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# SOLUTION OF PARAXIAL WAVE EQUATION FOR INHOMOGENEOUS MEDIA IN LINEAR AND QUADRATIC APPROXIMATION

ALEX MAHALOV AND SERGEI K. SUSLOV

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ABSTRACT. We construct explicit solutions of the inhomogeneous parabolic wave equation in a linear and quadratic approximation. As examples, oscillating laser beams in a $1D$ parabolic waveguide, spiral light beams in $2D$ varying media and an effect of superfocusing of particle beams in a thin monocrystal film are briefly discussed. Transformations of nonlinear equations into the corresponding autonomous and homogeneous forms are found and a review of important applications is also given.

## 1. INTRODUCTION

The inverse scattering method is a standard approach to several completely integrable nonlinear partial differential equations, such as the Korteweg-de Vries, nonlinear Schrödinger, Sine-Gordon and Kadomtsev–Petviashvili equations [1], [2], [60], [81]. At the same time, the physical situations in which these equations arise are usually highly idealized. The inclusion of damping, external forces, an inhomogeneous medium with variable density and a higher order of the nonlinearity may provide a more realistic model. Yet, the addition of these perturbation effects could mean that the system is no longer completely integrable (see an example in Refs. [1], [14], [35] and [70]). Hence it is of interest to determine under what conditions the perturbation preserves the integrability. Some of the nonintegrable systems in question possess important classes of exact solutions (see, for instance, [18], [20], [21], [19], [58], [55], [64] and the references therein). These solutions may serve as a starting point of perturbation methods and provide a useful testing ground for numerical investigation of more complicated models including stochastic differential equations [8], [28], [62], [72].

In this paper, we show that the following nonlinear PDE in the two-dimensional space-time continuum:

$$\frac{\partial \psi}{\partial t} + Q\left(\frac{\partial}{\partial x}, x\right)\psi = P\left(\psi, \psi^*, \frac{\partial \psi}{\partial x}, \frac{\partial \psi^*}{\partial x}\right), \tag{1.1}$$

where $Q$ is a quadratic of two (noncommuting) operators $\partial/\partial x$ and $x$ with time-dependent coefficients, the asterisk denotes the complex conjugation, and $P$ is a

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gauge invariant nonlinearity, can be reduced by a certain change of variables to the autonomous form:

$$\frac{\partial \chi}{\partial \tau} + \frac{\partial^2 \chi}{\partial \xi^2} = R\left(\chi, \chi^*, \frac{\partial \chi}{\partial \xi}, \left(\frac{\partial \chi}{\partial \xi}\right)^*\right), \tag{1.2}$$

which is somewhat easier to analyze by a variety of available tools. In general, the new time variable $\tau$ may be complex-valued and the new "spacial" variable $\xi$ is a linear complex-valued function of $x$ (the precise form of this transformation is established by Theorem 1). Our approach can also be used for nonlinear equations with some stochastic coefficients and random initial data. Among important applications are the nonlinear parabolic equation of diffraction theory, the derivative nonlinear Schrödinger equation, which describes the propagation of circular polarized nonlinear Alfvén waves in plasma physics, the quintic complex Ginzburg–Landau equation, and the Gerdjikov–Ivanov equation. (We mainly concentrate on variants of the nonlinear Schrödinger equation for which the unperturbed models are known to be completely integrable and/or have explicit solutions.)

The paper is organized as follows. In sections 2 to 4, we study the inhomogeneous linear parabolic equation of the theory of electromagnetic wave diffraction in varying media. In quadratic approximation with respect to the spacial variables perpendicular to the direction of the wave propagation and their gradient, we solve this equation in terms of the corresponding Fresnel integral. Explicit solutions are constructed from the given fields on the boundary by using the Green function of generalized harmonic oscillators extensively studied in quantum mechanics. Certain oscillating laser beams in a $1D$ parabolic waveguide and spiral beams in $2D$ varying media are briefly discussed as examples. In section 5, we transform equation (1.1) into the standard form (1.2). An overview of integrable autonomous cases and solutions of some nonintegrable PDE models are given. Section 6 deals with the $2D$ nonlinear Schrödinger equations. Applications to varying inhomogeneous media including fiber optics, ionospheric plasma, hydrodynamics, crystal film channeling, theory of open cavities, and Bose condensation will be discussed elsewhere.

## 2. Nonlinear parabolic equation

The propagation of electromagnetic waves in a medium, like ionospheric plasma, can be described by the phenomenological Maxwell equations [77]. The corresponding electric field satisfies

$$\nabla^2 \boldsymbol{E} - \frac{4\pi\mu}{c^2}\frac{\partial}{\partial t}\left(\widehat{\sigma}\boldsymbol{E}\right) - \frac{\mu}{c^2}\frac{\partial^2}{\partial t^2}\left(\widehat{\varepsilon}\boldsymbol{E}\right) = \nabla\left(\nabla\cdot\boldsymbol{E}\right) - \frac{\nabla\mu}{\mu}\times\left(\nabla\times\boldsymbol{E}\right), \tag{2.1}$$

