

## MULTIPLE LATTICE TILES AND RIESZ BASES OF EXPONENTIALS

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**ABSTRACT.** Suppose  $\Omega \subseteq \mathbb{R}^d$  is a bounded and measurable set and  $\Lambda \subseteq \mathbb{R}^d$  is a lattice. Suppose also that  $\Omega$  tiles multiply, at level  $k$ , when translated at the locations  $\Lambda$ . This means that the  $\Lambda$ -translates of  $\Omega$  cover almost every point of  $\mathbb{R}^d$  exactly  $k$  times. We show here that there is a set of exponentials  $\exp(2\pi it \cdot x)$ ,  $t \in T$ , where  $T$  is some countable subset of  $\mathbb{R}^d$ , which forms a Riesz basis of  $L^2(\Omega)$ . This result was recently proved by Grepstad and Lev under the extra assumption that  $\Omega$  has boundary of measure 0, using methods from the theory of quasicrystals. Our approach is rather more elementary and is based almost entirely on linear algebra. The set of frequencies  $T$  turns out to be a finite union of shifted copies of the dual lattice  $\Lambda^*$ . It can be chosen knowing only  $\Lambda$  and  $k$  and is the same for all  $\Omega$  that tile multiply with  $\Lambda$ .

*Notation.* We write  $e(x) = e^{2\pi ix}$ . If  $E$  is a set, then  $\chi_E$  is its indicator function. If  $A$  is a non-singular  $d \times d$  matrix and  $\Lambda = AZ^d$  is a lattice in  $\mathbb{R}^d$ , then  $\Lambda^* = A^{-T}Z^d$  denotes the dual lattice.

### 1. INTRODUCTION

**1.1. Riesz bases.** In this paper we deal with the question of the existence of a Riesz (unconditional) basis of exponentials

$$e_t(x) := e(t \cdot x) = e^{2\pi it \cdot x}, \quad t \in L,$$

for the space  $L^2(\Omega)$ , where  $\Omega \subseteq \mathbb{R}^d$  is a domain of finite Lebesgue measure and  $L \subseteq \mathbb{R}^d$  is a countable set of frequencies. By Riesz basis we mean that every  $f \in L^2(\Omega)$  can be written uniquely in the form

$$(1) \quad f(x) = \sum_{t \in L} a_t \cdot e(t \cdot x)$$

with the coefficients  $a_t \in \mathbb{C}$  satisfying

$$(2) \quad C_1 \|f\|_2^2 \leq \sum_{t \in L} |a_t|^2 \leq C_2 \|f\|_2^2,$$

for some positive and finite constants  $C_1, C_2$ .

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**1.2. Orthogonal bases.** One very special example of a Riesz basis occurs when the exponentials  $e(t \cdot x)$ ,  $t \in L$ , can be chosen to be orthogonal and complete for  $L^2(\Omega)$ . One can then choose  $a_t = |\Omega|^{-1/2} \langle f, e_t \rangle$  and  $C_1 = C_2 = 1/|\Omega|$  for (2) to hold as an equality. For instance, if  $\Omega = (0, 1)^d$  is the unit cube in  $\mathbb{R}^d$ , then one can take  $L = \mathbb{Z}^d$  and obtain such an orthogonal basis of exponentials. This case, where an orthogonal basis of exponentials exists, is a very rigid situation though and many “reasonable” domains do not have such a basis (a ball is one example [4, 11], or any other smooth convex body or any non-symmetric convex body [7]).

The problem of which domains admit an orthogonal basis of exponentials has been studied intensively. The so-called Fuglede or Spectral Set Conjecture [4] (claiming that for  $\Omega$  to have such a basis it is necessary and sufficient that it can tile space by translations) was eventually proved to be false in dimension at least 3 [2, 3, 9, 12, 20], in both directions. Yet the conjecture may still be true in several important special cases such as convex bodies [8], and it generated many interesting results even after the disproof of its general validity (a rather dated account may be found in [11]).

It is expected that the existence of a Riesz basis for a domain  $\Omega$  is a much more general, and perhaps even generic, phenomenon, although proofs of existence of a Riesz basis for specific domains are still rather rare, especially in higher dimension [13, 14, 16]. Also no domain is known not to have a Riesz basis of exponentials [13].

