

## REGULARITY BOUNDS FOR KOSZUL CYCLES

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ABSTRACT. We study the Castelnuovo-Mumford regularity of the module of Koszul cycles  $Z_t(I, M)$  of a homogeneous ideal  $I$  in a polynomial ring  $S$  with respect to a graded module  $M$  in the homological position  $t \in \mathbb{N}$ . Under mild assumptions on the base field we prove that  $\text{reg } Z_t(I, S)$  is a subadditive function of  $t$  when  $\dim S/I = 0$ . For Borel-fixed ideals  $I, J$  we prove that  $\text{reg } Z_t(I, S/J) \leq t(1 + \text{reg } I) + \text{reg } S/J$ , a result already announced by Bruns, Conca and Römer.

### INTRODUCTION

Let  $S$  be a polynomial ring over a field  $K$ , say of characteristic 0 for simplicity. Let  $I \subset S$  be a homogeneous ideal of  $S$  and  $M$  a finitely generated graded module. Denote by  $\text{reg } M$  the Castelnuovo-Mumford regularity of  $M$ . Denote by  $K(I, M)$  the Koszul complex associated to a minimal system of generators of  $I$  with coefficients in  $M$ . Let  $Z_t(I, M)$  be the  $S$ -module of cycles of homological position  $t$  of  $K(I, M)$ . If there is no danger of confusion and both the ring  $S$  and the ideal  $I$  are clear from the context we simply set

$$Z_t = Z_t(I, S).$$

By construction  $Z_1$  is the first syzygy module of  $I$  and so by definition we have

$$\text{reg } Z_1 = 1 + \text{reg } I$$

unless  $I$  is principal (in that case  $Z_1 = 0$  and it has regularity  $-\infty$  by convention).

Our study of regularity bounds for the Koszul cycles and homology has its motivations and origin in the work of Green [G] who proved (among other things) a regularity bound for the Koszul homology of the powers of the maximal ideal in a polynomial ring. Green's result gives a bound for the degrees of the syzygies of the Veronese varieties. In [BCR1] and [BCR2] the authors made explicit the connection between the properties of the Koszul homology and cycles of the powers of the maximal ideal in a polynomial ring and the study of the so-called  $N_p$  property of the syzygies of the Veronese varieties. Green's regularity bounds for Koszul cycles and homology have been improved in [BCR1] and that led to an improvement of the understanding of the syzygies of Veronese varieties. In particular, generalizing results of [BCR1], in [BCR2, Prop.3.2] it is shown that

$$(1) \quad \text{reg } Z_t(I, M) \leq t(1 + \text{reg } I) + \text{reg } M$$

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holds for every  $t$  when  $\dim M/IM \leq 1$  and examples are given showing that (1) does not hold in general. It is also asked in [BCR2] whether the inequality

$$(2) \quad \operatorname{reg} Z_t \leq t(\operatorname{reg} I + 1)$$

does hold in general. In this paper we give examples showing that (2) does not hold in general, but we show that in two special cases variants of (1) and (2) do hold. In detail, we show that if  $\dim S/I = 0$ , then

$$(3) \quad \operatorname{reg} Z_{s+t} \leq \operatorname{reg} Z_s + \operatorname{reg} Z_t$$

holds for all  $s$  and  $t$ . And we also prove that

$$(4) \quad \operatorname{reg} Z_t(I, S/J) \leq t(\operatorname{reg} I + 1) + \operatorname{reg} S/J$$

holds whenever  $I$  and  $J$  are Borel-fixed ideals. This result was already announced in [BCR2, Thm.3.8].

### 1. GENERALITIES

In this section we collect notation and discuss general properties of Koszul complexes. As a general reference for information concerning the Koszul complex and homology the reader can consult for instance Bruns and Herzog [BH, Chap.1].

Let  $S = K[x_1, \dots, x_n]$  be a polynomial ring over a field  $K$ . The maximal homogeneous ideal  $(x_1, \dots, x_n)$  of  $S$  is denoted by  $\mathfrak{m}_S$  or just by  $\mathfrak{m}$ . Let  $I \subset S$  be an ideal minimally generated by homogeneous polynomials  $f_1, \dots, f_m$ . Denote by  $K(I, S)$  the Koszul complex associated to the  $S$ -linear map  $\phi : F = \bigoplus S(-\deg f_i) \rightarrow S$  defined by  $\phi(e_i) = f_i$ . Given a graded  $S$ -module  $M$  we set  $K(I, M) = K(I, S) \otimes M$ . We consider both  $K(I, S)$  and  $K(I, M)$  graded complexes with maps of degrees 0. We have decompositions  $K(I, S) = \bigoplus_{t=0}^m K_t(I, S) = \bigwedge^\bullet F$  and  $K(I, M) = \bigoplus_{t=0}^m K_t(I, M) = \bigwedge^\bullet F \otimes M$ . The complex  $K(I, M)$  can be seen as a graded module over the exterior algebra  $K(I, S)$ . For  $a \in K(I, S)$  and  $b \in K(I, M)$  the multiplication will be denoted by  $a.b$ . The differential of  $K(I, S)$  and  $K(I, M)$  will be denoted simply by  $\phi$  and it satisfies

