HYPERBOLICITY AND EXPONENTIAL LONG-TIME CONVERGENCE FOR SPACE-TIME PERIODIC HAMILTON-JACOBI EQUATIONS

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Abstract. In this note we prove exponential convergence to time periodic states of the solutions of space-time periodic Hamilton-Jacobi equations, assuming that the Aubry set is the union of a finite number of hyperbolic periodic orbits of the Euler-Lagrange flow. The period of limiting solutions is the least common multiple of the periods of the orbits in the Aubry set. This extends a result that was obtained by Iturriaga and the author for the autonomous case.

1. Introduction

Let $M$ be a closed connected manifold and $TM$ its tangent bundle. Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a $C^k$, $k \geq 3$, Lagrangian. We assume the Lagrangian satisfies the hypotheses of Mather’s seminal paper [M]

Convexity: In linear coordinates, $L$ restricted to $T_x M$ has positive definite Hessian.

Superlinearity: For some Riemannian metric we have

$$\lim_{|v| \to \infty} \frac{L(x, v, t)}{|v|} = \infty,$$

uniformly on $x$ and $t$.

Periodicity: For all $x, v, t$: $L(x, v, t + 1) = L(x, v, t),$

Completeness: The Euler-Lagrange flow $\phi_t$ associated to $L$ is complete.

Let $H : T^* M \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated to the Lagrangian:

$$H(x, p, t) = \max_{v \in T_x M} pv - L(x, v, t).$$

For $c \in \mathbb{R}$ consider the Hamilton-Jacobi equation

$$u_t + H(x, d_x u, t) = c. \tag{1.1}$$

It is known ([CIS], [B1]) that there is only one value $c = c(L)$, the so-called critical value, such that [11] has time periodic viscosity solutions. In this note we prove

Theorem 1.1. Assume $M = \mathbb{T}^d$ and the Aubry set is the union of a finite number of hyperbolic periodic orbits of the Euler-Lagrange flow and let $N$ be the least common multiple of their periods. There is $\mu > 0$ such that for any viscosity solution
v : \mathbb{T}^d \times [s, \infty) \to \mathbb{R} \text{ of } (1.1) \text{ with } c = c(L), \text{ there is an } N\text{-periodic viscosity solution } u : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R} \text{ of } (1.1) \text{ and } K > 0 \text{ such that for any } t > 0
\|v(x, t) - u(x, t)\| \leq Ke^{-nt}.

In [IS] we proved a similar result in the autonomous case under the assumption that the Aubry set consists in a finite number of hyperbolic critical points of the Euler-Lagrange flow and claimed wrongly, as observed in [WY], that it also holds changing some critical points by periodic orbits. Although the present note follows ideas that we used in the autonomous case, we had to provide proofs of some statements such as Lemma 4.2 and Propositions 5.1 and 5.2.

2. Preliminaries on weak KAM theory

Define the action of an absolutely continuous curve \( \gamma : [a, b] \to M \) as
\[ A_L(\gamma) := \int_a^b L(\gamma(\tau), \dot{\gamma}(\tau), \tau) d\tau. \]
A curve \( \gamma : [a, b] \to M \) is closed if \( \gamma(a) = \gamma(b) \) and \( b - a \in \mathbb{Z} \). The critical value can be defined as
\[ c(L) := \min \{ k \in \mathbb{R} : \forall \gamma \text{ closed } A_{L+k}(\gamma) \geq 0 \}. \]
Let \( \mathcal{P}(L) \) be the set of probabilities on the Borel-\( \sigma \)-algebra of \( TM \times \mathbb{S}^1 \) that have compact support and are invariant under the Euler-Lagrange flow. Then
\[ c(L) = -\min \{ \int L d\mu : \mu \in \mathcal{P}(L) \}. \]
The Mather set is defined as
\[ \widetilde{M} := \bigcup \{ \text{supp} \mu : \mu \in \mathcal{P}(L), \int L d\mu = -c(L) \}. \]
For \( a \leq b, x, y \in M \) let \( C(a, y, b) \) be the set of absolutely continuous curves \( \gamma : [a, b] \to M \) with \( \gamma(a) = x \) and \( \gamma(b) = y \). For \( a \leq b \) define \( F_{a, b} : M \times M \to \mathbb{R} \) by
\[ F_{a, b}(x, y) := \min \{ A_L(\gamma) : \gamma \in C(x, a, y, b) \}. \]
Define \( \mathcal{L}_t : C(M, \mathbb{R}) \to C(M, \mathbb{R}) \) by
\[ \mathcal{L}_t u(x) = \inf_{y \in M} u(y) + F_{0, t}(y, x). \]
Then \( v(x, t) = \mathcal{L}_t u(x) \) is the viscosity solution to (1.1) for \( c = 0 \) with \( v(x, 0) = u(x) \).