where $\widehat{\sigma}$ and $\widehat{\varepsilon}$ are operators of the conductivity and of the electric permittivity of plasma [37]. In what follows, the magnetic permeability $\mu$ of the nonionized medium is equal to unity. (In general, this inhomogeneous equation is highly nonlinear since the tensors $\widehat{\sigma}$ and $\widehat{\varepsilon}$ depend on the field magnitude $|\boldsymbol{E}|$ and the spacial variable $\boldsymbol{r}$.) When the wave frequency $\omega$ is large enough and the time $1/\omega$ is much less than the characteristic time of establishing the average temperature of the electrons in plasma, one may use successive approximations in powers of the small parameter $\delta\nu_{\mathrm{e}}/\omega$. Assuming that the conductivity $\widehat{\sigma}$ and the electric permittivity $\widehat{\varepsilon}$ are time-independent in the first-order approximation, one obtains the familiar nonlinear wave equation [37], [77]:

$$\Delta \boldsymbol{E}^{(1)} - \operatorname{grad}\operatorname{div}\boldsymbol{E}^{(1)} + \frac{\omega^2}{c^2}\widehat{\epsilon}\boldsymbol{E}^{(1)} = 0, \qquad \widehat{\epsilon} = \widehat{\varepsilon}^{(1)} + 4\pi i\widehat{\sigma}^{(1)}/\omega. \tag{2.2}$$

In this approximation, $\Lambda\left(e^{-i\omega t}f(\boldsymbol{r})\right) = e^{-i\omega t}\Lambda f(\boldsymbol{r})$ for an operator $\Lambda = \left\{\widehat{\varepsilon}^{(1)}, \widehat{\sigma}^{(1)}\right\}$ and a wave arriving at the plasma boundary preserves its frequency $\omega$ [37].

Equation (2.2) can be significantly simplified when the properties of the medium vary slowly over the wave length and the paraxial approximation is used. In the ionosphere, this is valid at $\omega^2 \gg \omega_H^2 = (eH/mc)^2$ (Larmor frequency), when the complex dielectric tensor is a scalar and can be represented as follows:

$$\widehat{\epsilon} = \varepsilon_0(s) + \widehat{\varepsilon}_n\left(\boldsymbol{r}, i^{-1}\partial/\partial\boldsymbol{r}, \boldsymbol{E}\right) + i\varepsilon_1. \tag{2.3}$$

Here, $\varepsilon_0$ and $\varepsilon_1 = 4\pi\sigma_0/\omega$ are the real and imaginary parts of the dielectric constant of inhomogeneous medium, while the operator $\widehat{\varepsilon}_n$ depends on the physical mechanism of the nonlinearity [8], [37].

Upon assumption that the operator $\widehat{\epsilon}$ is not essentially altered over the wave length by the inhomogeneity of the plasma and by the nonlinear effect, the electric field can be written as

$$\boldsymbol{E}^{(1)} = \boldsymbol{e}E(\boldsymbol{r})\exp\left(i\int_0^s k(s')\ ds'\right), \qquad k = \frac{\omega}{c}\sqrt{\varepsilon_0(s)}, \tag{2.4}$$

where the field amplitude changes much slower than the phase. (Here, $k$ is the modulus of the wave vector in the linear approximation and $\boldsymbol{e}$ is the unit polarization vector, which is assumed to be a constant [30], [37]). As a result, the propagation of a wave of frequency $\omega$ in plasma, with self-action taken into account, is described by the nonlinear parabolic equation [37]:

$$\Delta_\perp E + 2ik\frac{\partial E}{\partial s} + iE\frac{dk}{ds} + \frac{\omega^2}{c^2}\left(\widehat{\varepsilon}_n + i\varepsilon_1\right)E = 0 \qquad \left(\left|\frac{\partial^2 E}{\partial s^2}\right| \ll \Delta_\perp E\right), \tag{2.5}$$

where $\Delta_\perp$ is the Laplace operator in the plane perpendicular to the beam propagation direction $s$ (see [8], [27], [37] for more details on the derivation of parabolic equation (2.5) from Maxwell's equations; see also [8], [62] for a generic case of turbulent medium and [6], [7], [30], [40], [41], [73], [78] and the references therein for applications of the parabolic equation in nonlinear optics, theory of open cavities and waveguides, plasma physics, acoustics and hydrodynamics). In the case of plane wave ($\Delta_\perp E = 0$) and in the absence of nonlinearity ($\widehat{\varepsilon}_n = 0$) or of inhomogeneity $(dk/ds = 0)$, the solution of parabolic equation is well-known [30], [37].

In this paper, we consider an important in practice case of plane-layered medium, when the properties of plasma may vary in one direction, say $s = z$ with $\widehat{\sigma} = Q\left(z, \boldsymbol{r}_\perp, i^{-1}\partial/\partial\boldsymbol{r}_\perp\right)$ and $\widehat{\varepsilon} = P\left(z, \boldsymbol{r}_\perp, i^{-1}\partial/\partial\boldsymbol{r}_\perp\right)$ being two quadratic forms of the variables $\boldsymbol{r}_\perp$ in perpendicular direction and its gradient $\nabla_\perp$ at any $z$ under consideration. The wave propagates in the $z$ direction and the boundary condition is given by $E|_{z=0} = E_0(\boldsymbol{r}_\perp)$. In the linear case, this wave propagation in geometrical optics is mathematically equivalent to the problem of two-dimensional generalized driven harmonic oscillator in quantum mechanics (see, for example, [27]). The intrinsic connection between Hamiltonian mechanics and the process of wave propagation is anything but a new idea [30], [78]. Yet, in view of this analogy between optics and quantum mechanics, the diffraction model under consideration is integrable in terms of the quadratures. The linear parabolic equation (2.5) can be thought of as the time-dependent Schrödinger equation. (The $z$ coordinate in the direction of the wave propagation plays a role of time with the perpendicular variables $\boldsymbol{r}_\perp$ being two spatial coordinates [27].)