$$\phi(a.b) = \phi(a).b + (-1)^s a.\phi(b)$$

for all  $a \in K_s(I, S)$  and  $b \in K(I, M)$ . We let  $Z_t(I, M)$ ,  $B_t(I, M)$ ,  $H_t(I, M)$  denote the cycles, the boundaries and the homology in homological position  $t$  and set  $Z(I, M) = \bigoplus Z_t(I, M)$  and so on. One knows that  $Z(I, S)$  is a (graded-commutative)  $S$ -subalgebra of  $K(I, S)$  and that  $B(I, R)$  is a homogeneous ideal of  $Z(I, S)$  so that the homology  $H(I, S)$  is itself a (graded-commutative)  $S$ -algebra. More generally,  $Z(I, M)$  is a  $Z(I, S)$ -module. We will denote by  $Z_s(I, S)Z_t(I, M)$  the image of the multiplication map  $Z_s(I, S) \otimes Z_t(I, M) \rightarrow Z_{s+t}(I, M)$ . Similarly,  $Z_1(I, S)^t$  will denote the image of the map  $\bigwedge^t Z_1(I, S) \rightarrow Z_t(I, S)$ .

By construction, Koszul cycles, boundaries and homology have an induced graded structure. An index on the left of a graded module always denotes the selection of the homogeneous component of that degree.

Denote by  $\{e_1, \dots, e_m\}$  the canonical basis of the free  $S$ -module  $F = \bigoplus S(-\deg f_i)$ , so that  $\deg e_i = \deg f_i$ . Given  $\mathbf{u} = \{u_1, \dots, u_s\} \subset [m]$  with  $u_1 < u_2 < \dots < u_s$  we write  $e_{\mathbf{u}}$  for the corresponding basis element  $e_{u_1} \wedge \dots \wedge e_{u_s}$  of  $\bigwedge^s F$ . Alternatively we use the symbol  $[f_{u_1}, \dots, f_{u_s}]$  to denote  $e_{\mathbf{u}}$  which is a homogeneous element of degree  $\sum_i \deg f_{u_i}$ .

Any element  $g \in \bigwedge^s F \otimes M$  can be written uniquely as  $g = \sum e_{\mathbf{u}} \otimes m_{\mathbf{u}}$  with  $m_{\mathbf{u}} \in M$  where the sum is over the subsets of cardinality  $s$  of  $[m]$ . If  $m_{\mathbf{u}} = 0$ , then we will say that  $e_{\mathbf{u}}$  does not appear in  $g$ . For every  $g \in K_{s+t}(I, M)$  and for every  $\mathbf{u} \subset [m]$  with  $s = \#\mathbf{u}$  we have a unique decomposition

$$(5) \quad g = a_{\mathbf{u}}(g) + e_{\mathbf{u}} \cdot b_{\mathbf{u}}(g)$$

with  $a_{\mathbf{u}}(g) \in K_{s+t}(I, M)$  and  $b_{\mathbf{u}}(g) \in K_t(I, M)$  provided we require that  $e_{\mathbf{j}}$  does not appear in  $a_{\mathbf{u}}(g)$  whenever  $\mathbf{j} \supset \mathbf{u}$  and that  $e_{\mathbf{v}}$  does not appear in  $b_{\mathbf{u}}(g)$  whenever  $\mathbf{v} \cap \mathbf{u} \neq \emptyset$ . With the notation above in [BCR2, Lemma 2.2 and 2.4] it is proved that:

**Lemma 1.1.**

- (1) If  $g \in Z_{s+t}(I, M)$ , then  $b_{\mathbf{u}}(g) \in Z_t(I, M)$  for every  $\mathbf{u}$  with  $s = \#\mathbf{u}$ .
- (2) The assignment

$$\beta_t(g) = \sum_{\mathbf{u}} e_{\mathbf{u}} \otimes b_{\mathbf{u}}(g)$$

where the sum is over the  $\mathbf{u} \subset [m]$  with  $\#\mathbf{u} = s$  gives a homomorphism

$$\beta_t : Z_{s+t}(I, M) \rightarrow Z_s(I, Z_t(I, M))$$

of  $S$ -modules.

- (3) The assignment

$$\alpha_t \left( \sum_i a_i \otimes g_i \right) = \sum_i a_i \cdot g_i$$

gives a homomorphism

$$\alpha_t : Z_s(I, Z_t(I, M)) \rightarrow Z_{s+t}(I, M)$$

of  $S$ -modules.

- (4) The composition  $\alpha_t \circ \beta_t$  is the multiplication by  $\binom{t+s}{s}$ . Hence  $Z_{s+t}(I, M)$  is a direct summand of  $Z_s(I, Z_t(I, M))$  as an  $S$ -module provided  $\binom{t+s}{s}$  is invertible in  $K$ .

An easy but interesting fact follows:

**Lemma 1.2.** Let  $s, t \in \mathbb{N}$ . With the notation introduced above one has

$$Z_s(I, Z_t(I, S)) = Z_t(I, Z_s(I, S))$$

where both sets are interpreted as subsets of  $\bigwedge^s F \otimes \bigwedge^t F$ .

