline{x} = \int_M x dM$. Since $\langle x, x\rangle = 1$, then it is elementary to conclude that $\langle \overline{x}, \overline{x}\rangle > 0$. We choose $a_0 = \frac{\overline{x}}{|\overline{x}|}$, then

$$1 = \langle a_0, a_0\rangle = -\langle a_0, x\rangle^2 + \sum_{A=1}^{n}\langle a_0, e_i\rangle^2 - \langle a_0, N\rangle^2 \geq -g_0^2 - f_0^2,$$

since x is stable. Then, for each A, $J''(0)(g_A) \leq 0$. On the other hand,

$$\begin{aligned}J''(0)(g_A) = \sum_{A=0}^{n+1}\int_M \Big\{ &-(2\Box + a\Delta)g_A + 2[S_1S_2 - 3S_3 + (n-1)S_1]g_A\\ &+ a(S_1^2 - 2S_2 + n)g_A\Big\}g_A dM\\ = \int_M \Big\{ &-4S_2f_0g_0 - aS_1f_0g_0 + 2[S_1S_2 - 3S_3]g_0^2 + a(S_1^2 - 2S_2)g_0^2\\ &-2(n-1)S_1 - an + 2[S_1S_2 - 3S_3 + (n-1)S_1] + a(S_1^2 - 2S_2 + n)\Big\}dM\\ \geq \int_M \Big\{ &-4S_2f_0g_0 - aS_1f_0g_0 + 2[S_1S_2 - 3S_3](-1 - f_0^2) + a(S_1^2 - 2S_2)(-1 - f_0^2)\\ &-2(n-1)S_1 - an + 2[S_1S_2 - 3S_3 + (n-1)S_1] + a(S_1^2 - 2S_2 + n)\Big\}dM\\ = -\int_M & f_0(2\Box + a\Delta)f_0 dM\\ = \int_M & (\langle 2P_1\nabla f_0, \nabla f_0\rangle + \langle \nabla f_0, \nabla f_0\rangle)dM \geq 0,\end{aligned}$$

where we use the fact that $2D_{\nu^T} S_2 + aD_{\nu^T} S_1 = D_{\nu^T}(nb) = 0$ and $P_1 = S_1 I - A$.

Therefore $J''(0)(g_A)$ must be zero. Then $\nabla f_0 = 0$ and $1 + g_0^2 = -f_0^2$. This implies that g_0 is constant and then M is totally umbilical. □

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