For \( t \in \mathbb{R} \) let \( \lfloor t \rfloor \) be the corresponding point in \( \mathbb{S}^1 \) and \( \langle t \rangle \) be its integer part.
Define the action potential \( \Phi : M \times \mathbb{S}^1 \times M \times \mathbb{S}^1 \to \mathbb{R} \) by
\[ \Phi(x, [s], y, \lfloor t \rfloor) := \inf \{ F_{a, b}(x, y) + c(L)(b-a) : [a] = [s], [b] = [t] \}, \]
and the Peierls barrier \( h : M \times \mathbb{S}^1 \times M \times \mathbb{S}^1 \to \mathbb{R} \) by
\[ h(x, [s], y, \lfloor t \rfloor) := \liminf_{\langle b-a \rangle \to \infty} (F_{a, b}(x, y) + c(L)(b-a))_{[a]=[s],[b]=[t]}. \]
We have \(-\infty < \Phi \leq h < \infty \).

The critical value is the unique number \( c \) such that (1.1) has viscosity solutions \( u : M \times \mathbb{S}^1 \to \mathbb{R} \). In fact (CIS), for any \( (p, [s]) \) the functions \( (x, \lfloor t \rfloor) \mapsto h(p, [s], x, \lfloor t \rfloor) \)
and \((x, [t]) \mapsto -h(x, [t], p, [s])\) are respectively backward and forward viscosity solutions of \((1.1)\). Set \(c = c(L)\) and let \(S^-(S^+)\) be the set of backward (forward) viscosity solutions of \((1.1)\).

The Lagrangian is called regular if the \(\lim\inf\) in \((2.1)\) is a \(\lim\) and in that case, for each \(s, t \in \mathbb{R}\) the convergence of the sequence \((F_{a,b} + c(L)(b - a))([a]=[s],[b]=[t])\) is uniform. The Lagrangian is regular if and only if \((L_n + c(L)n)_{n \in \mathbb{N}}\) converges \((\mathbb{B}_1)\), so that for each \(u \in C(M, \mathbb{R})\) there exists \(\bar{u} \in C(M \times S^1, \mathbb{R})\) such that

\[
\lim_{k \to \infty} \mathcal{L}_{t+k} u(x) + c(t + k) = \bar{u}(x, [t])
\]

and in fact

\[
\bar{u}(x, [t]) = \inf_{y \in M} u(y) + h(y, [0], x, [t]).
\]

A subsolution of \((1.1)\) always means a viscosity subsolution. A curve \(\gamma : [a, b] \to M\) calibrates a subsolution \(u : M \times S^1 \to \mathbb{R}\) of \((1.1)\) if

\[
u(\gamma([a, b])) - u(\gamma([a, b])) = A_{L+c}(\gamma|_{[a, b]})
\]

for any \([a, b] \subset I\). If \(u \in S^-(S^+)\), for any \((x, [s]) \in M \times S^1\) there is \(\gamma : [s, \infty[ \to M\) that calibrates \(u\) and \(\gamma(s) = x\).

A pair \((u_-, u_+) \in S^- \times S^+\) is called conjugated if \(u_- = u_+\) on \(M = \pi(\tilde{M})\). For such a pair \((u_-, u_+)\), we define \(I(u_-, u_+)\) as the set where \(u_-\) and \(u_+\) coincide. If \((x, [s]) \in I(u_-, u_+), then \(u_\pm\) is differentiable at \((x, [s])\) and \(d_x u_\pm(x, [s]) = d_x u_\pm(x, [s])\). Let

\[
I^*(u_-, u_+) = \{(x, d_x u_\pm(x, [s]), [s]) : (x, [s]) \in I(u_-, u_+)\}.
\]

We may define the Aubry set either as the set \((\mathbb{B}_3)\)

\[
\mathcal{A}^* := \bigcap \{I^*(u_-, u_+) : (u_-, u_+) \text{ conjugated} \} \subset T^* M \times S^1
\]

or as its pre-image under the Legendre transformation \((\mathbb{F})\)

\[
\mathcal{A} := \{(x, H_p(x, p, t), [t]) : (x, p, [t]) \in \mathcal{A}^*\}.
\]

The projection of either Aubry set in \(M \times S^1\) is

\[
\mathcal{A} = \{(x, [t]) \in M \times S^1 : h(x, [t], x, [t]) = 0\}.
\]

An important tool for our proof of the result is the existence of strict \(C^k\) critical subsolutions in our setting, that extends the result of Bernard \((\mathbb{E})\) for the autonomous case.

**Theorem 2.1** \((GS)\). Assume the Aubry set \(\mathcal{A}\) is the union of a finite number of hyperbolic periodic orbits of the Euler-Lagrange flow. Then there is a \(C^k\) subsolution \(f\) of \((1.1)\) such that

\[
f_t + H(x, d_x f(x, [t]), t) < c
\]

for any \((x, [t]) \notin \mathcal{A}\).
3. Reduction to a regular Lagrangian

We assume that \( M = \mathbb{T}^d \) and the Aubry set \( \mathcal{A} \) is the union of the hyperbolic periodic orbits

\[
\Gamma_i(t) = \phi_t(x_i, v_i, [0]) = (\gamma_i(t), \dot{\gamma}_i(t), [t]) \quad i \in [1, m]
\]

of the Euler-Lagrange flow with periods \( N_i, i \in [1, m] \). In this case the projected Aubry and Mather set coincide. Let \( N \) be the least common multiple of \( N_1, \ldots, N_m \). Define

\[
P_N : \mathbb{T}^d \times \mathbb{R}^d \times S^1 \to \mathbb{T}^d \times \mathbb{R}^d \times S^1
\]

\[
(x, v, [t]) \mapsto (x, \frac{v}{N}, [Nt]),
\]

and the Lagrangian \( L_N = L \circ P_N \). The corresponding Hamiltonian is given by

\[
H_N(x, p, t) = H(x, Np, Nt).
\]