**1.3. Lattice tiles.** One general class of domains for which an orthogonal basis of exponentials is known to exist is the class of *lattice tiles*. A domain  $\Omega \in \mathbb{R}^d$  is said to *tile* space when translated at the locations of the lattice  $L$  (a discrete additive subgroup of  $\mathbb{R}^d$  containing  $d$  linearly independent vectors) if

$$(3) \quad \sum_{t \in L} \chi_{\Omega}(x - t) = 1, \quad \text{for almost all } x \in \mathbb{R}^d.$$

Intuitively this condition means that one can cover  $\mathbb{R}^d$  with the  $L$ -translates of  $\Omega$ , with no overlaps, except for a set of measure zero (usually the translates of  $\partial\Omega$ , for “nice” domains  $\Omega$ ).

It is not hard to see that when  $\Omega$  has finite and non-zero measure the set  $L$  has density equal to  $1/|\Omega|$ . If  $L$  is a lattice, then we call  $\Omega$  an *almost fundamental domain* of  $L$  and  $|\Omega| = (\text{dens } L)^{-1}$ . A *fundamental domain* of  $L$  is any set which contains exactly one element of each coset mod  $L$ , for instance a fundamental parallelepiped. There are of course many others, as indicated in Figure 1.

It is easy to see [4, 11] that every lattice tile by the lattice  $L$  has an orthogonal basis of exponentials, namely those with frequencies  $t \in L^*$ , where  $L^*$  is the dual lattice.

**1.4. Multiple tiling by a lattice.** We say that a domain tiles multiply when its translates cover space the same number of times, almost everywhere.

**Definition 1.1.** Let  $\Omega \subseteq \mathbb{R}^d$  be measurable and  $L \subseteq \mathbb{R}^d$  be a countable set. We say that  $\Omega$  tiles  $\mathbb{R}^d$  when translated by  $L$  at level  $k \in \mathbb{N}$  if

$$(4) \quad \sum_{t \in L} \chi_{\Omega}(x - t) = k,$$

for almost every  $x \in \mathbb{R}^d$ . If we do not specify  $k$ , then we mean  $k = 1$ .

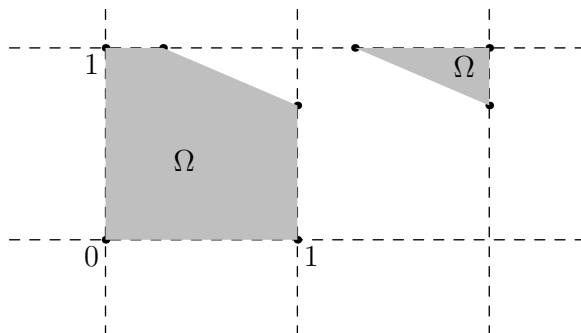


FIGURE 1. Shaded  $\Omega$  is a fundamental domain of  $\mathbb{R}^2/\mathbb{Z}^2$

Multiple tiles are a much wider class of domains than level-one tiles. For instance [1, 10], any centrally symmetric convex polygon in the plane whose vertices have integer coordinates tiles multiply by the lattice  $\mathbb{Z}^2$  at some level  $k \in \mathbb{N}$ . In contrast, only parallelograms or symmetric hexagons can tile at level one.

Another difference is the fact that if two *disjoint* domains  $\Omega_1$  and  $\Omega_2$  both tile multiply when translated at the locations  $L$ , then so does their union. In the case of multiple lattice tiling this operation gives essentially the totality of multiple tiles starting from level-one tiles, according to the following easy lemma.

**Lemma 1.** *Suppose  $\Omega \subseteq \mathbb{R}^d$  is a measurable set which tiles  $\mathbb{R}^d$  at level  $k$  when translated by the lattice  $\Lambda \subseteq \mathbb{R}^d$ . Then we can partition*

$$(5) \quad \Omega = \Omega_1 \cup \dots \cup \Omega_k \cup E,$$

where  $E$  has measure 0 and the  $\Omega_j$  are measurable, mutually disjoint and each  $\Omega_j$  is an almost fundamental domain of the lattice  $\Lambda$ .

