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HYPERBOLICITY AND EXPONENTIAL LONG-TIME CONVERGENCE FOR SPACE-TIME PERIODIC HAMILTON-JACOBI EQUATIONS

HÉCTOR SÁNCHEZ-MORGADO

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Abstract. In this note we prove exponential convergence to time-periodic states of the solutions of space-time periodic Hamilton-Jacobi equations, assuming that the Aubry set is the union of a finite number of hyperbolic periodic orbits of the Euler-Lagrange flow. The period of limiting solutions is the least common multiple of the periods of the orbits in the Aubry set. This extends a result that was obtained by Iturriaga and the author for the autonomous case.

1. Introduction

Let M be a closed connected manifold and TM its tangent bundle. Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a $C^k, k \geq 3$, Lagrangian. We assume the Lagrangian satisfies the hypotheses of Mather's seminal paper [M]

Convexity: In linear coordinates, L restricted to T_xM has positive definite Hessian.

Superlinearity: For some Riemannian metric we have

$$\lim_{|v|\to\infty} \frac{L(x,v,t)}{|v|} = \infty,$$

uniformly on x and t.

Periodicity: For all x, v, t; $L(x,v,t+1) = L(x,v,t)$,

Completeness: The Euler-Lagrange flow ϕ_t associated to L is complete.

Let $H : T^*M \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated to the Lagrangian:

$$H(x,p,t) = \max_{v\in T_xM} pv - L(x,v,t).$$

For $c \in \mathbb{R}$ consider the Hamilton-Jacobi equation

$$u_t + H(x, d_xu, t) = c. \tag{1.1}$$

It is known ([CIS], [B1]) that there is only one value $c = c(L)$, the so-called critical value, such that (1.1) has time periodic viscosity solutions. In this note we prove

Theorem 1.1. *Assume $M = \mathbb{T}^d$ and the Aubry set is the union of a finite number of hyperbolic periodic orbits of the Euler-Lagrange flow and let N be the least common multiple of their periods. There is $\mu > 0$ such that for any viscosity solution*

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$v : \mathbb{T}^d \times [s, \infty[\to \mathbb{R}$ *of* (1.1) *with* $c = c(L)$, *there is an* N*-periodic viscosity solution* $u : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ *of* (1.1) *and* $K > 0$ *such that for any* $t > 0$

$$|v(x,t) - u(x,t)| \leq Ke^{-\mu t}.$$

In [IS] we proved a similar result in the autonomous case under the assumption that the Aubry set consists in a finite number of hyperbolic critical points of the Euler-Lagrange flow and claimed wrongly, as observed in [WY], that it also holds changing some critical points by periodic orbits. Although the present note follows ideas that we used in the autonomus case, we had to provide proofs of some statements such as Lemma 4.2 and Propositions 5.1 and 5.2.

2. Preliminaries on weak KAM theory

Define the action of an absolutely continuous curve $\gamma : [a,b] \to M$ as

$$A_L(\gamma) := \int_a^b L(\gamma(\tau), \dot\gamma(\tau), \tau) d\tau.$$

A curve $\gamma : [a,b] \to M$ is *closed* if $\gamma(a) = \gamma(b)$ and $b - a \in \mathbb{Z}$. The critical value can be defined as

$$c(L) := \min\{k \in \mathbb{R} : \forall \gamma \text{ closed } A_{L+k}(\gamma) \geq 0\}.$$

Let $\mathcal{P}(L)$ be the set of probabilities on the Borel σ-algebra of $TM \times \mathbb{S}^1$ that have compact support and are invariant under the Euler-Lagrange flow. Then

$$c(L) = -\min\{\int L d\mu : \mu \in \mathcal{P}(L)\}.$$

The Mather set is defined as

$$\widetilde{\mathcal{M}} := \overline{\bigcup \{\operatorname{supp} \mu : \mu \in \mathcal{P}(L), \int L d\mu = -c(L)\}}.$$

For $a \leq b$, $x, y \in M$ let $\mathcal{C}(x,a,y,b)$ be the set of absolutely continuous curves $\gamma : [a,b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$. For $a \leq b$ define $F_{a,b} : M \times M \to \mathbb{R}$ by

$$F_{a,b}(x,y) := \min\{A_L(\gamma) : \gamma \in \mathcal{C}(x,a,y,b)\}.$$

Define $\mathcal{L}_t : C(M, \mathbb{R}) \to C(M, \mathbb{R})$ by

$$\mathcal{L}_t u(x) = \inf_{y \in M} u(y) + F_{0,t}(y,x).$$

Then $v(x,t) = \mathcal{L}_t u(x)$ is the viscosity solution to (1.1) for $c = 0$ with $v(x,0) = u(x)$. $(\mathcal{L}_n)_{n \in \mathbb{N}}$ is a semi-group known as the Lax-Oleinik semi-group.