For a curve \( \gamma : [a, b] \to \mathbb{T}^d \) define \( \gamma^N : [a/N, b/N] \to \mathbb{T}^d, t \mapsto \gamma(Nt) \); then \( NA_{L_N}(\gamma^N) = A_L(\gamma) \). A curve \( \gamma \) is an extremal (minimizer) of \( L \) if and only if the curve \( \gamma^N \) is an extremal (minimizer) of \( L_N \).

Let \( \gamma^N_{i,j}(t) = \gamma_i(j + Nt), j \in [1, N_i], i \in [1, m] \). According to sections 3 and 5 of \([B1]\), the Aubry set of \( L_N \) is the union of the hyperbolic 1-periodic orbits \( \Gamma^N_{i,j}(t) = (\gamma^N_{i,j}(t), \dot{\gamma}^N_{i,j}(t), [t]) \) and \( L_N \) is regular. Observe that a function \( u : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R} \) is a viscosity solution of \((1.1)\) if and only if \( w(x, t) = \frac{1}{N} v(x, Nt) \) is a viscosity solution of

\[
w_t + H_N(x, d_x w, t) = c.
\]

Thus our main theorem is reduced to the case in which the Lagrangian is regular and the Aubry set is the union of finite number of hyperbolic 1-periodic orbits.

Let \( f : \mathbb{T}^{d+1} \to \mathbb{R} \) be a \( C^k \) subsolution of \((1.1)\) strict outside \( \mathcal{A} \). Consider the Lagrangian

\[
\mathbb{L}(x, v, t) = L(x, v, t) - d_x f(x, [t])v - f_t(x, [t]) + c
\]

with Hamiltonian \( \mathbb{H}(x, p, t) = H(x, p + d_x f, [t]) + f_t(x, [t]) - c \).

If \( \alpha \in C(x, s, y, t), A_L(\alpha) = A_{L+c}(\alpha) + f(x, [s]) - f(y, [t]) \). Thus \( L \) and \( \mathbb{L} \) have the same Euler-Lagrange flow and projected Aubry set. Moreover,

\[
\forall (x, v, t) \mathbb{L}(x, v, t) \geq 0, \mathcal{A} = \{(x, v, [t]) : \mathbb{L}(x, v, t) = 0\}
\]

and \( u \) is a viscosity solution of \((1.1)\) if and only if \( u - f \) is a viscosity solution of

\[
w_t + \mathbb{H}(x, d_x w, t) = 0.
\]

We can therefore assume that \( c = 0, L \) is regular and has the property \((3.2)\) of \( \mathbb{L} \), and the Aubry set is the union of hyperbolic 1-periodic orbits \( \Gamma_i, i \in [1, m] \) of the Euler-Lagrange flow.

Thus, for any \( u \in C(\mathbb{T}^d, \mathbb{R}) \) we have

\[
\lim_{k \to \infty} \mathcal{L}_{\tau+k} u(x) = \bar{u}(x, [\tau]) := \min_{y \in \mathbb{T}^d} u(y) + h(y, [0], x, [\tau]).
\]

We get our result by proving that the convergence in \((3.3)\) is exponentially fast.
4. Useful lemmas

We assume that $L(x, v, t) \geq 0$ for any $(x, v, t)$ and $\bar{A} = \{(x, v, [t]) : L(x, v, t) = 0\}$.
For $\gamma : I \to \mathbb{T}^d$, we write $\Gamma(s) = (\gamma(s), \dot{\gamma}(s), [s])$.

Lemma 4.1. Let $W = \bigcup_{i=1}^{m} W_i$ be a neighborhood of the Aubry set in $\mathbb{T}^d \times \mathbb{R}^d \times S^1$.
Then, there exist $K > 0$ such that if $\gamma : [0, t] \to \mathbb{T}^d$, $t > 1$ is a minimizer, then the time that $\Gamma$ remains outside $W$ is less than $K$.

Proof. From our hypotheses, there is $\eta > 0$ such that $L(x, v, t) \geq \eta$ for $(x, v, [t]) \notin \bar{A}$. For $t > 1$ the action of minimizers $\gamma : [0, t] \to \mathbb{T}^d$ is bounded from above independently of $t$. The lemma follows. \hfill $\square$

Lemma 4.2. Let $V = \bigcup_{i=1}^{m} V_i$ be a neighborhood of the Aubry set in $\mathbb{T}^d \times \mathbb{R}^d \times S^1$.
There is $N = N(V) \in \mathbb{N}$ such that if $\gamma : [0, t] \to \mathbb{T}^d$, $t > 1$ is minimizer, then $\Gamma$ stays in one $V_i$ during an interval larger than $\frac{t}{N} - 1$.