*Proof.* Let  $D \subseteq \mathbb{R}^d$  be a measurable fundamental domain of  $\Lambda$ , for instance one of its fundamental parallelepipeds. For almost every  $x \in D$  (call the exceptional set  $E \subseteq D$ ) it follows from our tiling assumption that  $\Omega \cap (x + \Lambda)$  contains exactly  $k$  points, which we denote by

$$p_1(x) < p_2(x) < \dots < p_k(x),$$

ordered according to the lexicographical ordering in  $\mathbb{R}^d$ . We also have that almost every point of  $\Omega$  belongs to exactly one such list.

Let then  $\Omega_j = \bigcup_{x \in D \setminus E} p_j(x)$ , for  $j = 1, 2, \dots, k$ . In other words, for (almost) each one of the classes mod  $\Lambda$  we distribute its  $k$  occurrences in  $\Omega$  into the sets  $\Omega_j$ . It is easy to see that the  $\Omega_j$  are disjoint and measurable and that they are almost fundamental domains of  $\Lambda$ . □

**1.5. Multiple lattice tiles have Riesz bases of exponentials.** It is not true that domains that tile multiply by a lattice have an orthogonal basis of exponentials. For instance, it is known [8] that the only convex polygons that have such a basis are parallelograms and symmetric hexagons, yet every symmetric convex polygon with integer vertices is a multiple tile, a much wider class.

It is however true that multiple tiles have a Riesz basis of exponentials. The main result of this paper is the following theorem.

**Theorem 1.** *Suppose  $\Omega \subseteq \mathbb{R}^d$  is bounded, measurable and tiles  $\mathbb{R}^d$  multiply at level  $k$  with the lattice  $\Lambda$ . Then there are vectors  $a_1, \dots, a_k \in \mathbb{R}^d$  such that the exponentials*

$$(6) \quad e((a_j + \lambda^*) \cdot x), \quad j = 1, 2, \dots, k, \lambda^* \in \Lambda^*$$

*form a Riesz basis for  $L^2(\Omega)$ .*

*The vectors  $a_1, \dots, a_k$  depend on  $\Lambda$  and  $k$  only, not on  $\Omega$ .*

Theorem 1 was proved by Grepstad and Lev [6] with the additional topological assumption that the boundary  $\partial\Omega$  has Lebesgue measure 0.

In [6] the result is proved following the method of [17, 18] on quasicrystals. Our approach is more elementary and almost entirely based on linear algebra. The authors of [6] have pointed out that there are similarities in the method of this paper and the methods in [14–16]. The method essentially appears also in [19, §3.2].

As an interesting corollary of Theorem 1 let us mention, as is done in [6], that, according to the recent result of [5], if  $\Omega$  is a centrally symmetric polytope in  $\mathbb{R}^d$ , whose codimension 1 faces are also centrally symmetric and whose vertices all have rational coordinates, then  $L^2(\Omega)$  has a Riesz basis of exponentials.

**Open Problem 1.** Is Theorem 1 still true if  $\Omega$  is of finite measure but unbounded?

## 2. PROOF OF THE MAIN RESULT

The essence of the proof is contained in the following lemma.

**Lemma 2.** *Suppose  $\Omega \subseteq \mathbb{R}^d$  is bounded, measurable and tiles  $\mathbb{R}^d$  multiply at level  $k$  with the lattice  $\Lambda$ . Then there exist vectors  $a_1, a_2, \dots, a_k \in \mathbb{R}^d$  such that the following is true.*

*For any  $f \in L^2(\Omega)$  there are unique measurable functions  $f_j : \mathbb{R}^d \rightarrow \mathbb{C}$  such that*

- (1) *The  $f_j$  are  $\Lambda$ -periodic.*
- (2) *The  $f_j$  are in  $L^2$  of any almost fundamental domain of  $\Lambda$ .*
- (3) *We have the decomposition*

$$(7) \quad f(x) = \sum_{j=1}^k e(a_j \cdot x) f_j(x), \quad \text{for a.e. } x \in \Omega.$$

*Finally we have*

$$(8) \quad C_1 \|f\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^k \|f_j\|_{L^2(\Omega)}^2 \leq C_2 \|f\|_{L^2(\Omega)}^2,$$