*Proof.* Let

$$g = \sum_{\alpha, \beta} a_{\alpha, \beta} e_{\alpha} \otimes e_{\beta} \in \bigwedge^s F \otimes \bigwedge^t F$$

where  $a_{\alpha, \beta} \in S$  and  $\alpha$  varies in the set of subsets of cardinality  $s$  on  $[m]$  and  $\beta$  varies in the set of subsets of cardinality  $t$  on  $[m]$ . One has  $g \in Z_s(I, \bigwedge^t F)$  if and only if

$$\sum_{\alpha, \beta} a_{\alpha, \beta} \phi(e_{\alpha}) \otimes e_{\beta} = 0,$$

that is,

$$\sum_{\alpha} a_{\alpha, \beta} \phi(e_{\alpha}) = 0 \text{ for every } \beta,$$

equivalently,

$$(6) \quad \sum_{\alpha} a_{\alpha,\beta} e_{\alpha} \in Z_s(I, R) \text{ for every } \beta.$$

Furthermore,  $g \in \bigwedge^s F \otimes Z_t(I, S)$  if and only if

$$(7) \quad \sum_{\beta} a_{\alpha,\beta} e_{\beta} \in Z_t(I, S) \text{ for every } \alpha.$$

It follows that  $g \in Z_s(I, Z_t(I, S))$  if and only if (6) and (7) hold. Symmetrically,  $g \in Z_t(I, \bigwedge^s F)$  if and only if (7) holds and  $g \in Z_t(I, S) \otimes \bigwedge^t F$  if and only if (6) holds. □

*Remark 1.3.* As the proof shows, the statement of Lemma 1.2 holds for every Noetherian ring.

The following result allows us, when studying (1), to assume that the ideals we deal with have a linear resolution.

**Proposition 1.4.** *Let  $I$  be a homogeneous ideal and  $M$  a graded  $S$ -module. Let  $d = \text{reg } I$  and set  $J = (I_d)$  (so that  $\text{reg } J = d$ ). Then (1) holds for  $I$  and  $M$  and every  $i$  if it holds for  $J$  and  $M$  and every  $i$ .*

In order to prove Proposition 1.4 we need some auxiliary results. To this end we introduce a piece of notation. Given a sequence of homogeneous polynomials  $\mathbf{f} = f_1, \dots, f_m$  we will denote by  $K(\mathbf{f}, M)$  the Koszul complex associated to the sequence  $\mathbf{f}$  with coefficients in  $M$ . And we denote by  $Z(\mathbf{f}, M)$  the cycles and so on. Note that here we do not assume that the  $f_i$  are a minimal system of generators of the ideal they generate.

We have:

**Lemma 1.5.** *Let  $I = (\mathbf{f})$  and  $g_1, \dots, g_v \in I$ . Set  $\mathbf{g} = g_1, \dots, g_v$ . Then  $\text{reg } Z_i(\mathbf{f}, M) \leq \text{reg } Z_i(\mathbf{f}, \mathbf{g}, M)$ .*

*Proof.* By induction on  $v$ , it is enough to prove the statement for  $v = 1$ . The assertion is obvious since  $Z_i(\mathbf{f}, \mathbf{g}, M) \simeq Z_i(\mathbf{f}, M) \oplus Z_{i-1}(\mathbf{f}, M)(-\text{deg } g_1)$ . □

**Lemma 1.6.** *Let  $I = (\mathbf{f})$  and let  $a_1, \dots, a_v \in \mathbb{N}$ . Let  $g_i \in I : \mathfrak{m}_S^{a_i}$ . Set  $\mathbf{g} = g_1, \dots, g_v$ . Then*

$$\text{reg } Z_i(\mathbf{f}, \mathbf{g}, M) \leq \max\{\text{reg } Z_{i-\#D}(\mathbf{f}, M) + \sum_{j \in D} (a_j + \text{deg } g_j) : D \subseteq \{1, \dots, v\} \text{ and } \#D \leq i\}.$$

*Proof.* By induction on  $v$ , it is enough to prove the statement for  $v = 1$ . Set  $a = a_1$  and  $g = g_1$ . Let  $\alpha : Z_i(\mathbf{f}, \mathbf{g}, M) \rightarrow Z_{i-1}(\mathbf{f}, M)(-\text{deg } g)$  be the map defined by  $\alpha(h) = h_1$  where  $h = h_0 + e_{m+1}h_1$  with  $h_0 \in K_i(\mathbf{f}, M)$ . Set  $p = a + \text{deg } g + \text{reg } Z_{i-1}(\mathbf{f}, M)$ . We claim that  $\alpha$  is surjective in degrees  $\geq p$ . Let  $h_1 \in Z_{i-1}(\mathbf{f}, M)(-\text{deg } g)$  with  $\text{deg } h_1 \geq p$ . In other words,  $h_1$  has degree  $\geq a + \text{reg } Z_{i-1}(\mathbf{f}, M)$  as an element of  $Z_{i-1}(\mathbf{f}, M)$ . That is,  $h_1 \in \mathfrak{m}^a Z_{i-1}(\mathbf{f}, M)$ . Hence

$$gh_1 \in g\mathfrak{m}^a Z_{i-1}(\mathbf{f}, M) \subset IZ_{i-1}(\mathbf{f}, M) \subset B_{i-1}(\mathbf{f}, M).$$

Therefore it exists  $w \in K_i(\mathbf{f}, M)$  such that  $\phi(w) = gh_1$ . This implies that  $-w + e_{m+1}h_1 \in Z_i(\mathbf{f}, \mathbf{g}, M)$  and hence  $\alpha(-w + e_{m+1}h_1) = h_1$ . It follows that the complex

$$0 \rightarrow Z_i(\mathbf{f}, M) \rightarrow Z_i(\mathbf{f}, \mathbf{g}, M) \rightarrow Z_{i-1}(\mathbf{f}, M)(-\deg g) \rightarrow 0$$

is exact in degrees  $\geq p$ . We deduce that

$$\begin{aligned} \operatorname{reg} Z_i(\mathbf{f}, \mathbf{g}, M) &\leq \max\{p, \operatorname{reg} Z_i(\mathbf{f}, M), \operatorname{reg} Z_{i-1}(\mathbf{f}, M) + \deg g\} \\ &= \max\{p, \operatorname{reg} Z_i(\mathbf{f}, M)\}. \end{aligned}$$