For $t \in \mathbb{R}$ let $[t]$ be the corresponding point in $\mathbb{S}^1$ and $\langle t \rangle$ be its integer part. Define the *action potential* $\Phi : M \times \mathbb{S}^1 \times M \times \mathbb{S}^1 \to \mathbb{R}$ by

$$\Phi(x, [s], y, [t]) := \inf\{F_{a,b}(x,y) + c(L)(b-a) : [a] = [s], [b] = [t]\},$$

and the *Peierls barrier* $h : M \times \mathbb{S}^1 \times M \times \mathbb{S}^1 \to \mathbb{R}$ by

$$h(x,[s],y,[t]) := \liminf_{\langle b-a \rangle \to \infty} \big(F_{a,b}(x,y) + c(L)(b-a)\big)_{[a]=[s],[b]=[t]}. \tag{2.1}$$

We have $-\infty < \Phi \leq h < \infty$.

The critical value is the unique number c such that (1.1) has viscosity solutions $u : M \times \mathbb{S}^1 \to \mathbb{R}$. In fact ([CIS]), for any $(p,[s])$ the functions $(x,[t]) \mapsto h(p,[s],x,[t])$

and $(x,[t]) \mapsto -h(x,[t],p,[s])$ are respectively *backward* and *forward* viscosity solutions of (1.1). Set $c = c(L)$ and let $\mathcal{S}^-(\mathcal{S}^+)$ be the set of *backward* (*forward*) viscosity solutions of (1.1).

The Lagrangian is called regular if the lim inf in (2.1) is a lim and in that case, for each $s,t \in \mathbb{R}$ the convergence of the sequence $(F_{a,b} + c(L)(b-a))_{[a]=[s],[b]=[t]}$ is uniform. The Lagrangian is regular if and only if $(\mathcal{L}_n + c(L)n)_{n\in\mathbb{N}}$ converges ([B1]), so that for each $u \in C(M,\mathbb{R})$ there exists $\bar{u} \in C(M\times\mathbb{S}^1,\mathbb{R})$ such that

$$\lim_{k\to\infty} \mathcal{L}_{t+k}u(x) + c(t+k) = \bar{u}(x,[t])$$

and in fact

$$\bar{u}(x,[t]) = \inf_{y\in M} u(y) + h(y,[0],x,[t]).$$

A subsolution of (1.1) always means a viscosity subsolution. A curve $\gamma : I \to M$ *calibrates* a subsolution $u : M\times\mathbb{S}^1 \to \mathbb{R}$ of (1.1) if

$$u(\gamma(b),[b]) - u(\gamma(a),[a]) = A_{L+c}(\gamma|[a,b])$$

for any $[a,b] \subset I$. If $u \in \mathcal{S}^-(\mathcal{S}^+)$, for any $(x,[s]) \in M\times\mathbb{S}^1$ there is $\gamma :]-\infty,s] \to M$ ($\gamma : [s,\infty[\to M$) that calibrates u and $\gamma(s) = x$.

A pair $(u_-,u_+) \in \mathcal{S}^-\times\mathcal{S}^+$ is called *conjugated* if $u_- = u_+$ on $\mathcal{M} = \pi(\widetilde{\mathcal{M}})$. For such a pair (u_-,u_+), we define $I(u_-,u_+)$ as the set where u_- and u_+ coincide. If $(x,[s]) \in I(u_-,u_+)$, then $u_\pm$ is differentiable at $(x,[s])$ and $d_xu_-(x,[s]) = d_xu_+(x,[s])$. Let

$$I^*(u_-,u_+) = \{(x, d_xu_\pm(x,[s]),[s]) : (x,[s]) \in I(u_-,u_+)\}.$$

We may define the *Aubry set* either as the set ([B])

$$\mathcal{A}^* := \bigcap\{I^*(u_-,u_+) : (u_-,u_+) \text{ conjugated}\} \subset T^*M\times\mathbb{S}^1$$

or as its pre-image under the Legendre transformation ([F])

$$\tilde{\mathcal{A}} := \{(x, H_p(x,p,t),[t]) : (x,p,[t]) \in \mathcal{A}^*\}.$$

The projection of either Aubry set in $M\times\mathbb{S}^1$ is

$$\mathcal{A} = \{\,(x,[t]) \in M\times\mathbb{S}^1 : h(x,[t],x,[t]) = 0\}.$$

An important tool for our proof of the result is the existence of strict C^k critical subsolutions in our setting, that extends the result of Bernard [B] for the autonomous case.

Theorem 2.1 ([GS]). *Assume the Aubry set $\tilde{\mathcal{A}}$ is the union of a finite number of hyperbolic periodic orbits of the Euler-Lagrange flow. Then there is a C^k subsolution f of* (1.1) *such that*

$$f_t + H(x, d_xf(x,[t]),t) < c$$

for any $(x,[t]) \notin \mathcal{A}$.