*where  $0 < C_1, C_2 < \infty$  do not depend on  $f$ .*

*Proof.* Using Lemma 1 we can write  $\Omega$  as the disjoint union

$$\Omega = \Omega_1 \cup \dots \cup \Omega_k,$$

where each  $\Omega_k$  is a measurable almost fundamental domain of  $\Lambda$ . We can now define for  $j = 1, 2, \dots, k$  and for almost every  $x \in \mathbb{R}^d$

$$(9) \quad \omega_j(x) \text{ as the unique point in } \Omega_j \text{ s.t. } x - \omega_j(x) \in \Lambda, \text{ and}$$

$$(10) \quad \lambda_j(x) = x - \omega_j(x).$$

(The maps  $\omega_j$  are clearly measurable and measure-preserving when restricted to a fundamental domain of  $\Lambda$ .) Since the sought-after  $f_j$  are to be  $\Lambda$ -periodic it is

enough to define them on  $\Omega_1$  and extend them to  $\mathbb{R}^d$  by their  $\Lambda$ -periodicity. We may therefore rewrite our target decomposition (7) equivalently as follows.

(11)

$$\text{For each } x \in \Omega_1 \text{ and } r = 1, 2, \dots, k: f(\omega_r(x)) = \sum_{j=1}^k e(a_j \cdot (x - \lambda_r(x))) f_j(x).$$

We view (11) as a  $k \times k$  linear system

$$(12) \quad M\tilde{F} = F$$

whose right-hand side is the column vector

$$F = (f(\omega_1(x)), f(\omega_2(x)), \dots, f(\omega_k(x)))^\top$$

and the unknowns form the column vector

$$\tilde{F} = (f_1(x), f_2(x), \dots, f_k(x))^\top.$$

We have a different linear system for each  $x \in \Omega_1$  and its matrix is  $M = M(x) \in \mathbb{C}^{k \times k}$  with

$$(13) \quad M_{r,j} = M_{r,j}(x) = e(a_j \cdot (x - \lambda_r(x))), \quad r, j = 1, 2, \dots, k.$$

Factoring we can write this matrix as

$$(14) \quad M(x) = N(x) \text{diag}(e(a_1 \cdot x), e(a_2 \cdot x), \dots, e(a_k \cdot x)),$$

with the matrix  $N = N(x)$  given by

$$N_{r,j} = N_{r,j}(x) = e(-a_j \cdot \lambda_r(x)), \quad r, j = 1, 2, \dots, k.$$

The key observation here is that when varying  $x \in \Omega_1$  the number of different  $N(x)$  matrices that arise (the  $a_j$  are fixed) is finite and bounded by a quantity that depends on  $\Omega$  and  $\Lambda$  only. The reason for this is that the vectors  $\lambda_r(x)$  are among the  $\Lambda$  vectors in the bounded set  $\Omega - \Omega$ , hence they take values in a finite set. (This is the only place where the boundedness of  $\Omega$  is used.)

Let us now see that the vectors  $a_1, \dots, a_k$  can be chosen so that all the (finitely many) possible matrices  $N$  are invertible. We have

$$(15) \quad \det N(x) = \sum_{\pi \in S_k} \text{sgn}(\pi) e\left(-\sum_{j=1}^k a_j \cdot \lambda_{\pi_j}(x)\right),$$

where  $S_k$  denotes the permutation group on  $\{1, 2, \dots, k\}$ . By the definition of the vectors  $\lambda_r(x)$  and the disjointness of the sets  $\Omega_r$  it follows that for each  $x$  no two  $\lambda_r(x)$  can be the same. View now the expression (15) as a function of the vector  $a = (a_1, \dots, a_k) \in \mathbb{R}^{dk}$ . Clearly it is a trigonometric polynomial and it is not identically zero as all the frequencies (for  $\pi$  in the symmetric group  $S_k$ )

$$(16) \quad \lambda_\pi(x) = (\lambda_{\pi_1}(x), \dots, \lambda_{\pi_k}(x)) \in \mathbb{R}^{dk},$$

are distinct precisely because all the  $\lambda_r(x)$  are distinct. Since the zero-set of any trigonometric polynomial (that is not identically zero) is a set of codimension at least 1 it follows that the vectors  $a_1, \dots, a_k$  can be chosen so that all the  $N(x)$  matrices that arise are invertible.