□

We are now ready to prove Proposition 1.4:

*Proof of Proposition 1.4.* Choose a set  $G$  of minimal generators of  $I$  and let  $g_1, \dots, g_v$  be the elements in  $G$  of degree  $< d$ . Let  $f_1, \dots, f_m$  be a set of minimal generators of  $I_d$ . Set  $\mathbf{f} = f_1, \dots, f_m$ ,  $\mathbf{g} = g_1, \dots, g_v$ . Since the sequence  $\mathbf{f}, \mathbf{g}$  contains a minimal system of generators of  $I$  by Lemma 1.5 we have  $\operatorname{reg} Z_i(I, M) \leq \operatorname{reg} Z_i(\mathbf{f}, \mathbf{g}, M)$ . Since  $g_i \mathfrak{m}^{d-\deg g_i} \subset (I_d) = J$  we may use Lemma 1.6 and get

$$\operatorname{reg} Z_i(I, M) \leq \operatorname{reg} Z_i(\mathbf{f}, \mathbf{g}, M) \leq \max\{jd + \operatorname{reg} Z_{i-j}(J, M) : j \leq i\}.$$

But, by assumption,

$$\operatorname{reg} Z_{i-j}(J, M) \leq (i-j)(\operatorname{reg} J + 1) + \operatorname{reg} M = (i-j)(d+1) + \operatorname{reg} M.$$

It follows that

$$\operatorname{reg} Z_i(I, M) \leq \max\{jd + (i-j)(d+1) + \operatorname{reg} M : j \leq i\} = i(d+1) + \operatorname{reg} M.$$

□

## 2. EXAMPLES

We present in this section some examples of ideals which do not satisfy inequality (2). They are all defined by cubics, with a linear resolution and the failure of (2) comes from the fact that some boundaries are minimal generators of the module of second cycles. The Betti tables presented in Examples 2.1, 2.2 and 2.3 are computed with CoCoA [Co].

**Example 2.1.** Let  $I$  be the ideal of the minors of size 3 of a  $3 \times 5$  matrix  $X = (x_{ij})$  of variables and  $S = K[x_{ij}]$  so that  $\operatorname{reg} I = 3$ . The module  $Z_2 = Z_2(I, S)$  has 105 generators of degree 8 and 50 generators of degree 9. The generators of degree 9 are indeed boundaries (i.e. the homology is generated in degree 8). The Betti table of  $Z_2$  is

	0	1	2	3	4	5	6
8 :	105	90	21	—	—	—	—
9 :	50	225	420	420	240	75	10

So we have that  $\operatorname{reg} Z_2 = 9 > 2(\operatorname{reg} I + 1) = 8$ .

**Example 2.2.** Let  $J$  be the ideal of the leading terms of the ideal  $I$  of Example 2.1 with respect to a diagonal term order, i.e.  $J = (x_{1i_1}x_{2i_2}x_{3i_3} : 1 \leq i_1 < i_2 < i_3 \leq 5)$ .

Then  $\text{reg } J = 3$  and  $Z_2 = Z_2(J, S)$  has minimal generators of degree 9 that are boundaries (i.e. the homology is generated in degree  $\leq 8$ ). The Betti table of  $Z_2$  is

	0	1	2	3	4
7 :	3	-	-	-	-
8 :	102	101	42	12	2
9 :	6	21	27	15	3

So we have that  $\text{reg } Z_2 = 9 > 2(\text{reg } J + 1) = 8$ .

**Example 2.3.** Consider the ideals

$$J_1 = (x_1x_2x_3, x_1x_4x_6, x_3x_4x_5, x_4x_5x_6, x_1x_2x_6, x_1x_3x_4, x_2x_3x_5)$$

and

$$J_2 = (x_2x_3x_6, x_1x_2x_6, x_1x_3x_5, x_1x_4x_5, x_3x_5x_6, x_1x_2x_5, x_3x_4x_6).$$

They both have a linear resolution. The Betti tables of the corresponding  $Z_2(J_i, S)$  are, respectively,

	0	1	2	3
8 :	36	27	6	-
9 :	1	3	3	1

and

	0	1	2	3
7 :	2	-	-	-
8 :	30	21	4	-
9 :	1	3	3	1

so that  $\text{reg } Z_2(J_i, S) = 9 > 8 = 2(\text{reg } J_i + 1)$ . In both cases the generator of degree 9 of  $Z_2$  is a boundary, corresponding to the triplet  $\{x_2x_3x_5, x_1x_2x_6, x_4x_5x_6\}$  in the first case and  $\{x_3x_4x_6, x_1x_4x_5, x_1x_2x_6\}$  in the second.

### 3. THE 0-DIMENSIONAL CASE

The goal of this section is to prove the following.