3. Reduction to a regular Lagrangian

We assume that $M = \mathbb{T}^d$ and the Aubry set $\tilde{\mathcal{A}}$ is the union of the hyperbolic periodic orbits

$$\Gamma_i(t) = \phi_t(x_i, v_i, [0]) = (\gamma_i(t), \dot{\gamma}_i(t), [t]) \quad i \in [1, m]$$

of the Euler-Lagrange flow with periods $N_i, i \in [1, m]$. In this case the projected Aubry and Mather set coincide. Let N be the least common multiple of $N_1, \dots, N_m$. Define

$$P_N : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^1 \to \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^1 \tag{3.1}$$
$$(x, v, [t]) \mapsto (x, \frac{v}{N}, [Nt]),$$

and the Lagrangian $L_N = L \circ P_N$. The corresponding Hamiltonian is given by

$$H_N(x, p, t) = H(x, Np, Nt).$$

For a curve $\gamma : [a, b] \to \mathbb{T}^d$ define $\gamma^N : [a/N, b/N] \to \mathbb{T}^d, t \mapsto \gamma(Nt)$; then $NA_{L_N}(\gamma^N) = A_L(\gamma)$. A curve γ is an extremal (minimizer) of L if and only if the curve γ^N is an extremal (minimizer) of L_N.

Let $\gamma^N_{i,j}(t) = \gamma_i(j + Nt), j \in [1, N_i], i \in [1, m]$. According to sections 3 and 5 of [B1], the Aubry set of L_N is the union of the hyperbolic 1-periodic orbits $\Gamma^N_{i,j}(t) = (\gamma^N_{i,j}(t)\dot{\gamma}^N_{i,j}(t), [t])$ and L_N is regular. Observe that a function $u : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ is a viscosity solution of (1.1) if and only if $w(x, t) = \frac{1}{N} v(x, Nt)$ is a viscosity solution of

$$w_t + H_N(x, d_x w, t) = c.$$

Thus our main theorem is reduced to the case in which the Lagrangian is regular and the Aubry set is the union of finite number of hyperbolic 1-periodic orbits.

Let $f : \mathbb{T}^{d+1} \to \mathbb{R}$ be a C^k subsolution of (1.1) strict outside $\mathcal{A}$. Consider the Lagrangian

$$\mathbb{L}(x, v, t) = L(x, v, t) - d_x f(x, [t])v - f_t(x, [t]) + c$$

with Hamiltonian $\mathbb{H}(x, p, t) = H(x, p + d_x f, [t]) + f_t(x, [t]) - c$.

If $\alpha \in \mathcal{C}(x, s, y, t)$, $A_{\mathbb{L}}(\alpha) = A_{L+c}(\alpha) + f(x, [s]) - f(y, [t])$. Thus L and $\mathbb{L}$ have the same Euler-Lagrange flow and projected Aubry set. Moreover,

$$\forall (x, v, t)\, \mathbb{L}(x, v, t) \geq 0,\ \tilde{\mathcal{A}} = \{(x, v, [t]) : \mathbb{L}(x, v, t) = 0\} \tag{3.2}$$

and u is a viscosity solution of (1.1) if and only if $u - f$ is a viscosity solution of

$$w_t + \mathbb{H}(x, d_x w, t) = 0.$$

We can therefore assume that $c = 0$, L is regular and has the property (3.2) of $\mathbb{L}$, and the Aubry set is the union of hyperbolic 1-periodic orbits $\Gamma_i, i \in [1, m]$ of the Euler-Lagrange flow.

Thus, for any $u \in C(\mathbb{T}^d, \mathbb{R})$ we have

$$\lim_{k \to \infty} \mathcal{L}_{\tau + k} u(x) = \bar{u}(x, [\tau]) := \min_{y \in \mathbb{T}^d} u(y) + h(y, [0], x, [\tau]). \tag{3.3}$$

We get our result by proving that the convergence in (3.3) is exponentially fast.

4. Useful lemmas

We assume that $L(x,v,t) \geq 0$ for any (x,v,t) and $\tilde{\mathcal{A}} = \{(x,v,[t]) : L(x,v,t) = 0\}$. For $\gamma : I \to \mathbb{T}^d$, we write $\Gamma(s) = (\gamma(s), \dot\gamma(s), [s])$.

Lemma 4.1. *Let $W = \bigcup_{i=1}^m W_i$ be a neighborhood of the Aubry set in $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^1$. Then, there exist $K > 0$ such that if $\gamma : [0,t] \to \mathbb{T}^d$, $t > 1$ is a minimizer, then the time that Γ remains outside W is less than K.*

Proof. From our hypotheses, there is $\eta > 0$ such that $L(x,v,t) \geq \eta$ for $(x,v,[t]) \notin \tilde{\mathcal{A}}$. For $t > 1$ the action of minimizers $\gamma : [0,t] \to \mathbb{T}^d$ is bounded from above independently of t. The lemma follows. □

Lemma 4.2. *Let $V = \bigcup_{i=1}^m V_i$ be a neighborhood of the Aubry set in $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^1$. There is $N = N(V) \in \mathbb{N}$ such that if $\gamma : [0,t] \to \mathbb{T}^d, t > 1$ is minimizer, then Γ stays in one V_i during an interval larger than $\dfrac{t}{N} - 1$.*