Proof. Let $\delta \in [0, 1]$ be such that the $\delta$-neighborhood of $\Gamma_i$ is contained in $V_i$ for $i \in [1, m]$. Let $W_i$ be the $\frac{\delta}{2}$-neighborhood of $\Gamma_i$ and apply Lemma 4.1 to $W = \bigcup_{i=1}^{m} W_i$.
Since the velocity of any minimizer $\gamma : [0, t] \to \mathbb{T}^d$, $t > 1$ is bounded by a constant $\geq 1$, the number of times it can go from $W$ to the complement of $V$ is bounded by some $N \in \mathbb{N}$. The lemma follows. \hfill $\square$

Let $\lambda_{i,j} = 1, \ldots, d$ be the positive Lyapunov exponents of $\gamma_i$, set $\lambda = \min_{i,j} \lambda_{i,j}$ and write $\phi_t(x, v, [0]) = \psi_t(x, v)$. There is a splitting $\mathbb{R}^{2d} = E_i^+ \oplus E_i^-$, invariant under $D_i = d\psi_1(x_i, v_i)$, and a norm $\| \cdot \|$ such that $\|D_i^{\pm 1}\|_{E_i^{\mp} \to \mathbb{R}^d} \leq e^{-\lambda}$.

Proposition 4.3 ([BV], [BC]). There are $\alpha, \rho \in (0, 1)$, neighborhoods $A_i$ of $(x_i, v_i)$ in $\mathbb{T}^d \times \mathbb{R}^d$, and $\alpha$-Hölder maps $g_i : A_i \to B_{2\rho}(0) \subset \mathbb{R}^{2d}$ with $\alpha$-Hölder inverse such that $D_i \circ g_i = g_i \circ \psi_1$.

Set

$$U_i = g_i^{-1}(B_{\rho}), \quad V_i = \bigcup_{s \in [0, 1]} (\psi_s(U_i) \cap \psi_{s-1}(U_i)) \times [s], \quad V = \bigcup_{i=1}^{m} V_i.$$  

Apply Lemma 4.2 to $V$ given by (4.1) and let $N = N(V)$.

Lemma 4.4. There are $C, \mu > 0$ such that

1. If the Euler-Lagrange orbit $\Gamma$ stays in $V_i$ on an interval $[j, j+k]$ with $k \geq 2n$, then
   $$d(\Gamma(s), \Gamma_i(s)) \leq Ce^{-\alpha \lambda n}, s \in [j + n, j + k - n].$$

2. If $\gamma : [0, t] \to \mathbb{T}^d$, $t > 2N$ is a minimizer, then there are integers $i \in [1, m]$, $l \in [0, t]$ such that
   $$d(\gamma(l), x_i) \leq Ce^{-\mu t}.$$  

Proof. Let $Y_l = (\gamma(j + l), \dot{\gamma}(j + l))$ and write

$$g_i(Y_0) = z_+ + z_+, \quad g_i(Y_k) = w_- + w_+, \quad z_\pm, w_\pm \in E_i^{\pm}.$$
For \( l = n, \cdots, k - n \) we have
\[
g_l(Y_l) = D_l^l(z_-) + D_{l-k}^l(w_+),
\]
\[
\|g_l(Y_l)\| \leq e^{-\lambda t}\|z_-\| + e^{\lambda(l-k)}\|w_+\| \leq Ce^{-\lambda n},
\]
\[
d(Y_l,(x_l,v_l)) \leq Ce^{-\alpha \lambda n}.
\]

Now, if \( \gamma : [0,t] \to \mathbb{T}^d, \ t > 2N \) is a minimizer, let \( n = \left\lfloor \frac{t}{2N} \right\rfloor - 1 \). Then \( \Gamma \) stays in one \( V_i \) on an interval \( I \) of length \( 2n + 1 \). Let \( j \in \mathbb{N} \) be such that \([j,j+2n] \subseteq I\). Then
\[
d(\gamma(j+n),x_i) \leq d(Y_n,(x_i,v_i)) \leq Ce^{-\alpha \lambda n} \leq Ce^{-\alpha \lambda t/2N}.
\]

\[\square\]

\section{5. Proof of the result}

We assume that the Lagrangian is regular, \( L(x,v,t) \geq 0 \) for any \((x,v,t)\) and \( \bar{A} = \{(x,v,t) : L(x,v,t) = 0\} \). We have to prove that there are \( \mu > 0 \) depending only on \( L \), and \( C > 0 \) depending on \( u \), such that for any \((x,\tau) \in \mathbb{T}^d \times [0,1], k \in \mathbb{N}\)
\[
-Ce^{-\mu k} \leq L_{\tau+k}u(x) - \bar{u}(x,[\tau]) \leq Ce^{-\mu k}.
\]
First we find constants \( \mu, C > 0 \) giving right inequality. For brevity, for \( x \in \mathbb{T}^d \) we write \( \bar{x} = (x,[0]) \), and for \( \beta : I \to \mathbb{T}^d \) we write \( \bar{\beta}(t) = (\bar{\beta}(t),[t]), d\bar{\beta}(t) = (\bar{\beta}(t),\bar{\beta}(t)) \), \( B(t) = (\bar{\beta}(t),\bar{\beta}(t),[t]) \).