**Theorem 3.1.** *Assume  $\dim S/I = 0$  and the characteristic of  $K$  is either 0 or  $> s + t$ . Then*

$$\text{reg } Z_{s+t} \leq \text{reg } Z_s + \text{reg } Z_t$$

*holds.*

Indeed we prove

**Proposition 3.2.** *Let  $S$  be a polynomial ring of characteristic 0 or  $> s + t$ . Assume that  $M$  is graded, finitely generated with  $\text{depth } M > 0$  and  $\dim S/I = 0$ . Then*

$$\text{reg } Z_{s+t}(I, M) \leq \text{reg } Z_t + \text{reg } Z_s(I, M).$$

*Proof.* First note that since, by Lemma 1.1, the module  $Z_{s+t}(I, M)$  is a direct summand of  $Z_t(I, Z_s(I, M))$ , we have

$$\text{reg } Z_{s+t}(I, M) \leq \text{reg } Z_t(I, Z_s(I, M)).$$

The canonical map  $f : Z_t \otimes Z_s(I, M) \rightarrow Z_t(I, Z_s(I, M))$  becomes an isomorphism when localized at a relevant homogeneous prime because  $\dim S/I = 0$ . Hence  $f$  has 0-dimensional kernel and cokernel. Since  $Z_t(I, Z_s(I, M))$  is a submodule of a direct

sum of copies of  $M$  it has positive depth. It follows that  $\text{reg } Z_t(I, Z_s(I, M)) \leq \text{reg } Z_t \otimes Z_s(I, M)$ . We observe that  $\text{Tor}_1^S(Z_t, N)$  has Krull dimension 0 for every  $S$ -module  $N$  because  $Z_t$  is free when localized at a relevant homogeneous prime. So we may apply [C, Cor.3.4] or [EHU, Cor.3.1] and get  $\text{reg } Z_t \otimes Z_s(I, M) \leq \text{reg } Z_t + \text{reg } Z_s(I, M)$ . This concludes the proof.  $\square$

Now Theorem 3.1 is a special case ( $M = S$ ) of Proposition 3.2. We may deduce from Theorem 3.1 the following corollary concerning the regularity of Koszul homology.

**Corollary 3.3.** *Let  $S$  be a polynomial ring of characteristic 0 or  $> s + t$ . Assume  $I$  is a homogeneous ideal with  $\dim S/I = 0$ . Set  $h_i = \text{reg } H_i(I, S)$ . We then have*

$$h_{s+t} \leq s + t + 1 + \text{reg } I + \max\{h_j - j : j < s\} + \max\{h_j - j : j < t\}.$$

*Proof.* Set  $z_i = \text{reg } Z_i(I, S)$  and  $b_i = \text{reg } B_i(I, S)$ . Since  $I$  has dimension 0 and annihilates  $H_i(I, S)$  we have  $h_i \leq z_i + \text{reg } I - 1$ . On the other hand, the standard short exact sequences relating Koszul cycles, boundaries and homologies, give

$$z_i = b_{i-1} + 1 \leq \max\{z_{i-1} + 1, h_{i-1} + 2\}.$$

Hence

$$z_i \leq 1 + i + \max\{h_j - j : j < i\}.$$

It follows that

$$\begin{aligned} h_{s+t} &\leq z_{s+t} + \text{reg } I - 1 \leq z_s + z_t + \text{reg } I - 1 \\ &\leq s + 1 + \max\{h_j - j : j < s\} + r + 1 + \max\{h_j - j : j < t\} + \text{reg } I - 1. \end{aligned}$$

$\square$

*Remark 3.4.* In the proof of Proposition 3.2 it is shown that for every  $s, t$  one has

$$\text{reg } Z_{s+t} \leq \text{reg } Z_s(I, Z_t) \leq \text{reg}(Z_s \otimes Z_t) \leq \text{reg } Z_s + \text{reg } Z_t$$

provided  $\dim S/I = 0$ . The three inequalities are, in general, not equalities. This already happens for ideals generated by regular sequences.

- (1) For  $s = t = 1$ ,  $S = \mathbb{Q}[a, b, c, d, e]$ , and  $I = (a^2, b^2, c^2, d^2, e^2)$  one has  $\text{reg } Z_1 = 7$  and

$$\text{reg } Z_2 = 8 < \text{reg } Z_1(I, Z_1) = 11 < \text{reg}(Z_1 \otimes Z_1) = 13 < \text{reg } Z_1 + \text{reg } Z_1 = 14.$$

- (2) If  $\dim S < 5$ , then  $\text{Tor}_1^S(Z_s, Z_t) = \text{Tor}_5^S(C_{s-1}, C_{t-1}) = 0$  where  $C_i$  is the cokernel of  $K(I, S)$  in position  $i$ . Hence the resolution of  $Z_s \otimes Z_t$  is the tensor product of the resolution of  $Z_s$  with that of  $Z_t$ . It follows that  $\text{reg}(Z_s \otimes Z_t) = \text{reg } Z_s + \text{reg } Z_t$ . The other two inequalities can be strict also for  $\dim S < 5$ . For instance, with  $I = (a^2, b^2, c^2) \subset \mathbb{Q}[a, b, c]$  one has  $\text{reg } Z_2 = 6, \text{reg } Z_1(I, Z_1) = 9, \text{reg}(Z_1 \otimes Z_1) = 2 \text{reg } Z_1 = 10$ .

#### 4. BOREL-FIXED IDEALS

In this section, we prove (4) for Borel-fixed ideals. Throughout this section, we assume that the characteristic of  $K$  is 0. Let  $\text{GL}_n(K)$  be the general linear group with coefficients in  $K$ . Any  $\varphi = (a_{ij}) \in \text{GL}_n(K)$  induces an automorphism of  $S$ , again denoted by  $\varphi$ ,

$$\varphi(f(x_1, \dots, x_n)) = f\left(\sum_{k=1}^n a_{k1}x_k, \dots, \sum_{k=1}^n a_{kn}x_k\right)$$

for any  $f \in S$ . A monomial ideal  $I \subset S$  is said to be *Borel-fixed* if  $\varphi(I) = I$  for any upper triangular matrix  $\varphi \in \text{GL}_n(K)$ . It is well known that a monomial ideal  $I \subset S$  is Borel-fixed if and only if, for any monomial  $fx_j \in I$  and for any  $i < j$ , one has  $fx_i \in I$ . For a monomial ideal  $I$ , we write  $G(I)$  for the set of minimal monomial generators of  $I$ .