Proof. Let $\delta \in]0,1[$ be such that the δ-neighborhood of Γ_i is contained in V_i for $i \in [1,m]$. Let W_i be the $\dfrac{\delta}{2}$-neighborhood of Γ_i and apply Lemma 4.1 to $W = \bigcup_{i=1}^m W_i$. Since the velocity of any minimizer $\gamma : [0,t] \to \mathbb{T}^d, t > 1$ is bounded by a constant ≥ 1, the number of times it can go from W to the complement of V is bounded by some $N \in \mathbb{N}$. The lemma follows. □

Let $\lambda_{i,j}, j = 1, \dots, d$ be the positive Lyapunov exponents of γ_i, set $\lambda < \min\limits_{i,j} \lambda_{ij}$ and write $\phi_t(x,v,[0]) = \psi_t(x,v)$. There is a splitting $\mathbb{R}^{2d} = E_i^- \oplus E_i^+$, invariant under $D_i = d\psi_1(x_i,v_i)$, and a norm $\|\cdot\|$ such that $\|D_i^{\pm 1}|E_i^{\mp}\| \leq e^{-\lambda}$.

Proposition 4.3 ([BV], [Be])**.** *There are $\alpha, \rho \in (0,1)$, neighborhoods A_i of (x_i,v_i) in $\mathbb{T}^d \times \mathbb{R}^d$, and α-Hölder maps $g_i : A_i \to B_{2\rho}(0) \subset \mathbb{R}^{2d}$ with α-Hölder inverse such that*

$$D_i \circ g_i = g_i \circ \psi_1.$$

Set

$$U_i = g_i^{-1}(B_\rho), \quad V_i = \bigcup_{s\in[0,1]} (\psi_s(U_i) \cap \psi_{s-1}(U_i)) \times [s], \quad V = \bigcup_{i=1}^m V_i. \tag{4.1}$$

Apply Lemma 4.2 to V given by (4.1) and let $N = N(V)$.

Lemma 4.4. *There are $C, \mu > 0$ such that*

(1) *If the Euler-Lagrange orbit Γ stays in V_i on an interval $[j, j+k]$ with $k \geq 2n$, then*
$$d(\Gamma(s), \Gamma_i(s)) \leq Ce^{-\alpha\lambda n}, s \in [j+n, j+k-n].$$
(2) *If $\gamma : [0,t] \to \mathbb{T}^d$, $t > 2N$ is a minimizer, then there are integers $i \in [1,m]$, $l \in [0,t]$ such that*
$$d(\gamma(l), x_i) \leq Ce^{-\mu t}.$$

Proof. Let $Y_l = (\gamma(j+l), \dot\gamma(j+l))$ and write

$$g_i(Y_0) = z_- + z_+,\ g_i(Y_k) = w_- + w_+,\ z_\pm, w_\pm \in E_i^\pm.$$

For $l = n, \cdots, k-n$ we have

$$\begin{aligned} g_i(Y_l) &= D_i^l(z_-) + D_i^{l-k}(w_+), \\ \|g_i(Y_l)\| &\leq e^{-\lambda l}\|z_-\| + e^{\lambda(l-k)}\|w_+\| \leq Ce^{-\lambda n}, \\ d(Y_l, (x_i, v_i)) &\leq Ce^{-\alpha\lambda n}. \end{aligned}$$

Now, if $\gamma : [0, t] \to \mathbb{T}^d$, $t > 2N$ is a minimizer, let $n = \left\langle \frac{t}{2N} \right\rangle - 1$. Then Γ stays in one V_i on an interval I of length $2n+1$. Let $j \in \mathbb{N}$ be such that $[j, j+2n] \subset I$. Then

$$d(\gamma(j+n), x_i) \leq d(Y_n, (x_i, v_i)) \leq Ce^{-\alpha\lambda n} \leq Ce^{-\alpha\lambda t/2N}.$$

□

5. Proof of the result

We assume that the Lagrangian is regular, $L(x, v, t) \geq 0$ for any (x, v, t) and $\tilde{\mathcal{A}} = \{(x, v, t) : L(x, v, t) = 0\}$. We have to prove that there are $\mu > 0$ depending only on L, and $C > 0$ depending on u, such that for any $(x, \tau) \in \mathbb{T}^d \times [0, 1]$, $k \in \mathbb{N}$

$$-Ce^{-\mu k} \leq \mathcal{L}_{\tau+k}u(x) - \bar{u}(x, [\tau]) \leq Ce^{-\mu k}. \tag{5.1}$$

First we find constants $\mu, C > 0$ giving right inequality. For brevity, for $x \in \mathbb{T}^d$ we write $\bar{x} = (x, [0])$, and for $\beta : I \to \mathbb{T}^d$ we write $\bar{\beta}(t) = (\beta(t), [t])$, $d\beta(t) = (\beta(t), \dot{\beta}(t))$, $B(t) = (\beta(t), \dot{\beta}(t), [t])$.