For each \( z \in \mathbb{T}^{d+1} \) let \( y_z \) be such that
\[
\bar{u}(z) = u(y_z) + h(\bar{y}_z,z).
\]
In particular let \( y_j = y_{\bar{x}_j} \). Since \( z \to -h(z,\bar{x}) \in S^+ \), there is a semi-static curve \( \beta^j : [0,\infty[ \to \mathbb{T}^d \) such that \( \beta^j(0) = y_j \) and
\[
A_L(\beta^j|[0,t]) = h(\bar{y}_j,\bar{x}_j) - h(\bar{\beta}(t),\bar{x}_j), \quad t > 0.
\]

\begin{proposition}
Fix \( j \in [1,m] \). There are \( k_1, \ldots, k_j \) all different and semi-static curves \( \beta_r : \mathbb{R} \to \mathbb{T}^d, \ 0 \leq r < l, \ \beta_0 = \beta^j \), such that \( \Gamma_{k_{r+1}} \) is the \( \omega \) limit of \( B_r \) and if \( r > 0, \Gamma_{k_r} \) is the \( \alpha \) limit of \( B_r \),
\[
h(\bar{\beta}_0(t),\bar{x}_j) = h(\bar{\beta}_0(t),\bar{x}_{k_1}) + h(\bar{x}_{k_1},\bar{x}_j), \quad t \geq 0,
\]
\[
h(\bar{x}_{k_1},\bar{x}_j) = \sum_{r=1}^{l-1} h(\bar{x}_{k_r},\bar{x}_{k_{r+1}}),
\]
\[
A_L(\beta_0|[0,t]) = h(\bar{y}_j,\bar{x}_{k_1}) - h(\bar{\beta}_{0}(t),\bar{x}_{k_1}), \quad t \geq 0,
\]
\[
A_L(\beta_r|[t,s]) = h(\bar{x}_{k_r},\bar{x}_r) - h(\bar{x}_{k_r},\bar{x}_r), \quad t \leq s.
\]
\end{proposition}

\begin{proof}
Let \( \Gamma_{k_1} \) be the \( \omega \) limit of \( B_0 \). If \( k_1 = j \) we stop; otherwise we observe that by the regularity of \( L \)
\[
h(\bar{\beta}_0(t),\bar{x}_j) = h(\bar{\beta}_0(t),\bar{x}_{k_1}) + h(\bar{x}_{k_1},\bar{x}_j), \quad t \geq 0
\]
and then
\[
A_L(\beta_0|[0,t]) = h(\bar{y}_j,\bar{x}_{k_1}) - h(\bar{\beta}_{0}(t),\bar{x}_{k_1}), \quad t \geq 0.
\]
For each \( i \in [1,m] \) there is a neighborhood \( U'_i \) of \( \bar{\gamma}_i \) where \( h_i(z) = h(\bar{x}_i,z) \) is \( C^k \) and the local weak unstable manifold of \( \Gamma_i \) is the graph of \( H_p(x,d_xh_1(x,[t]),[t]) \).

Let $U_i$ be a neighborhood of $\bar{\gamma}_i$ with compact $\bar{U}_i \subset U'_i$. Let $\rho_n : [0, n] \to \mathbb{T}^d$ be a curve joining $x_{k_i}$ to $x_j$ such that

$$A_L(\rho_n) = F_{0,n}(x_i, x_j).$$

Let $t_n \in [0, n]$ be the first exit time of $\bar{\gamma}_n(t)$ out of $U_i$, and $\bar{\gamma}_n(t_n)$ be the first point of intersection with $\partial U_1$. As $n$ goes to infinity, $t_n$ and $n - t_n$ tend to infinity. This follows from the fact that $\rho_n(0)$ has to tend to $\bar{\gamma}_1(0)$, and $\rho_n(n)$ has to tend to $\bar{\gamma}_j(0)$. To justify this, consider $v$ a limit point of $\bar{\rho}_n(0)$, and $\gamma : \mathbb{R} \to \mathbb{T}^d$ the solution to the Euler-Lagrange equation such that $\gamma(0) = x_i$, $\dot{\gamma}(0) = v$. From the fact that

$$F_{0,n}(x_{k_1}, x_j) - F_{1,n}(\rho_n(1), x_j) = A_L(\rho_n|_{[0, 1]}),$$

and the regularity of $L$, taking limit $n \to \infty$ it follows

$$h(\bar{x}_{k_1}, \bar{x}_j) - h(\bar{\gamma}(1), \bar{x}_j) = A_L(\gamma|_{[0, 1]}).$$