The $2D$ wave equation in paraxial optics, which is of interest here, can be studied by separation of the variables in normal coordinates. The corresponding initial value problem for the Schrödinger equation, and therefore the boundary values problem for the original parabolic equation, can be solved with the help of the Green function/Feynman propagator or by eigenfunction expansion (see, for example, [22], [49], [50], [70] and the references therein regarding the $1D$ spatial case). Solution in terms of Airy functions [30] found in [53] are also generalized. The corresponding nearly singular solutions are related to strong scintillation regimes of wave propagation in inhomogeneous media. After discussion of these important examples from paraxial optics, explicit transformations of the nonlinear inhomogeneous parabolic equations into corresponding homogeneous forms will be analyzed.

## 3. $1D$ inhomogeneous media in quadratic approximation

In the linear case, when $\widehat{\varepsilon}_n$ is a quadratic function of spatial variables and of their gradient in the direction perpendicular to the wave propagation, equation (2.5) is integrable in quadratures (see, for example, [49] and the references therein). The average beam trajectory can be described by the corresponding Ehrenfest theorem thus stressing the analogy between geometrical optics and classical mechanics. Solution of the corresponding boundary value problem can be found in terms the Fresnel integral (or Green's function). In the context of quantum mechanics, the $1D$ linear Schrödinger equation, which describes the most general quadratic model of this kind,

$$i\psi_t = -a(t)\,\psi_{xx} + b(t)\,x^2\psi - ic(t)\,x\psi_x - id(t)\,\psi - f(t)\,x\psi + ig(t)\,\psi_x \tag{3.1}$$

($a$, $b$, $c$, $d$, $f$, and $g$ are suitable real-valued functions of time only), can be solved (formally) by the integral superposition principle:

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\,\psi(y,0)\;dy, \tag{3.2}$$

where

$$\begin{aligned} G(x,y,t) &= (2\pi\mu_0(t))^{-1/2} \\ &\times \exp\left[i\left(\alpha_0(t)\,x^2 + \beta_0(t)\,xy + \gamma_0(t)\,y^2 + \delta_0(t)\,x + \varepsilon_0(t)\,y + \kappa_0(t)\right)\right], \end{aligned} \tag{3.3}$$

for suitable initial data $\psi(x,0) = \varphi(x)$ (see [22], [70] and the references therein for more details). In paraxial optics, when the time variable $t$ represents the coordinate, say $z$, in the direction of wave propagation, expressions (3.2)–(3.3) are known as the Fresnel integral [3], [77].

The coefficients $\alpha_0$, $\beta_0$, $\gamma_0$, $\delta_0$, $\varepsilon_0$, $\kappa_0$ are given by [22], [70]:

$$\alpha_0(t) = \frac{1}{4a(t)}\frac{\mu_0'(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \tag{3.4}$$

$$\beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \qquad \lambda(t) = \exp\left(-\int_0^t (c(s) - 2d(s))\;ds\right), \tag{3.5}$$

$$\gamma_0(t) = \frac{1}{2\mu_1(0)}\frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)} \tag{3.6}$$

and

$$\delta_0(t)=\frac{\lambda(t)}{\mu_0(t)}\int_0^t\left[\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)\mu_0(s)+\frac{g(s)}{2a(s)}\mu_0'(s)\right]\frac{ds}{\lambda(s)}, \tag{3.7}$$

$$\begin{aligned}\varepsilon_0(t) &= -\frac{2a(t)\lambda(t)}{\mu_0'(t)}\delta_0(t)+8\int_0^t\frac{a(s)\sigma(s)\lambda(s)}{(\mu_0'(s))^2}(\mu_0(s)\delta_0(s))\,ds\\ &\quad +2\int_0^t\frac{a(s)\lambda(s)}{\mu_0'(s)}\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)ds,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\kappa_0(t) &= \frac{a(t)\mu_0(t)}{\mu_0'(t)}\delta_0^2(t)-4\int_0^t\frac{a(s)\sigma(s)}{(\mu_0'(s))^2}(\mu_0(s)\delta_0(s))^2\,ds\\ &\quad -2\int_0^t\frac{a(s)}{\mu_0'(s)}(\mu_0(s)\delta_0(s))\left(f(s)-\frac{d(s)}{a(s)}g(s)\right)ds\end{aligned} \tag{3.9}$$

($\delta_0(0)=-\varepsilon_0(0)=g(0)/(2a(0))$ and $\kappa_0(0)=0$) provided that $\mu_0$ and $\mu_1$ are the standard (real-valued) solutions of the characteristic equation:

$$\mu''-\tau(t)\mu'+4\sigma(t)\mu=0 \tag{3.10}$$

with varying coefficients

$$\tau(t)=\frac{a'}{a}-2c+4d,\qquad \sigma(t)=ab-cd+d^2+\frac{d}{2}\left(\frac{a'}{a}-\frac{d'}{d}\right), \tag{3.11}$$

subject to the initial conditions $\mu_0(0)=0$, $\mu_0'(0)=2a(0)\neq 0$ and $\mu_1(0)\neq 0$, $\mu_1'(0)=0$. Here, for applications to turbulent ionospheric plasma ([62], [72]), the integrals are treated in the most general way which includes stochastic calculus; see, for example, [28], [61].