For each $z \in \mathbb{T}^{d+1}$ let y_z be such that

$$\bar{u}(z) = u(y_z) + h(\bar{y}_z, z).$$

In particular let $y_j = y_{\bar{x}_j}$. Since $z \mapsto -h(z, \bar{x}_j) \in \mathcal{S}^+$, there is a semi-static curve $\beta^j : [0, \infty[\to \mathbb{T}^d$ such that $\beta^j(0) = y_j$ and

$$A_L(\beta^j|[0, t]) = h(\bar{y}_j, \bar{x}_j) - h(\bar{\beta}^j(t), \bar{x}_j), \quad t > 0.$$

Proposition 5.1. *Fix $j \in [1, m]$. There are $k_1, \ldots, k_l = j$ all different and semi-static curves $\beta_r : \mathbb{R} \to \mathbb{T}^d$, $0 \leq r < l$, $\beta_0 = \beta^j$, such that $\Gamma_{k_{r+1}}$ is the ω limit of B_r and if $r > 0$, Γ_{k_r} is the α limit of B_r,*

$$\begin{aligned} h(\bar{\beta}_0(t), \bar{x}_j) &= h(\bar{\beta}_0(t), \bar{x}_{k_1}) + h(\bar{x}_{k_1}, \bar{x}_j), \quad t \geq 0, \\ h(\bar{x}_{k_1}, \bar{x}_j) &= \sum_{r=1}^{l-1} h(\bar{x}_{k_r}, \bar{x}_{k_{r+1}}), \\ A_L(\beta_0|[0, t]) &= h(\bar{y}_j, \bar{x}_{k_1}) - h(\bar{\beta}_0(t), \bar{x}_{k_1}), \quad t \geq 0, \\ A_L(\beta_r|_{[t,s]}) &= h(\bar{x}_{k_r}, \bar{\beta}_r(s)) - h(\bar{x}_{k_r}, \bar{\beta}_r(t)), \quad t \leq s. \end{aligned}$$

Proof. Let Γ_{k_1} be the ω-limit of B_0. If $k_1 = j$ we stop; otherwise we observe that by the regularity of L

$$h(\bar{\beta}_0(t), \bar{x}_j) = h(\bar{\beta}_0(t), \bar{x}_{k_1}) + h(\bar{x}_{k_1}, \bar{x}_j), \quad t \geq 0$$

and then

$$A_L(\beta_0|[0, t]) = h(\bar{y}_j, \bar{x}_{k_1}) - h(\bar{\beta}_0(t), \bar{x}_{k_1}), \quad t \geq 0.$$

For each $i \in [1, m]$ there is a neighborhood U'_i of $\bar{\gamma}_i$ where $h_i(z) = h(\bar{x}_i, z)$ is C^k and the local weak unstable manifold of Γ_i is the graph of $H_p(x, d_x h_i(x, [t]), [t])$.

Let U_i be a neighborhood of $\bar{\gamma}_i$ with compact $\bar{U}_i \subset U_i'$. Let $\rho_n : [0,n] \to \mathbb{T}^d$ be a curve joining x_{k_1} to x_j such that

$$A_L(\rho_n) = F_{0,n}(x_i, x_j).$$

Let $t_n \in [0,n]$ be the first exit time of $\bar{\gamma}_n(t)$ out of U_i, and $\bar{\gamma}_n(t_n)$ be the first point of intersection with ∂U_{k_1}. As n goes to infinity, t_n and $n - t_n$ tend to infinity. This follows from the fact that $\dot{\rho}_n(0)$ has to tend to $\dot{\gamma}_{k_1}(0)$, and $\dot{\rho}_n(n)$ has to tend to $\dot{\gamma}_j(0)$. To justify this, consider v a limit point of $\dot{\rho}_n(0)$, and $\gamma : \mathbb{R} \to \mathbb{T}^d$ the solution to the Euler-Lagrange equation such that $\gamma(0) = x_i, \dot{\gamma}(0) = v$. From the fact that

$$F_{0,n}(x_{k_1}, x_j) - F_{1,n}(\rho_n(1), x_j) = A_L(\rho_n|_{[0,1]})$$

and the regularity of L, taking limit $n \to \infty$ it follows

$$h(\bar{x}_{k_1}, \bar{x}_j) - h(\bar{\gamma}(1), \bar{x}_j) = A_L(\gamma|_{[0,1]}).$$

Since $\gamma_{k_1}(-1) = x_{k_1}$ and $L = 0$ on $\tilde{\mathcal{A}}$

$$h(\bar{\gamma}_{k_1}(-1), \bar{x}_j) - h(\bar{\gamma}(1), \bar{x}_j) = A_L(\gamma_{k_1}|_{[-1,0]}) + A_L(\gamma_{[0,1]})$$

so that the curve obtained by gluing $\gamma_{k_1}|_{[-1,0]}$ with $\gamma|_{[0,1]}$ minimizes the action between its endpoints. In particular, it has to be differentiable, thus $v = \dot{\gamma}(0) = \dot{\gamma}_{k_1}(0)$. Define $\alpha_n : [-\langle t_n \rangle, n - \langle t_n \rangle] \to \mathbb{T}^d$ by $\alpha_n(t) = \rho_n(t + \langle t_n \rangle)$ and let (y, w, τ) be a cluster point of $(d\rho_n(t_n), t_n - \langle t_n \rangle)$. Then there is a sequence (α_{n_l}) converging uniformly on compact intervals to the solution $\beta_1 : \mathbb{R} \to \mathbb{T}^d$ of the Euler-Lagrange equation such that $d\beta_1(\tau) = (y, w)$. Since for any $t \le s$ we have