Since $\gamma_k(1) = x_{k_1}$ and $L = 0$ on $\tilde{A}$

$$h(\bar{\gamma}_k(1), \bar{x}_j) - h(\bar{\gamma}(1), \bar{x}_j) = A_L(\gamma_k|_{[-1, 0]} + A_L(\gamma|_{[0, 1]}),$$

so that the curve obtained by gluing $\gamma_k|_{[-1, 0]}$ with $\gamma|_{[0, 1]}$ minimizes the action between its endpoints. In particular, it has to be differentiable, thus $v = \dot{\gamma}(0) = \dot{\gamma}_k(0)$. Define $\alpha_n : [-t_n, n - t_n] \to \mathbb{T}^d$ by $\alpha_n(t) = \rho_n(t + t_n)$ and let $(y, w, \tau)$ be a cluster point of $(d\rho_n(t_n), t_n - (t_n))$. Then there is a sequence $(\alpha_n)$ converging uniformly on compact intervals to the solution $\beta_1 : \mathbb{R} \to \mathbb{T}^d$ of the Euler-Lagrange equation such that $d\beta_1(\tau) = (y, w)$. Since for any $t \leq s$ we have

$$F_{-\tau_n,1}(x_k, \alpha_n(s)) - F_{-\tau_n,1,t}(\bar{x}_k, \alpha_n(t)) = A_L(\alpha_n|_{[t, s]}),$$

$$F_{-\tau_n,1,t}(x_k, \alpha_n(t)) + F_{1,\tau_n}(\alpha_n(t), x_j) = F_{0,\tau_n}(x_k, x_j),$$

from the uniform convergence of $F_{a,b}$ when $|b - a| \to \infty$, we obtain for any $t \leq s$

$$h(\bar{x}_k, \bar{\beta}_1(s)) - h(\bar{x}_k, \bar{\beta}_1(t)) = A_L(\beta_1|_{[t, s]}),$$

$$h(\bar{x}_k, \bar{\beta}_1(t)) + h(\bar{\beta}_1(t), \bar{x}_j) = h(\bar{x}_k, \bar{x}_j).$$

Since $\alpha_n([-t_n, n - t_n]) \subset \bar{U}_k$, we have that $\Gamma_k$ is the $\alpha$-limit of $B_1$ and let $\Gamma_k$ be its $\omega$-limit. If $k_2 = j$ we stop; otherwise we observe that

$$h(\bar{x}_{k_1}, \bar{x}_{k_2}) + h(\bar{x}_{k_2}, \bar{x}_j) = h(\bar{x}_{k_1}, \bar{x}_j),$$

$$h(\bar{x}_{k_1}, \bar{\beta}_1(t)) + h(\bar{\beta}_1(t), \bar{x}_{k_2}) = h(\bar{x}_{k_1}, \bar{x}_{k_2}).$$

Therefore $k_1 \neq k_2$. We proceed in the same way to find a solution to the Euler-Lagrange equation $\beta_2 : \mathbb{R} \to \mathbb{T}^d$ such that $\Gamma_k$ is the $\alpha$-limit of $B_2$ and for any $t, s \in \mathbb{R}$, $t < s$

$$h(\bar{x}_{k_2}, \bar{\beta}_2(s)) - h(\bar{x}_{k_2}, \bar{\beta}_2(t)) = A_L(\beta_2|_{[t, s]}),$$

$$h(\bar{x}_{k_2}, \bar{\beta}_2(t)) + h(\bar{\beta}_2(t), \bar{x}_j) = h(\bar{x}_{k_2}, \bar{x}_j).$$

Let $\Gamma_k$ be the $\omega$-limit of $B_2$. If $k_3 = j$ we stop; otherwise we observe that

$$h(\bar{x}_{k_2}, \bar{x}_j) + h(\bar{x}_{k_3}, \bar{x}_j) = h(\bar{x}_{k_2}, \bar{x}_j),$$

$$h(\bar{x}_{k_2}, \bar{\beta}_3(t)) + h(\bar{\beta}_3(t), \bar{x}_{k_3}) = h(\bar{x}_{k_2}, \bar{x}_{k_3}).$$

Therefore $k_2 \neq k_3$. Since

$$h(\bar{x}_{k_1}, \bar{x}_{k_2}) + h(\bar{x}_{k_2}, \bar{x}_{k_3}) = h(\bar{x}_{k_1}, \bar{x}_{k_3}),$$

we have that $k_1 \neq k_3$. We continue until we get $k_r = j$. \qed
Let $V$ be given by (4.1). There is $T > 0$ such that for $t \geq T$, we have $B_r(t), B_r(-t) \in V$ and then
\[
 d(d\beta_r(t), d\gamma_{k+1}(t)) \leq C_1 e^{-\lambda t}, \quad t > T, \\
 d(d\beta_r(t), d\gamma_k(t)) \leq C_1 e^{\lambda t}, \quad t < -T.
\]

**Proposition 5.2.** Given $(x, \tau) \in \mathbb{T}^d \times [0, 1]$, there are $j \in [1, m]$ and a semi-static curve $\beta_{x, \tau} : -\infty, \tau] \to \mathbb{T}^d$ such that $\beta_{x, \tau}(\tau) = x$, $\Gamma_j$ is the $\alpha$-limit of $B_{x, \tau}$ and
\[
 \bar{u}(x, [\tau]) = u(x, t) = h(\bar{x}_j, x, [\tau]), \\
 A_L(\beta_{x, \tau}[t, \tau]) = h(\bar{x}_j, x, [\tau]) - h(\bar{x}_j, \beta_{x, \tau}(t)), \quad t < \tau.
\]