An important particular solution (generalized Gaussian–Hermite beams in optics) is given by [49]:

$$\psi_n(x,t)=\frac{e^{i\left(\alpha x^2+\delta x+\kappa\right)+i(2n+1)\gamma}}{\sqrt{2^n n!\mu\sqrt{\pi}}}\ e^{-(\beta x+\varepsilon)^2/2}\ H_n(\beta x+\varepsilon), \tag{3.12}$$

where $H_n(x)$ are the Hermite polynomials [59]. Here,

$$\mu=\mu(0)\mu_0\sqrt{\beta^4(0)+4(\alpha(0)+\gamma_0)^2}, \tag{3.13}$$

$$\alpha=\alpha_0-\beta_0^2\frac{\alpha(0)+\gamma_0}{\beta^4(0)+4(\alpha(0)+\gamma_0)^2}, \tag{3.14}$$

$$\beta=-\frac{\beta(0)\beta_0}{\sqrt{\beta^4(0)+4(\alpha(0)+\gamma_0)^2}}=\frac{\beta(0)\mu(0)}{\mu(t)}\lambda(t), \tag{3.15}$$

$$\gamma=\gamma(0)-\frac{1}{2}\arctan\frac{\beta^2(0)}{2(\alpha(0)+\gamma_0)},\quad a(0)>0 \tag{3.16}$$

and

$$\delta=\delta_0-\beta_0\frac{\varepsilon(0)\beta^3(0)+2(\alpha(0)+\gamma_0)(\delta(0)+\varepsilon_0)}{\beta^4(0)+4(\alpha(0)+\gamma_0)^2}, \tag{3.17}$$

$$\varepsilon=\frac{2\varepsilon(0)(\alpha(0)+\gamma_0)-\beta(0)(\delta(0)+\varepsilon_0)}{\sqrt{\beta^4(0)+4(\alpha(0)+\gamma_0)^2}}, \tag{3.18}$$

$$\kappa=\kappa(0)+\kappa_0-\varepsilon(0)\beta^3(0)\frac{\delta(0)+\varepsilon_0}{\beta^4(0)+4(\alpha(0)+\gamma_0)^2} \tag{3.19}$$
$$+(\alpha(0)+\gamma_0)\frac{\varepsilon^2(0)\beta^2(0)-(\delta(0)+\varepsilon_0)^2}{\beta^4(0)+4(\alpha(0)+\gamma_0)^2}$$

in terms of the fundamental solution (3.4)–(3.9) subject to the arbitrary initial data $\mu(0)\neq 0,\ \alpha(0),\ \beta(0)\neq 0,\ \gamma(0),\ \delta(0),\ \varepsilon(0),\ \kappa(0)$. (Equations (3.14)–(3.19) solve the Ermakov-type system (5.7)–(5.12) below with $c_0=1$. The complex form of these solutions is found in [46].)

By the superposition principle, solutions (3.12) can be used for the corresponding eigenvalue expansions. In our approach, functions $f$ and $g$ are treated as two stochastic processes and equations (3.7)–(3.9) and (3.17)–(3.19) can be analyzed by statistical methods [5], [62] (which may include random initial data).

A solution in terms of Airy function [30] (generalized Airy beams) has the form:

$$\psi(x,t)=\frac{e^{i\left(\alpha x^2+\delta x+\kappa\right)-i\left(\beta x+\varepsilon-2\gamma^2/3\right)\gamma}}{\sqrt{\mu}}\ \mathrm{Ai}\left(\beta x+\varepsilon-\gamma^2\right), \tag{3.20}$$

where

$$\mu(t)=2\mu(0)\mu_0(t)(\alpha(0)+\gamma_0(t)), \tag{3.21}$$

$$\alpha(t)=\alpha_0(t)-\frac{\beta_0^2(t)}{4(\alpha(0)+\gamma_0(t))}, \tag{3.22}$$

$$\beta(t)=-\frac{\beta(0)\beta_0(t)}{2(\alpha(0)+\gamma_0(t))}=\frac{\beta(0)\mu(0)}{\mu(t)}\lambda(t), \tag{3.23}$$

$$\gamma(t)=\gamma(0)-\frac{\beta^2(0)}{4(\alpha(0)+\gamma_0(t))} \tag{3.24}$$

and

$$\delta(t)=\delta_0(t)-\frac{\beta_0(t)(\delta(0)+\varepsilon_0(t))}{2(\alpha(0)+\gamma_0(t))}, \tag{3.25}$$

$$\varepsilon(t)=\varepsilon(0)-\frac{\beta(0)(\delta(0)+\varepsilon_0(t))}{2(\alpha(0)+\gamma_0(t))}, \tag{3.26}$$

$$\kappa(t)=\kappa(0)+\kappa_0(t)-\frac{(\delta(0)+\varepsilon_0(t))^2}{4(\alpha(0)+\gamma_0(t))}. \tag{3.27}$$

(These equations solve the Riccati-type system (5.7)–(5.12) with $c_0=0$.) It is worth noting that our solution resembles the main features, in the linear approximation, of rogue waves [44], [53], [68]. Similar solutions can be obtained with the help of Airy beams found in [9], [66], and [67] (see [53] and the references therein). Once again, functions $f$ and $g$ may be treated as stochastic processes.