$$F_{-\langle t_n \rangle, s}(x_{k_1}, \alpha_n(s)) - F_{-\langle t_n \rangle, t}(\bar{x}_{k_1}, \alpha_n(t)) = A_L(\alpha_n|_{[t,s]}),$$
$$F_{-\langle t_n \rangle, t}(x_{k_1}, \alpha_n(t)) + F_{t, n-\langle t_n \rangle}(\alpha_n(t), x_j) = F_{0,n}(x_{k_1}, x_j),$$

from the uniform convergence of $F_{a,b}$ when $\langle b - a \rangle \to \infty$, we obtain for any $t \le s$

$$h(\bar{x}_{k_1}, \bar{\beta}_1(s)) - h(\bar{x}_{k_1}, \bar{\beta}_1(t)) = A_L(\beta_1|_{[t,s]}),$$
$$h(\bar{x}_{k_1}, \bar{\beta}_1(t)) + h(\bar{\beta}_1(t), \bar{x}_j) = h(\bar{x}_{k_1}, \bar{x}_j).$$

Since $\bar{\alpha}_n([-\langle t_n \rangle, t_n - \langle t_n \rangle]) \subset \bar{U}_{k_1}$ we have that Γ_{k_1} is the α-limit of B_1 and let Γ_{k_2} be its ω-limit. If $k_2 = j$ we stop; otherwise we observe that

$$h(\bar{x}_{k_1}, \bar{x}_{k_2}) + h(\bar{x}_{k_2}, \bar{x}_j) = h(\bar{x}_{k_1}, \bar{x}_j),$$
$$h(\bar{x}_{k_1}, \bar{\beta}_1(t)) + h(\bar{\beta}_1(t), \bar{x}_{k_2}) = h(\bar{x}_{k_1}, \bar{x}_{k_2}).$$

Therefore $k_1 \neq k_2$. We proceed in the same way to find a solution to the Euler-Lagrange equation $\beta_2 : \mathbb{R} \to \mathbb{T}^d$ such that Γ_{k_2} is the α-limit of B_2 and for any $t, s \in \mathbb{R}$, $t < s$

$$h(\bar{x}_{k_2}, \bar{\beta}_2(s)) - h(\bar{x}_{k_2}, \bar{\beta}_2(t)) = A_L(\beta_2|_{[t,s]}),$$
$$h(\bar{x}_{k_2}, \bar{\beta}_2(t)) + h(\bar{\beta}_2(t), \bar{x}_j) = h(\bar{x}_{k_2}, \bar{x}_j).$$

Let Γ_{k_3} be the ω-limit of B_2. If $k_3 = j$ we stop; otherwise we observe that

$$h(\bar{x}_{k_2}, \bar{x}_{k_3}) + h(\bar{x}_{k_3}, \bar{x}_j) = h(\bar{x}_{k_2}, \bar{x}_j),$$
$$h(\bar{x}_{k_2}, \bar{\beta}_2(t)) + h(\bar{\beta}_2(t), \bar{x}_{k_3}) = h(\bar{x}_{k_2}, \bar{x}_{k_3}).$$

Therefore $k_2 \neq k_3$. Since

$$h(\bar{x}_{k_1}, \bar{x}_{k_2}) + h(\bar{x}_{k_2}, \bar{x}_{k_3}) = h(\bar{x}_{k_1}, \bar{x}_{k_3}),$$

we have that $k_1 \neq k_3$. We continue until we get $k_r = j$. □

Let V be given by (4.1). There is $T > 0$ such that for $t \geq T$, we have $B_r(t), B_r(-t) \in V$ and then

$$\begin{aligned} d(d\beta_r(t), d\gamma_{k_{r+1}}(t)) &\leq C_1 e^{-\lambda t}, \qquad t > T, \\ d(d\beta_r(t), d\gamma_{k_r}(t)) &\leq C_1 e^{\lambda t}, \qquad t < -T. \end{aligned}$$

Proposition 5.2. *Given $(x,\tau) \in \mathbb{T}^d \times [0,1]$, there are $j \in [1,m]$ and a semi-static curve $\beta_{x,\tau} :]-\infty,\tau] \to \mathbb{T}^d$ such that $\beta_{x,\tau}(\tau) = x$, Γ_j is the α-limit of $B_{x,\tau}$ and*

$$\begin{aligned} \bar{u}(x,[\tau]) &= \bar{u}(\bar{x}_j) + h(\bar{x}_j, x, [\tau]), \\ A_L(\beta_{x,\tau}|[t,\tau]) &= h(\bar{x}_j, x, [\tau]) - h(\bar{x}_j, \bar{\beta}_{x,\tau}(t)), \quad t < \tau. \end{aligned}$$