From now on, we fix Borel-fixed ideals  $I$  and  $J$  with  $G(I) = \{f_1, f_2, \dots, f_m\}$  and consider the Koszul complex  $K(I, S/J) = \bigwedge^\bullet F \otimes S/J$ , where  $F = \bigoplus_{i=1}^m S(-\deg f_i)$ . Since Proposition 1.4 says that we may assume that  $I$  is generated in a single degree to prove (4), we assume  $\deg f_1 = \dots = \deg f_m$ .

Let  $\varphi \in \text{GL}_n(K)$  be an upper triangular matrix. Since  $\varphi(J) = J$ ,  $\varphi$  induces an automorphism of  $S/J$  defined by  $\varphi(h + J) = \varphi(h) + J$ . Also, for each  $f_i \in G(I)$ , since  $\phi(I) = I$  we can write  $\varphi(f_i) = \sum_{j=1}^m c_{ij}f_j$ , where  $c_{ij} \in K$ , in a unique way. We define  $\varphi(e_i) = \sum_{j=1}^m c_{ij}e_j$ , and define the  $K$ -linear map

$$\tilde{\varphi} : K_t(I, S/J) \rightarrow K_t(I, S/J)$$

by  $\tilde{\varphi}(e_{u_1} \wedge \dots \wedge e_{u_t} \otimes h) = \varphi(e_{u_1}) \wedge \dots \wedge \varphi(e_{u_t}) \otimes \varphi(h)$ . Then, it is clear that  $\tilde{\varphi} \circ \phi = \phi \circ \tilde{\varphi}$ , where  $\phi$  is the differential of  $K(I, S/J)$ . Thus we have

**Lemma 4.1.** *With the same notation as above,  $\tilde{\varphi}(Z_t(I, S/J)) \subseteq Z_t(I, S/J)$ .*

Note that  $\tilde{\varphi}$  is actually bijective and  $\tilde{\varphi}(Z_t(I, S/J)) = Z_t(I, S/J)$ . But we do not use this fact in the proof.

Next, we introduce a term order on  $\bigwedge^t F$ . We refer the readers to [CLO] for the basics on Gröbner basis theory for submodules of free modules. Let  $>_{\text{rev}}$  be the degree reverse lexicographic order induced by the ordering  $x_1 > \dots > x_n$ . We consider the ordering  $\succ$  for the basis elements of  $\bigwedge^t F$  defined by  $e_{i_1} \wedge \dots \wedge e_{i_t} \succ e_{j_1} \wedge \dots \wedge e_{j_t}$ , where  $i_1 < \dots < i_t$  and  $j_1 < \dots < j_t$ , if (i)  $f_{i_1} \dots f_{i_t} <_{\text{rev}} f_{j_1} \dots f_{j_t}$  or (ii)  $f_{i_1} \dots f_{i_t} = f_{j_1} \dots f_{j_t}$  and  $x_{i_1} \dots x_{i_t} >_{\text{rev}} x_{j_1} \dots x_{j_t}$ . Then we define the term order  $>$  on the free  $S$ -module  $\bigwedge^t F$  defined by  $e_{\mathbf{u}}v > e_{\mathbf{u}'}v'$ , where  $v$  and  $v'$  are monomials, if (i)  $e_{\mathbf{u}} \succ e_{\mathbf{u}'}$ , or (ii)  $e_{\mathbf{u}} = e_{\mathbf{u}'}$  and  $v >_{\text{rev}} v'$ . For  $g = \sum_{k=1}^t c_k e_{\mathbf{u}_k} v_k$ , where each  $c_k \in K \setminus \{0\}$  and each  $v_k$  is a monomial, let  $\text{in}_{>}(g) = \max_{>}\{v_1 e_{\mathbf{u}_1}, \dots, v_t e_{\mathbf{u}_t}\}$  be the initial term of  $g$  with respect to  $>$ , and for a submodule  $N \subset \bigwedge^t F$ , let  $\text{in}_{>}(N) = \langle \text{in}_{>}(g) : g \in N \rangle$  be the initial module of  $N$  with respect to  $>$ .

Since  $J$  is a monomial ideal, we can extend the above order  $>$  to the free  $S/J$ -module  $K_t(I, S/J) = \bigwedge^t F \otimes S/J$  in a natural way. Thus, we call an element  $v + J$  such that  $v$  is a monomial of  $S$  which is not in  $J$  a monomial of  $S/J$ , and extend the term order on  $S$  to  $S/J$  by identifying  $v + J$  and  $v$ . Since  $Z_t(I, S/J)$  is a submodule of  $K_t(I, S/J)$ , its initial module can be written as

$$(8) \quad \text{in}_{>}(Z_t(I, S/J)) = \bigoplus_{\mathbf{u} \subset [m], \#\mathbf{u}=t} e_{\mathbf{u}} \otimes (L_{\mathbf{u}}/J),$$

where  $L_{\mathbf{u}} \subset S$  is a monomial ideal which contains  $J$ .