Now let  $x \in \Omega_1$  and consider the solution of the linear system (12) at  $x$  that now takes the form

$$(17) \quad \tilde{F}(x) = \text{diag}(e(-a_1 \cdot x), e(-a_2 \cdot x), \dots, e(-a_k \cdot x)) N(x)^{-1} F(x).$$

Since  $N(x)$  runs through a finite number of invertible matrices and the diagonal matrix in (17) is an isometry it follows that there are finite constants  $A_1, A_2 > 0$ , independent of  $f$ , such that for any  $x \in \Omega_1$  we have

$$(18) \quad A_1 \|F(x)\|_{\ell^2}^2 \leq \left\| \tilde{F}(x) \right\|_{\ell^2}^2 \leq A_2 \|F(x)\|_{\ell^2}^2.$$

Integrating (18) over  $\Omega_1$  we obtain

$$(19) \quad A_1 \|f\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^k \|f_j\|_{L^2(\Omega_1)}^2 \leq A_2 \|f\|_{L^2(\Omega)}^2.$$

This implies (8) with  $C_j = k \cdot A_j, j = 1, 2$ . To show the uniqueness of the decomposition (7) observe that any such decomposition must satisfy the linear system (17), whose non-singularity has been ensured by our choice of the  $a_j$ .  $\square$

We can now complete the proof of our main result.

*Proof of Theorem 1.* Let  $f \in L^2(\Omega)$ . By Lemma 2 we can write  $f$  as in (7). Since the  $f_j$  are  $\Lambda$ -periodic and are in  $L^2$  of any almost fundamental domain  $D$  of  $\Lambda$  it follows that we can expand each  $f_j$  in the frequencies of  $\Lambda^*$  (the dual lattice of  $\Lambda$ )

$$(20) \quad f_j(x) = \sum_{\lambda^* \in \Lambda^*} f_{j,\lambda^*} e(\lambda^* \cdot x), \quad j = 1, 2, \dots, k,$$

with

$$(21) \quad \|f_j\|_{L^2(D)}^2 = \sum_{\lambda^* \in \Lambda^*} |f_{j,\lambda^*}|^2,$$

since the exponentials  $e(\lambda^* \cdot x), \lambda^* \in \Lambda^*$ , form an orthogonal basis of  $L^2(D)$  (we assume without loss of generality that  $|D| = 1$ ).

The completeness of (6) follows from (7):

$$(22) \quad f(x) = \sum_{j=1}^k \sum_{\lambda^* \in \Lambda^*} f_{j,\lambda^*} e((a_j + \lambda^*) \cdot x).$$

The fact that (6) is a Riesz sequence follows from (8):

$$\frac{k}{C_2} \sum_{j,\lambda^*} |f_{j,\lambda^*}|^2 \leq \left\| \sum_{j,\lambda^*} f_{j,\lambda^*} e((a_j + \lambda^*) \cdot x) \right\|_{L^2(\Omega)}^2 \leq \frac{k}{C_1} \sum_{j,\lambda^*} |f_{j,\lambda^*}|^2.$$

As is clear from the proof above, the  $k$ -tuples of vectors  $a_1, \dots, a_k$  that appear in Theorem 1 are a generic choice: almost all  $k$ -tuples will do. The exceptional set in  $\mathbb{R}^{dk}$  is a set of lower dimension.

With a little more care one can see that one can choose the vectors  $a_1, \dots, a_k$  to depend on  $\Lambda$  and  $k$  only and not on  $\Omega$ . In the proof of Lemma 2 the  $a_j$  were chosen to ensure that the trigonometric polynomials (15) are all non-zero. Fix  $\Lambda$  and  $k$  and form the set of all polynomials of the form (15) which are not identically zero. This set of polynomials is countable and each such polynomial vanishes on a set of codimension at least 1 in  $\mathbb{R}^{dk}$ . It follows that the union of their zero-sets cannot possibly exhaust  $\mathbb{R}^{dk}$  and we only have to choose the  $a_j$  to avoid that union.

Thus there is a choice of  $a_j$  that works for all  $\Omega$  of the same lattice. This proof does not give uniform values for the constants  $C_1$  and  $C_2$  in (8) though.  $\square$

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