Proof. Consider a directed graph with vertices at the points $\bar{x}_1, \cdots, \bar{x}_m$ of the Aubry set, and a directed edge from $\bar{x}_j$ to $\bar{x}_k$ if and only if

$$h(\bar{x}_k, x, [\tau]) = h(\bar{x}_k, \bar{x}_j) + h(\bar{x}_j, x, [\tau]).$$

We call a point $\bar{x}_k$ a root of this graph if there is no edge arriving to this point, which means that for $j \neq k$

$$h(\bar{x}_k, x, [\tau]) < h(\bar{x}_k, \bar{x}_j) + h(\bar{x}_j, x, [\tau]).$$

Notice that the graph contains no cycles, and so each point $\bar{x}_k$ belongs to a branch starting at a root. Take $k \in [1,m]$ such that

$$\bar{u}(x,[\tau]) = \bar{u}(\bar{x}_k) + h(\bar{x}_k, x, [\tau]).$$

If there is an edge from x_l to x_k, then

$$\begin{aligned} \bar{u}(x,[\tau]) \leq \bar{u}(\bar{x}_l) + h(\bar{x}_l, x, [\tau]) &\leq \bar{u}(\bar{x}_k) + h(\bar{x}_k, \bar{x}_l) + h(\bar{x}_l, x, [\tau]) \\ &= \bar{u}(\bar{x}_k) + h(\bar{x}_k, x, [\tau]) = \bar{u}(x,[\tau]). \end{aligned}$$

Therefore, if $\bar{x}_j$ is the root of a branch containing $\bar{x}_k$ we have

$$\bar{u}(x,[\tau]) = \bar{u}(\bar{x}_j) + h(\bar{x}_j, x, [\tau]).$$

Since $z \mapsto h(\bar{x}_j, z) \in \mathcal{S}^-$, there is a semi-static curve $\beta_{x,\tau} :]-\infty,\tau] \to \mathbb{T}^d$ with $\beta_{x,\tau}(\tau) = x$ such that

$$A_L(\beta_{x,\tau}|[t,\tau]) = h(\bar{x}_j, x, [\tau]) - h(\bar{x}_j, \bar{\beta}_{x,\tau}(t)), \quad t < \tau.$$

Let Γ_i be the α-limit of $B_{x,\tau}$. If $(x,[\tau]) = \bar{\gamma}_j(\tau)$, then $\beta_{x,\tau} = \gamma_i = \gamma_j$. Otherwise, by the regularity of L

$$h(\bar{x}_j, x, [\tau]) = h(\bar{x}_j, \bar{x}_i) + h(\bar{x}_i, x, [\tau]).$$

Since $\bar{x}_j$ is a root, $i = j$ and so Γ_j is the α-limit of $B_{x,\tau}$. □

From Propositions 5.1 and 5.2, for any $(x,[\tau]) \in \mathbb{T}^d \times [0,1]$ there are $j \in [1,m]$ and a chain of semi-static curves $\beta_0, \ldots, \beta_{l-1}, \beta_{x,\tau}$ such that

$$\begin{aligned} \bar{u}(x,[\tau]) &= u(y_j) + h(\bar{y}_j, \bar{x}_j) + h(\bar{x}_j, (x,[\tau])) \\ &= u(y_j) + \sum_{r=0}^{l-1} A_L(\beta_r) + A_L(\beta_{x,\tau}). \end{aligned}$$

For k large we next define a curve whose action approximates this sum of actions. There is $R > 0$ such that $B_{x,\tau}(t) \in V_j$ for $t < -R$ and then

$$d(d\beta_{x,\tau}(t), d\gamma_j(t)) \leq C_1 e^{\lambda t}.$$

If $B_{x,\tau}$ stays during a long interval inside $V_i, i \neq j$, then number R will be very large. However, according to Lemma 4.4, in a very long subinterval $[s,t]$ it is exponentially close to Γ_i, so we may jump from $B_{x,\tau}(s)$ to $B_{x,\tau}(t)$ in an interval shorter than 2. Let $K = K(V), N = N(V)$ be as in Lemmas 4.1, 4.4.

Consider the partition $t_0 = -R < s_1 < t_1 < \cdots < s_p \leq t_p$ of the interval $[-R,\tau]$ such that $B_{x,\tau}(]s_h,t_h[) \subset V_{i(h)}$, $B_{x,\tau}([t_h,s_{h+1}]) \cap V = \emptyset$. From Lemma 4.1, we know that $\sum_{h=0}^{p-1} s_{h+1} - t_h$ is bounded by the constant $K > 0$.

For $k \in \mathbb{N}$, $k > 2(N+m+1)+K$ let $n+1 = \left\langle \dfrac{k-K}{2(p+l+1)} \right\rangle$. Set $h_0 = 0$,

$$\{h_1 < \cdots < h_q\} = \{h \in [1,p] : t_h - s_h > 2n+1\},\ a_r = \langle s_{h_r}\rangle,\ b_r = \langle t_{h_r}\rangle,\ a_{q+1} = 0,$$
$$d_r = a_{r+1} - b_r + \cdots + a_{q+1} - b_q + q(2n+1) + n - k, \quad r \in [0,q].$$