**Examples.** Use of (3.19)–(3.25) from [50] in equation (3.12) provides a new family of oscillating Gaussian–Hermite beams in $1D$ homogeneous medium (see also [4], [5], [77] and the next section for an extension to the $2D$ case). The "missing" solutions for the simple harmonic oscillator [51] provide new examples of oscillating Gaussian–Hermite beams in parabolic (self-focusing fiber) waveguides discussed in [4], [30], [52], [78]. Particular solutions in terms of Airy functions for the plane-layered medium can be obtained in analogy with [53] and [30]. All of these solutions are derived with the help of Theorem 1, where the simplest possible nonlinear terms are also analyzed.

## 4. $2D$ INHOMOGENEOUS MEDIA IN QUADRATIC APPROXIMATION

The Schrödinger (parabolic) equation is given by

$$i\psi_t(\boldsymbol{r},t) = H\psi(\boldsymbol{r},t), \qquad H = H_1(x) + H_2(y), \tag{4.1}$$

where $H_{1,2}$ are the Hamiltonians in $x$ and $y$ directions similar to one in (3.1) but, in general, with two different sets of suitable functions $a_{1,2}(t)$, $b_{1,2}(t)$, $c_{1,2}(t)$, $d_{1,2}(t)$, $f_{1,2}(t)$ and $g_{1,2}(t)$. (We assume that the nondiagonal terms are eliminated by passing to normal coordinates.) The Green function can be obtained as the product:

$$G(\boldsymbol{r},\boldsymbol{r}',t) = G_1(x,\xi,t)\,G_2(y,\eta,t), \tag{4.2}$$

where the kernels $G_{1,2}$ are given by (3.3) with a simple change of notation: The coefficients $\alpha_0^{(1,2)}(t)$, $\beta_0^{(1,2)}(t)$, $\gamma_0^{(1,2)}(t)$, $\delta_0^{(1,2)}(t)$, $\varepsilon_0^{(1,2)}(t)$, $\kappa_0^{(1,2)}(t)$ are defined, in general, in terms of two sets of the fundamental solutions (3.4)–(3.9).

Solution of the corresponding boundary-values problem can be found by the integral superposition principle (or generalized Fresnel integral):

$$\psi(\boldsymbol{r},t) = \int_{\boldsymbol{R}^2} G(\boldsymbol{r},\boldsymbol{r}',t)\,\psi(\boldsymbol{r}',t=0)\ d\boldsymbol{r}' \tag{4.3}$$

for suitable initial data.

The $2D$ generalized Gaussian–Hermite beams have the form

$$\begin{aligned}\psi_{nm}(\boldsymbol{r},t) = {} & \frac{e^{i(\kappa_1+\kappa_2)}}{\sqrt{2^{n+m}n!m!\mu^{(1)}\mu^{(2)}\pi}} \\ & \times e^{i\left(\alpha_1x^2+\delta_1x\right)+i(2n+1)\gamma_1}\ e^{i\left(\alpha_2y^2+\delta_2y\right)+i(2m+1)\gamma_2} \\ & \times e^{-(\beta_1x+\varepsilon_1)^2/2-(\beta_2y+\varepsilon_2)^2/2}\ \ H_n(\beta_1x+\varepsilon_1)\,H_m(\beta_2y+\varepsilon_2).\end{aligned} \tag{4.4}$$

Equations (3.13)–(3.19) are valid with a similar change of notation for given initial data $\mu^{(1,2)}(0)$, $\alpha_{1,2}(0)$, $\beta_{1,2}(0)\neq 0$, $\gamma_{1,2}(0)$, $\delta_{1,2}(0)$, $\varepsilon_{1,2}(0)$, $\kappa_{1,2}(0)$ (see also [4], [5], [77] for important special cases). In general, the product of any two $1D$ solutions (3.12) and (3.20), say

$$\psi_n(\boldsymbol{r},t) = \psi_n(x,t)\,\psi(y,t), \tag{4.5}$$

gives an important class of $2D$ solutions (Airy–Gaussian–Hermite beams in inhomogeneous media).

**Example: Spiral Laser Beams in Varying Media.** In the inhomogeneous $2D$ case under consideration, equation (4.1) can be reduced to the standard forms

$$-i\chi_\tau + \chi_{\xi\xi} + \chi_{\eta\eta} = c_0\left(\xi^2+\eta^2\right)\chi \qquad (c_0 = 0,1) \tag{4.6}$$

(see Lemma 1 with $h_0 = 0$; we have assumed that $a_1\beta_1^2 = a_2\beta_2^2$ [23]). By the Ansatz $\Psi(X,Y,T) = \chi(\xi,\eta,\tau)$, $T = -\tau$ and

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\omega\tau & -\sin\omega\tau \\ \sin\omega\tau & \cos\omega\tau \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad \omega = \text{constant}, \tag{4.7}$$

equation (4.6) with $c_0 = 1$ can be transformed to an equation of motion for the isotropic planar harmonic oscillator in a perpendicular uniform magnetic field:

$$i\Psi_T + \Psi_{XX} + \Psi_{YY} = i\omega\left(X\Psi_Y - Y\Psi_X\right) + \left(X^2+Y^2\right)\Psi. \tag{4.8}$$