**Proof.** Consider a directed graph with vertices at the points $\bar{x}_1, \ldots, \bar{x}_m$ of the Aubry set, and a directed edge from $\bar{x}_j$ to $\bar{x}_k$ if and only if
\[
 h(\bar{x}_k, x, [\tau]) = h(\bar{x}_k, \bar{x}_j) + h(\bar{x}_j, x, [\tau]).
\]

We call a point $\bar{x}_k$ a root of this graph if there is no edge arriving to this point, which means that for $j \neq k$
\[
 h(\bar{x}_k, x, [\tau]) < h(\bar{x}_k, \bar{x}_j) + h(\bar{x}_j, x, [\tau]).
\]

Notice that the graph contains no cycles, and so each point $\bar{x}_k$ belongs to a branch starting at a root. Take $k \in [1, m]$ such that
\[
 \bar{u}(x, [\tau]) = \bar{u}(\bar{x}_k) + h(\bar{x}_k, x, [\tau]).
\]

If there is an edge from $x_l$ to $x_k$, then
\[
 \bar{u}(x, [\tau]) \leq \bar{u}(\bar{x}_l) + h(\bar{x}_l, x, [\tau]) \leq \bar{u}(\bar{x}_k) + h(\bar{x}_k, \bar{x}_l) + h(\bar{x}_l, x, [\tau]) = \bar{u}(\bar{x}_k) + h(\bar{x}_k, x, [\tau]) = \bar{u}(x, [\tau]).
\]

Therefore, if $\bar{x}_j$ is the root of a branch containing $\bar{x}_k$ we have
\[
 \bar{u}(x, [\tau]) = \bar{u}(\bar{x}_j) + h(\bar{x}_j, x, [\tau]).
\]

Since $z \to h(\bar{x}_j, z) \in \mathcal{S}^-$, there is a semi-static curve $\beta_{x, \tau} : -\infty, \tau] \to \mathbb{T}^d$ with $\beta_{x, \tau}(\tau) = x$ such that
\[
 A_L(\beta_{x, \tau}[t, \tau]) = h(\bar{x}_j, x, [\tau]) - h(\bar{x}_j, \beta_{x, \tau}(t)), \quad t < \tau.
\]

Let $\Gamma_i$ be the $\alpha$-limit of $B_{x, \tau}$. If $(x, [\tau]) = \bar{\gamma}_j(\tau)$, then $\beta_{x, \tau} = \gamma_i = \gamma_j$. Otherwise, by the regularity of $L$
\[
 h(\bar{x}_j, x, [\tau]) = h(\bar{x}_j, \bar{x}_i) + h(\bar{x}_i, x, [\tau]).
\]

Since $\bar{x}_j$ is a root, $i = j$ and so $\Gamma_j$ is the $\alpha$-limit of $B_{x, \tau}$. \qed

From Propositions 5.1 and 5.2 for any $(x, [\tau]) \in \mathbb{T}^d \times [0, 1]$ there are $j \in [1, m]$ and a chain of semi-static curves $\beta_0, \ldots, \beta_{l-1}, \beta_{x, \tau}$ such that
\[
 \bar{u}(x, [\tau]) = u(y_j) + h(y_j, \bar{x}_j) + h(\bar{x}_j, (x, [\tau])) = u(y_j) + \sum_{r=0}^{l-1} A_L(\beta_r) + A_L(\beta_{x, \tau}).
\]

For $k$ large we next define a curve whose action approximates this sum of actions. There is $R > 0$ such that $B_{x, \tau}(t) \in V_j$ for $t < -R$ and then
\[
 d(d\beta_{x, \tau}(t), d\gamma_j(t)) \leq C_1 e^{\lambda t}.
\]
If $B_{x,r}$ stays during a long interval inside $V_i$, $i \neq j$, then number $R$ will be very large. However, according to Lemma 4.1 in a very long subinterval $[s, t]$ it is exponentially close to $\Gamma_i$, so we may jump from $B_{x,r}(s)$ to $B_{x,r}(t)$ in an interval shorter than 2. Let $K = K(V), N = N(V)$ be as in Lemmas 4.1, 4.4.

Consider the partition $t_0 = -R < s_1 < t_1 < \cdots < s_p \leq t_p$ of the interval $[-R, \tau]$ such that $B_{x,r}([s_h, t_h]) \subset V_i(h)$, $B_{x,r}([t_h, s_{h+1}]) \cap V = \emptyset$. From Lemma 4.1, we know that $\sum_{h=0}^{p-1} s_{h+1} - t_h$ is bounded by the constant $K > 0$.