Define the curve $\alpha_k : [0,\tau+k] \to \mathbb{T}^d$ by

$$\alpha_k(s) = \begin{cases} \beta_0(s), & s \in [0,n], \\ c_r(s - r(2n+1) + n + 1), & s \in [r(2n+1)-n-1, r(2n+1)-n], r \leq l, \\ \beta_r(s - r(2n+1)), & s \in [r(2n+1)-n, (r+1)(2n+1)-n-1], \\ \gamma_j(s), & s \in [l(2n+1)-n, -d_0-1], \\ c_{l^*}(s+d_0+1), & s \in [-d_0-1, -d_0], \\ \beta_{x,\tau}(s+b_r+d_r-r(2n+1)-n), & s \in [-d_r + r(2n+1), -d_{r+1}+(r+1)(2n+1)-1], \\ c_{l+r}(s+d_r-r(2n+1)+1), & s \in [-d_r+r(2n+1)-1, -d_r+r(2n+1)], \\ \beta_{x,\tau}(s-k), & s \in [k+b_q-n, k+\tau], \end{cases}$$

where $c_r : [0,1] \to \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is defined for n large by

$$c_r(s) = \begin{cases} (1-s)\beta_{r-1}(s+n) + s\beta_r(s-1-n), & r < l, \\ (1-s)\beta_{l-1}(s+n) + s\gamma_j(s), & r = l, \\ (1-s)\gamma_j(s) + s\beta_{x,\tau}(s-1-n-R), & r = l^*, \\ (1-s)\beta_{x,\tau}(s+a_h+n) + s\beta_{x,\tau}(s-1+b_h-n), & r = l+h. \end{cases}$$

Notice that if $(x,\tau) = \gamma_j(\tau)$, then $\alpha_k(s) = \gamma_j(s)$ for $s \geq l(2n+1)-n$.

Since $L \geq 0$,

$$\begin{aligned} A_L(\alpha_k) &= A_L(\beta_0|_{[0,n]}) + \sum_{r=1}^{l-1} A_L(\beta_r|_{[-n,n]}) + \sum_{r=1}^{l+q} A_L(c_r) + A_L(c_{l^*}) \\ &\quad + A_L(\beta_{x,\tau}|_{[-n-R,\tau]}) - \sum_{r=1}^{q} A_L(\beta_{x,\tau}|_{[a_r+n, b_r-n]}) \\ &\leq h(\bar{y}_j, \bar{x}_j) + h(\bar{x}_j, (x,[\tau])) + \sum_{r=1}^{l+q} A_L(c_r) + A_L(c_{l^*}). \end{aligned}$$

Thus

$$\mathcal{L}_{\tau+k}u(x) - \bar{u}(x,[\tau]) \leq \sum_{r=1}^{l+q} A_L(c_r) + A_L(c_{l^*}).$$

Since $L = 0$ on $\tilde{\mathcal{A}}$ and there is $C_2 \geq C_1$ such that

$$d(dc_r(s), d\gamma_{k_r}(s)),\ d(dc_{l^*}(s), d\gamma_j(s)),\ d(dc_{l+r}(s), d\gamma_{i(h_r)}(s)) \leq C_2 e^{-\lambda n},$$

we have

$$\mathcal{L}_{\tau+k}u(x) - \bar{u}(x,[\tau]) \leq C_3 e^{-\lambda n} \leq C_4 e^{-\lambda k/2(m+N+1)}.$$

Now we prove that there are constants $C, \mu > 0$ giving left inequality in (5.1).

For $x \in \mathbb{T}^d$, $t > 0$ let $\gamma = \gamma_{x,t} : [0,t] \to \mathbb{T}^d$ be such that $\gamma(t) = x$ and

$$\mathcal{L}_t u(x) = u(\gamma(0)) + A_L(\gamma_t) = u(\gamma(0)) + F_{0,t}(\bar\gamma(0), x, [t]).$$

For any integers $j \in [0,t]$, $i \in [1,m]$ we have

$$\begin{aligned}
\bar u(x,[t]) &\le \bar u(\bar x_i) + h(\bar x_i, x, [t]) \\
&\le u(\gamma(0)) + h(\bar\gamma(0), x_i) + h(\bar x_i, x, [t]) \\
&\le u(\gamma(0)) + \Phi(\bar\gamma(0), \bar\gamma(j)) + h(\bar\gamma(j), \bar x_i) + h(\bar x_i, \bar\gamma(j)) + \Phi(\bar\gamma(j), x, [t]) \\
&\le u(\gamma(0)) + A_L(\gamma) + h(\bar\gamma(j), \bar x_i) + h(\bar x_i, \bar\gamma(j)) \\
&= \mathcal{L}_t u(x) + h(\bar\gamma(j), \bar x_i) + h(\bar x_i, \bar\gamma(j)) \\
&\le \mathcal{L}_t u(x) + K d(\gamma(j), x_i).
\end{aligned}$$

From Lemma 4.4 there are constants $C, \mu > 0$ such that, if $t > 2N$, there are integers $i \in [1,m]$, $j \in [0,t]$ such that

$$d(\gamma(j), x_i) \le C e^{-\mu t}.$$

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Instituto de Matemáticas, Universidad Nacional Autónoma de México. México DF 04510, México

E-mail address: hector@math.unam.mx