The latter was solved in polar coordinates $X = R\cos\Theta$, $Y = R\sin\Theta$ (as an example of a simple degenerated unperturbed system) by Fock in the early days of quantum mechanics [29]:

$$\Psi(R,\Theta,T) = \sqrt{\frac{n!}{\pi(n+|m|)!}}\ e^{-iET}e^{im\Theta}R^{|m|}e^{-R^2/2}L_n^{|m|}\left(R^2\right), \tag{4.9}$$
$$E = 4n + 2(|m|+1) - m\omega \qquad (m = \pm 0, \pm 1, \dots\ n = 0, 1, \dots)$$

in terms of Laguerre polynomials [59]. (This wave function coincides, up to a simple factor, with the one for a flat isotropic oscillator without magnetic field. Therefore, its development in terms of (4.4) for standard harmonics is a $2D$ special case of the multidimensional expansions from [59].) By back substitution, one arrives at a general family of rotating solutions. Numerous examples of the spiral laser beams in the uniform medium are discussed in [3] (see also [4], [5], [77]). Similar to [50], one gets a multi-parameter family of solutions in a $2D$ medium [52].

A similar effect of the superfocusing of a proton beam in a thin monocrystal film was discussed in [25], [26] (the validity of the $2D$ harmonic crystal model had been confirmed through Monte Carlo computer experiments).

## 5. Transformation of related $1D$ nonlinear equations

In this section, we complexify all time-dependent coefficients of the linear equation (3.1) and analyze the simplest nonautonomous nonlinear terms. For a generic nonautonomous (derivative) nonlinear Schrödinger equation of the form

$$i\psi_t = H\psi + R(\psi), \tag{5.1}$$

where $H$ is an arbitrary varying quadratic Hamiltonian and $R(\psi) = P(\psi,\psi^*,\psi_x,\psi_x^*)$ is a polynomial in four variables, we assume the natural gauge invariance condition:

$$P\left(\psi e^{iS}, \psi^* e^{-iS}, \left(\psi e^{iS}\right)_x, \left(\psi^* e^{-iS}\right)_x\right) = Ce^{iS}P(\psi,\psi^*,\psi_x,\psi_x^*) \tag{5.2}$$

(also known as $U(1)$, or phase invariance). The lowest terms that satisfy this condition are given by

$$P(\psi,\psi^*,\psi_x,\psi_x^*) = h_0\psi + (h_1x+h_2)|\psi|^2\psi + ih_3|\psi|^2\psi_x + ih_4\psi^2\psi_x^* + h_5|\psi|^4\psi, \tag{5.3}$$

where $h_k = h_k(x,t)$, $k = 0, \dots, 5$ are some real or complex-valued functions. Our result is the following.

**Theorem 1.** *The substitution*

$$\psi = \frac{e^{i\left(\alpha(t)x^2+\delta(t)x+\kappa(t)\right)}}{\sqrt{\mu(t)}}\ \chi(\xi,\tau), \qquad \xi = \beta(t)x+\varepsilon(t), \quad \tau = \gamma(t) \tag{5.4}$$

*transforms the nonautonomous and inhomogeneous equation* (5.1) *with the lowest gauge invariant nonlinearities* (5.3), *namely,*

$$\begin{aligned} i\psi_t + a(t)\psi_{xx} - b(t)x^2\psi + ic(t)x\psi_x + id(t)\psi + f(t)x\psi - ig(t)\psi_x \\ = h_0\psi + (h_1 x + h_2)|\psi|^2\psi + ih_3|\psi|^2\psi_x + ih_4\psi^2\psi_x^* + h_5|\psi|^4\psi, \end{aligned} \tag{5.5}$$

*into the autonomous form*

$$\begin{aligned} -i\chi_\tau + \chi_{\xi\xi} - c_0\xi^2\chi \\ = d_0\chi + (d_1\xi + d_2)|\chi|^2\chi + id_3|\chi|^2\chi_\xi + id_4\chi^2(\chi_\xi)^* + d_5|\chi|^4\chi \end{aligned} \tag{5.6}$$

$(c_0 = 0, 1)$ *provided that*

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = c_0 a\beta^4, \tag{5.7}$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0, \tag{5.8}$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0 \tag{5.9}$$

*and*

$$\frac{d\delta}{dt} + (c + 4a\alpha)\delta = f + 2g\alpha + 2c_0 a\beta^3\varepsilon, \tag{5.10}$$

$$\frac{d\varepsilon}{dt} = (g - 2a\delta)\beta, \tag{5.11}$$

$$\frac{d\kappa}{dt} = g\delta - a\delta^2 + c_0 a\beta^2\varepsilon^2. \tag{5.12}$$

*Here*

$$\alpha = \frac{1}{4a}\frac{\mu'}{\mu} - \frac{d}{2a} \tag{5.13}$$

*and*

$$h_1 = a\beta^2|\mu|\left(d_1\beta + \frac{2\alpha}{\beta}d_3 - \frac{2\alpha^*}{\beta^*}d_4\right)e^{2\operatorname{Im}S}, \tag{5.14}$$

$$h_2 = a\beta^2|\mu|\left[d_1\varepsilon + d_2 + \frac{\delta}{\beta}d_3 - \frac{\delta^*}{\beta^*}d_4\right]e^{2\operatorname{Im}S}, \tag{5.15}$$

$$h_3 = d_3 a\beta|\mu|e^{2\operatorname{Im}S}, \qquad h_4 = d_4 a\frac{\beta^2}{\beta^*}|\mu|e^{2\operatorname{Im}S}, \tag{5.16}$$

$$h_5 = d_5 a\beta^2|\mu|^2 e^{4\operatorname{Im}S}, \qquad h_0 = d_0 a\beta^2 \tag{5.17}$$

($d_0$, $d_1$, $d_2$, $d_3$, $d_4$, $d_5$ *are constants and* $S = \alpha x^2 + \delta x + \kappa$).