**Lemma 4.2.** *The monomial ideal  $L_{\mathbf{u}}$  in (8) is Borel-fixed.*

*Proof.* Let  $vx_j \in L_{\mathbf{u}}$  be a monomial which is not in  $J$ . We prove that  $vx_i \in L_{\mathbf{u}}$  for any  $i < j$  with  $vx_i \notin J$ . Let  $\varphi \in \text{GL}_n(K)$  be a general upper triangular matrix and

let  $g \in Z_t(I, S/J)$  with  $\text{in}_>(g) = e_{\mathbf{u}} \otimes vx_j$ . Write

$$g = e_{\mathbf{u}} \otimes h + \sum_{e_{\mathbf{u}'} \prec e_{\mathbf{u}}} e_{\mathbf{u}'} \otimes h_{\mathbf{u}'},$$

where each  $h_{\mathbf{u}'} \in S/J$  and  $\text{in}_{>\text{rev}}(h) = vx_j$ . Then, since  $\varphi$  is upper triangular,  $\tilde{\varphi}(g)$  can be written as

$$\tilde{\varphi}(g) = e_{\mathbf{u}} \otimes c\varphi(h) + \sum_{e_{\mathbf{u}'} \prec e_{\mathbf{u}}} e_{\mathbf{u}'} \otimes h'_{\mathbf{u}'},$$

where  $c \in K \setminus \{0\}$  and each  $h'_{\mathbf{u}'} \in S/J$ . By Lemma 4.1,  $\tilde{\varphi}(g) \in Z_t(I, S/J)$ . Since  $Z_t(I, S/J)$  is  $\mathbb{Z}^n$ -graded, for each monomial  $w$  which appears in  $\varphi(h)$ , there is an element  $g_w \in Z_t(I, S/J)$  such that  $\text{in}_>(g_w) = e_{\mathbf{u}} \otimes w$ . On the other hand, since  $\varphi$  is general,  $vx_i$  appears in  $\varphi(h)$  for any  $i < j$  with  $vx_i \notin J$ . These facts prove the desired statement.  $\square$

**Lemma 4.3.**  $\text{in}_>(Z_t(I, S/J))$  is generated by monomials of degree  $\leq t(\text{reg } I + 1) + \text{reg } S/J$ .

*Proof.* We say that  $e_{\mathbf{u}} \otimes v \in K_t(I, S/J)$  divides  $e_{\mathbf{u}'} \otimes v' \in K_t(I, S/J)$  if  $e_{\mathbf{u}} = e_{\mathbf{u}'}$  and  $v$  divides  $v'$ . We prove the statement by induction on  $t$ . In this proof, we assume that all the elements are homogeneous with respect to the  $\mathbb{Z}^n$ -grading.

We first consider the case  $t = 1$ . For a monomial  $v \in S$ , we write  $\max(v)$  (resp.  $\min(v)$ ) for the maximal (resp. minimal) integer  $k$  such that  $x_k$  divides  $v$ . Let

$$A = \{[f_i] \otimes x_k - [f_i(x_k/x_{\max(f_i)})] \otimes x_{\max(f_i)} : i = 1, 2, \dots, m, k < \max(f_i)\} \\ \subset Z_1(I, S/J)$$

and

$$B = \{[f_i] \otimes v : i = 1, 2, \dots, m, v \in G(J : f_i)\} \subset Z_1(I, S/J).$$

Note that any element in  $A \cup B$  has degree  $\leq \text{reg } I + \text{reg } J = \text{reg } I + 1 + \text{reg } S/J$ . We claim that, for any  $g \in Z_1(I, S/J)$ ,  $\text{in}_>(g)$  is divisible by the initial term of an element in  $A \cup B$ .

Let  $g \in Z_1(I, S/J)$  with  $\text{in}_>(g) = [f_i] \otimes v$ . If  $\min(v) < \max(f_i)$ , it is clear that  $[f_i] \otimes v$  is divisible by the initial term of an element in  $A$ . Suppose  $\min(v) \geq \max(f_i)$ . Then, by the definition of the ordering  $\succ$ ,  $g$  itself is divisible by  $v$ . Then  $\text{in}_>(g/v) = [f_i] \otimes 1$ , which implies that  $g/v = [f_i] \otimes 1$  since  $g$  is homogeneous and  $f_i \in G(I)$ . Then, since  $\phi(g) = 0$ , we have  $vf_i \in J$ . This fact says that  $g$  is divisible by an element in  $B$ .

Now we consider the case  $t > 1$ . Suppose the statement holds for  $Z_{t-1}(I, S/J)$ . Let  $g \in Z_t(I, S/J)$  with  $\text{deg } g > t(\text{reg } I + 1) + \text{reg } S/J$ , and let  $\text{in}_>(g) = e_{u_1} \wedge \dots \wedge e_{u_t} \otimes v$ , where  $e_{u_1} \succ \dots \succ e_{u_t}$ . We will show that there is an element  $g' \in Z_t(I, S/J)$  with  $\text{deg } g' < \text{deg } g$  such that  $\text{in}_>(g')$  divides  $\text{in}_>(g)$ .

*Case 1.* Suppose  $\max(v) \geq \max(f_{u_1})$ . Let  $\ell = \max(v)$ . Then, by the definition of the ordering  $\succ$ ,  $\ell = \max(vf_{u_1} \dots f_{u_s})$  and the element  $g$  is divisible by  $x_\ell$ . We claim that  $g/x_\ell \in Z_t(I, S/J)$ .