For $k \in \mathbb{N}$, $k > 2(N + m + 1) + K$ let $n + 1 = \left( \frac{k - K}{2(p + l + 1)} \right)$. Set $h_0 = 0$, $\{h_1 < \cdots < h_q\} = \{h \in [1, p] : t_h - s_h > 2n + 1\}$, $a_r = \langle s_{h_r} \rangle, b_r = \langle t_{h_r} \rangle, a_{q+1} = 0$.

Define the curve $\alpha_k : [0, \tau + k] \to \mathbb{T}^d$ by

$$\alpha_k(s) = \begin{cases} \beta_0(s), & s \in [0, n], \\ c_r(s-r(2n+1)+n+1), & s \in [r(2n+1) - n - 1, r(2n+1) - n - 1], r \leq l, \\ \beta_x(s-r(2n+1)), & s \in [r(2n+1) - n, (r+1)(2n+1) - n - 1], \\ c_r(s + d_0 + 1), & s \in [(2n+1) - n - d_0 - 1], \\ \beta_x(r(2n+1) - 1 - n), & s \in [-d_0 - 1, -d_0], \\ c_{r+1}(s + d_r - r(2n+1) + 1), & s \in [-d_r + r(2n+1) - 1 - d_r + r(2n+1) - 1], \\ \beta_x(s - k), & s \in [k + b_q - n, k + \tau], \end{cases}$$

where $c_r : [0, 1] \to \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ is defined for $n$ large by

$$c_r(s) = \begin{cases} (1 - s)\beta_{r-1}(s + n) + s\beta_r(s - 1 - n), & r < l, \\ (1 - s)\beta_{r-1}(s + n) + s\gamma_j(s), & r = l, \\ (1 - s)\beta_x(s + a_h + n) + s\beta_x(s - 1 + b_h - n), & r = l + h. \end{cases}$$

Notice that if $(x, \tau) = \gamma_j(\tau)$, then $\alpha_k(s) = \gamma_j(s)$ for $s \geq l(2n + 1) - n$.

Since $L \geq 0$,

$$\begin{align*}
A_L(\alpha_k) &= A_L(\beta_0|_{[0, n]}) + \sum_{r=1}^{l-1} A_L(\beta_r|_{[-n, n]}) + \sum_{r=1}^{l+q} A_L(c_r) + A_L(c_{l+q}) \\
&+ A_L(\beta_x|_{[-n - R, \tau]}) - \sum_{r=1}^{q} A_L(\beta_x|_{[a_r + n, b_r - n]}) \\
&\leq h(\bar{y}_j, \bar{x}_j) + h(\bar{x}_j, (x, [\tau])) + \sum_{r=1}^{l+q} A_L(c_r) + A_L(c_{l+q}).
\end{align*}$$

Thus

$$\mathcal{L}_{\tau+k} u(x) - \bar{u}(x, [\tau]) \leq \sum_{r=1}^{l+q} A_L(c_r) + A_L(c_{l+q}).$$

Since $L = 0$ on $\tilde{A}$ and there is $C_2 \geq C_1$ such that

$$d(dc_r(s), d\gamma_{k_r}(s)), d(dc_1(s), d\gamma_j(s)), d(dc_{l+q}(s), d\gamma_{i(h_r)}(s)) \leq C_2 e^{-\lambda n},$$

we have

$$\mathcal{L}_{\tau+k} u(x) - \bar{u}(x, [\tau]) \leq C_3 e^{-\lambda n} \leq C_4 e^{-\lambda k/2(m + N + 1)}.$$

Now we prove that there are constants $C, \mu > 0$ giving left inequality in (5.1).
For $x \in \mathbb{T}^d$, $t > 0$ let $\gamma = \gamma_{x,t} : [0, t] \to \mathbb{T}^d$ be such that $\gamma(t) = x$ and
$$\mathcal{L}_t u(x) = u(\gamma(0)) + A_L(\gamma_t) = u(\gamma(0)) + F_{0,t}(\gamma(0), x, [t]).$$

For any integers $j \in [0, t]$, $i \in [1, m]$ we have
$$\bar{u}(x, [t]) \leq \bar{u}(\bar{x}_i) + h(\bar{x}_i, x, [t])$$
$$\leq u(\gamma(0)) + h(\gamma(0), x, \bar{x}_i) + h(\bar{x}_i, x, [t])$$
$$\leq u(\gamma(0)) + \Phi(\gamma(0), \gamma(j)) + h(\gamma(j), \bar{x}_i) + h(\bar{x}_i, \gamma(j)) + \Phi(\gamma(j), x, [t])$$
$$\leq u(\gamma(0)) + A_L(\gamma) + h(\gamma(j), \bar{x}_i) + h(\bar{x}_i, \gamma(j))$$
$$= \mathcal{L}_t u(x) + h(\gamma(j), \bar{x}_i) + h(\bar{x}_i, \gamma(j))$$
$$\leq \mathcal{L}_t u(x) + Kd(\gamma(j), x, [t]).$$

From Lemma 4.4 there are constants $C, \mu > 0$ such that, if $t > 2N$, there are integers $i \in [1, m]$, $j \in [0, t]$ such that
$$d(\gamma(j), x_i) \leq C e^{-\mu t}.$$

**REFERENCES**