*Proof.* For the case of complex-valued coefficients, the transformation of the linear part is similar to [49] and [70] (with the use of contour integration if needed).

Changing the variables in the nonlinear part,

$$\mu^{1/2}e^{-iS}P = h_0\chi + ih_3\frac{\beta}{|\mu|}e^{-2\,\mathrm{Im}\,S}|\chi|^2\chi_\xi + ih_4\frac{\beta^*}{|\mu|}e^{-2\,\mathrm{Im}\,S}\chi^2(\chi_\xi)^* + \frac{h_5}{|\mu|^2}e^{-4\,\mathrm{Im}\,S}|\chi|^4\chi + \left[h_2 - h_3\delta + h_4\delta^* + (h_1 - 2\alpha h_3 + 2\alpha^* h_4)\frac{\xi-\varepsilon}{\beta}\right]e^{-2\,\mathrm{Im}\,S}\frac{|\chi|^2}{|\mu|}\chi \tag{5.18}$$

and substituting into (5.5), one completes the proof with the aid of our conditions (5.14)–(5.17). □

*Remark* 1. The systems (5.7)–(5.12) can be solved with the help of variants of a nonlinear superposition principle established in [49] and [69]; see equations (3.22)–(3.27) and (3.14)–(3.19) for $c_0 = 0, 1$, respectively.

The autonomous cubic nonlinear Schrödinger equation is completely integrable by the inverse scattering method [1], [2], [60], [81]. When $c_0 = d_0 = d_1 = d_3 = d_4 = d_5 = 0$, $d_2 = h_0 \neq 0$ and $d_0 = d_1 = d_2 = d_3 = d_4 = 0$, $d_5 = h_0 \neq 0$, we reproduce the results of [69] and [70] (for $p = 2$ and $p = 4$, respectively), where the nonautonomous integrability condition was established (see also the references therein). The derivative nonlinear Schrödinger equation, which describes the propagation of circular polarized nonlinear Alfvén waves in plasma physics [56], [57], [79], arises when $c_0 = d_0 = d_1 = d_2 = d_5 = 0$ and $d_3 = 2d_4$ (see, for example, [7], [8], [10], [17] and the references therein for methods of solution of this equation; our theorem identifies the corresponding nonautonomous integrable model). The amplitude equations of cubic-quintic type have been derived via asymptotic analysis of the governing equations of fluid mechanics near the onset of instability [40], [44]. Among other special cases are the Chen–Lee–Lui derivative nonlinear Schrödinger equation [10] and the Gerdjikov–Ivanov equation. Generic autonomous equations of the type (5.6) and some of their extensions such as quintic complex Ginzburg–Landau equation are discussed, for example, in [14], [15], [17], [20], [21], [31], [32], [33], [34], [35], [40], [43], [47], [55], [63], [64], [80] (see also the references therein; an extensive bibliography will be given elsewhere).

In the case $c_0 = d_0 = d_1 = 0$, a detailed Painlevé analysis of equation (5.6) is performed by Clarkson and Cosgrove [17] (see also [15] for the extension to the case of complex parameters). They have shown that this equation possesses the Painlevé property for partial differential equations only if $d_5 = d_4(2d_4 - d_3)/4$. When this relation holds, the latter is equivalent under a gauge transformation [43], [47]:

$$\chi = \phi \exp\left(-i\nu\int_{\xi_0}^{\xi}|\phi|^2\ d\eta\right), \tag{5.19}$$

to a hybrid of the nonlinear Schrödinger equation and the derivative nonlinear Schrödinger equation [7], [8],

$$-i\phi_\tau + \phi_{\xi\xi} + \lambda|\phi|^2\phi + i\mu\left(|\phi|^2\phi\right)_\xi = 0 \tag{5.20}$$

($\lambda$, $\mu$ and $\nu$ are constants), where one can assume that $\lambda = 0$ without loss of generality [80]. The corresponding derivative nonlinear Schrödinger equation is known to be completely integrable [2], [10] (a detailed bibliography will be provided elsewhere). Then solutions of the original equation are constructed by the gauge transformation (5.19) in principle (see, for example, Refs. [17], [15] and [43] for more details). Explicit solitary wave solutions can be found in [43], [55], [64] (review also the references therein). Theorem 1 allows us to extend these results to a larger class of nonautonomous and inhomogeneous nonlinear Schrödinger equations.

Exact solitary wave solutions of the one-dimensional quintic complex Ginzburg–Landau equation are obtained in [55], [64]. In their notation and terminology,

$$\frac{\partial A}{\partial t} = \varepsilon A + (b_1 + ic_1) \frac{\partial^2 A}{\partial x^2} - (b_3 - ic_3) |A|^2 A - (b_5 - ic_5) |A|^4 A, \tag{5.21}$$

where $\varepsilon$, $b_1$, $c_1$, $b_3$, $c_3$, $b_3$, $c_3$ are real constants and the field $A(x,t)$ is complex-valued (see also [13], [15]). These solutions are expressed in terms of hyperbolic functions and include coherent structures with a strong spatial localization such as pulses and fronts, as well as sources and sinks. Equation (5.21) is a one-dimensional model of the large-scale behavior of many nonequilibrium pattern-forming systems (see, for example, [55], [63], [64] and the references therein). A systematic method for obtaining analytic solitary wave solutions of nonintegrable PDEs has been introduced by Conte and Musette [19], [58] and further developed by Hone [38], [39] and Vernov [75], [76] (see also [15] for another approach and [20], [21], [31], [34] for exact solutions). The unique elliptic traveling wave solution of (5.21) is found in [76]. One may extend some of these results to nonautonomous equations.