Suppose  $g/x_\ell \notin Z_t(I, S/J)$ . Write  $\phi(g/x_\ell) = \sum_{k=1}^p e_{\mathbf{u}_k} \otimes c_k v_k$ , where  $c_k \in K \setminus \{0\}$  and  $v_k$  is a monomial in  $S$  which is not in  $J$ . Then  $x_\ell v_k \in J$  and  $\max(v_k) \leq \ell$  by the choice of  $\ell$ . Also, since  $\text{deg } g > t(\text{reg } I + 1) + \text{reg } S/J$ ,  $\text{deg } v_k > \text{reg } I + t + \text{reg } S/J \geq \text{reg } J$ . Hence there is a monomial  $w_k \in J$  which strictly divides  $x_\ell v_k$ . Since  $J$  is Borel-fixed and  $\max(v_k) \leq \ell$ , such a monomial  $w_k$  can be chosen so that  $w_k$  divides  $v_k$ , which contradicts  $v_k \notin J$ .

Case 2. Suppose  $\max(v) < \max(f_{u_1})$ . We write

$$g = a + e_1.b,$$

as in (5). Then  $b \in Z_{t-1}(I, S/J)$  and  $\text{in}_>(b) = e_{u_2} \wedge \cdots \wedge e_{u_s} \otimes v$ . By the induction hypothesis, there is an  $h \in Z_{t-1}(I, S/J)$  with  $\deg h \leq (t-1)(\text{reg } I + 1) + \text{reg } S/J$  such that  $\text{in}_>(h)$  divides  $\text{in}_>(b)$ . Let  $\text{in}_>(h) = e_{u_2} \wedge \cdots \wedge e_{u_s} \otimes \delta$ . Take an element  $x_i$  which divides  $v/\delta$ .

Observe that  $[f_{u_1}] \otimes x_i - [f_{u_1}(x_i/x_{\max(f_{u_1})})] \otimes x_{\max(f_{u_1})} \in Z_1(I, S/J)$  since  $i \leq \max(v) < \max(f_{u_1})$ . Then the element

$$g' = ([f_{u_1}] \otimes x_i - [f_{u_1}(x_i/x_{\max(f_{u_1})})] \otimes x_{\max(f_{u_1})}).h \in Z_1(I, S/J)Z_{t-1}(I, S/J) \subset Z_t(I, S/J)$$

satisfies the desired conditions. Indeed,  $\text{in}_>(g') = e_{u_1} \wedge \cdots \wedge e_{u_t} \otimes \delta x_i$  divides  $\text{in}_>(g)$  and

$\deg g' \leq \text{reg } I + 1 + (t-1)(\text{reg } I + 1) + \text{reg } S/J = t(\text{reg } I + 1) + \text{reg } S/J < \deg g$ , as desired. □

**Theorem 4.4.** *Let  $I$  and  $J$  be Borel-fixed ideals. Then*

$$\text{reg } Z_t(I, S/J) \leq t(\text{reg } I + 1) + \text{reg } S/J.$$

*Proof.* By Proposition 1.4, we may assume that  $I$  is generated in a single degree. Consider the decomposition (8) before Lemma 4.2. Then we have

$$(9) \quad \text{reg } Z_t(I, S/J) \leq \text{reg in}_>(Z_t(I, S/J)) = \max\{\text{reg } e_{\mathbf{u}} \otimes L_{\mathbf{u}}/J : \mathbf{u} \subset [m], \#\mathbf{u} = t\}.$$

On the other hand, by Lemma 4.3, each  $e_{\mathbf{u}} \otimes L_{\mathbf{u}}/J$  is generated by elements of degree  $\leq t(\text{reg } I + 1) + \text{reg } S/J$ . Thus  $L_{\mathbf{u}}$  is generated by monomials of degree  $\leq t + \text{reg } S/J$ . Since  $L_{\mathbf{u}}$  is Borel-fixed by Lemma 4.2, the main result of Eliahou and Kervaire [EK] implies that  $\text{reg } L_{\mathbf{u}} \leq t + \text{reg } S/J$ . Also the short exact sequence

$$0 \longrightarrow J \longrightarrow L_{\mathbf{u}} \longrightarrow L_{\mathbf{u}}/J \longrightarrow 0$$

shows  $\text{reg } L_{\mathbf{u}}/J \leq \max\{\text{reg } J - 1, \text{reg } L_{\mathbf{u}}\} \leq t + \text{reg } S/J$ . Then the desired statement follows from (9). □

From the above theorem, we get the next bound for the regularity of Koszul homology.

**Corollary 4.5.** *Let  $I$  and  $J$  be Borel-fixed ideals. Then*

$$\text{reg } H_t(I, S/J) \leq (t+1)(\text{reg } I + 1) + \text{reg } S/J - 2.$$

*Proof.* Let  $k_i = \text{reg } K_i(I, S/J)$ ,  $b_i = \text{reg } B_i(I, S/J)$ ,  $z_i = \text{reg } Z_i(I, S/J)$  and  $h_i = \text{reg } H_i(I, S/J)$ . Then,  $k_i \leq i \text{reg } I + \text{reg } S/J$  for all  $i$ . Also, the standard short exact sequences relating Koszul cycles, boundaries and homologies show that  $b_i \leq \max\{z_{i+1} - 1, k_{i+1}\}$  and  $h_i \leq \max\{b_i - 1, z_i\}$  for all  $i$ . Hence, by Theorem 4.4, we have

$\text{reg } H_t(I, S/J) = h_t \leq \max\{z_{t+1} - 2, k_{t+1} - 1, z_t\} \leq (t+1)(\text{reg } I + 1) + \text{reg } S/J - 2$ , as desired. □

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