If conditions (5.14)–(5.17) are not satisfied in a certain application, yet we can find the corresponding coefficients $d_k$ as functions of time and spatial variables thus moving the time dependence into the nonlinear part only. Then one may use perturbation methods and/or some parameter control when possible. The Feshbach resonance in Bose–Einstein condensation provides a classical example of such nonlinearity control [24], [42]. For instance, a justification of the so-called local density approximation [45] can be obtained from Theorem 1 with $c_0 = 0$ and $p = 2, 4$, when a classical motion of the corresponding quadratic system is already taken into account, under the following adiabatic condition:

$$\frac{d}{dt}\left(\frac{h}{a(t)\,\beta^2(t)\,\mu^{p/2}(t)}\right) = \mathrm{o}(1)\ (\text{or } \ll 1). \tag{5.22}$$

It is worth noting that Theorem 1 can be used for suitable stochastic process coefficients $f$ and $g$ with the corresponding random initial data in (3.7)–(3.9) and (3.17)–(3.19). Ansatz (5.4) transforms a stochastic field PDE (5.5) into the deterministic one with respect to stochastic spacial variable $\xi$. Then a stochastic process solution of the Ermakov-type system (5.7)–(5.12) with the proper averages can be found (see also [5], [8], [62] for an advanced study).

Last but not least, in several critical autonomous cases, Theorem 1 allows us to derive the maximum symmetry groups and the similarity transformations for the corresponding equations (see, for example, [50], [54] and the references therein). These symmetries result in explicit blow up solutions for those critical cases and variations of initial data can be used for a nonlinear stability analysis.

## 6. 2D inhomogeneous nonlinear Schrödinger equation

In a similar fashion, we prove the following.

**Lemma 1.** *The inhomogeneous nonlinear Schrödinger equation*

$$
\begin{aligned}
(6.1)\qquad i\psi_t &= -a\left(\psi_{xx}+\psi_{yy}\right)+b\left(x^2+y^2\right)\psi-ic\left(x\psi_x+y\psi_y\right)-2id\psi \\
&\quad -\left(xf_1+yf_2\right)\psi+i\left(g_1\psi_x+g_2\psi_y\right)+h\left|\psi\right|^p\psi,
\end{aligned}
$$

*where $a$, $b$, $c$, $d$, $f_{1,2}$ and $g_{1,2}$ are real-valued functions of $t$, can be reduced to the autonomous form*

$$
(6.2)\qquad -i\chi_\tau+\chi_{\xi\xi}+\chi_{\eta\eta}=c_0\left(\xi^2+\eta^2\right)\chi+h_0\left|\chi\right|^p\chi \qquad (c_0=0,1)
$$

*by the following Ansatz:*

$$
(6.3)\qquad \psi=\mu^{-1}e^{i\left(\alpha\left(x^2+y^2\right)+\left(\delta_1x+\delta_2y\right)+\kappa_1+\kappa_2\right)}\ \chi\left(\xi,\eta,\tau\right),
$$

*where $\xi=\beta x+\varepsilon_1$, $\eta=\beta y+\varepsilon_2$, $\tau=\gamma$, and $h=h_0a\beta^2\mu^p$ ($h_0$ is a constant). Here, solutions of the system* (5.7)–(5.12) *are given by* (3.21)–(3.27) *and* (3.13)–(3.19) *for $c_0=0$ and $c_0=1$, respectively.*

In the $1D$ linear case, where nonspreading Airy beams were introduced [9] (see also [66], [67]), the symmetry of free Schrödinger equation can be used in order to obtain new exact solutions [53]. Although the corresponding $1D$ cubic nonlinear Schrödinger equation is no longer preserved under the expansion transformation (but has a similarity reduction to the second Painlevé equation [34], [36], [53], [68], [71]), the same symmetry holds for the quintic nonlinear Schrödinger equation, which is thus invariant under the action of this group. Here, the blow up, namely a singularity such that the wave amplitude tends to infinity in a finite time, occurs (see [54], [70], [74] and the references therein).

As is well known, a similar symmetry holds for the homogeneous $2D$ cubic nonlinear Schrödinger equation [48], [73] (in optics this symmetry is known as Talanov's transformation [78]). The most general symmetry of this kind follows immediately from our lemma when $p=2$ with the aid of (3.1)–(3.4) of [50]. This is another classical example of blow up phenomenon. The stationary $2D$ waveguides in homogeneous quadratic Kerr media are unstable [48]. Under certain condition, self-focusing of light beams occur on a finite distance despite diffraction spreading. Moreover, for parabolic channels in a monocrystal film, the cubic nonlinearity may further enhance superfocusing of particle beams predicted in [25], [26]. The corresponding inhomogeneous medium effects deserve a detailed study. An extension to randomly varying media is also of interest (cf. [5], [28], [62], [72]).

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School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287–1804

*E-mail address*: `mahalov@asu.edu`

School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287–1804

*E-mail address*: `sks@asu.edu`

*URL*: `http://hahn.la.asu.edu/~suslov/index.html`