

BENEDETTO'S TRICK AND EXISTENCE OF RATIONAL PREPERIODIC STRUCTURES FOR QUADRATIC POLYNOMIALS

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ABSTRACT. We refine a result of R. Benedetto in p -adic analysis in order to exhibit infinitely many quadratic polynomials over \mathbb{Q} with a specified graph of rational preperiodic points.

1. INTRODUCTION

Let K be a field. For $c \in K$, the quadratic polynomial $f_c(z) = z^2 + c$ is an endomorphism of the affine line \mathbb{A}_K^1 . Via iteration, we may view it as a dynamical system on $\mathbb{A}^1(K) = K$. The set of K -rational preperiodic points — those with finite forward orbit under f_c — will be denoted by $\text{PrePer}(f_c, K)$. We equip this set with the structure of a directed graph by drawing an arrow from x to $f_c(x)$ for each $x \in \text{PrePer}(f_c, K)$. By Northcott's theorem, $\text{PrePer}(f_c, K)$ is a *finite* graph when K is a number field. A well-known conjecture of Morton and Silverman [9] implies that there are only finitely many isomorphism types of directed graphs that can arise as one varies over all $c \in K$ and all number fields K/\mathbb{Q} of bounded degree. The problem of which isomorphism types can occur has been investigated extensively when $K = \mathbb{Q}$ [5, 8, 10, 11, 13]. Poonen has conjectured that there are exactly 12 isomorphism types that may arise over \mathbb{Q} [10]. More recently, the case of quadratic extensions K/\mathbb{Q} has been studied by the author and others [4, 6].

Poonen has remarked [10, p. 18] that the graphs shown in Figure 1 occur for infinitely many rational parameters c . This statement follows from Conjecture 2 and the results in [10], but an unconditional proof is desirable. The goal of this note is to provide one.

Theorem 1.1. *For each of the graphs G shown below, there exist infinitely many parameters $c \in \mathbb{Q}$ such that $\text{PrePer}(f_c, \mathbb{Q}) \cong G$.*

Remark 1.2. We follow the labeling convention in [4]. Write $N(\ell_1, \ell_2, \dots)$ for a graph with N vertices and directed cycles of lengths ℓ_1, ℓ_2, \dots , in nonincreasing order.

We briefly explain a heuristic for the theorem. Let us say that a finite directed graph G is **admissible** if there is a parameter $c \in \bar{\mathbb{Q}}$ such that $\text{PrePer}(f_c, \bar{\mathbb{Q}}) \supset G$. For each admissible graph G , there exists a smooth connected projective algebraic curve X_G over \mathbb{Q} such that all but finitely many of the algebraic points $x \in X_G(\bar{\mathbb{Q}})$

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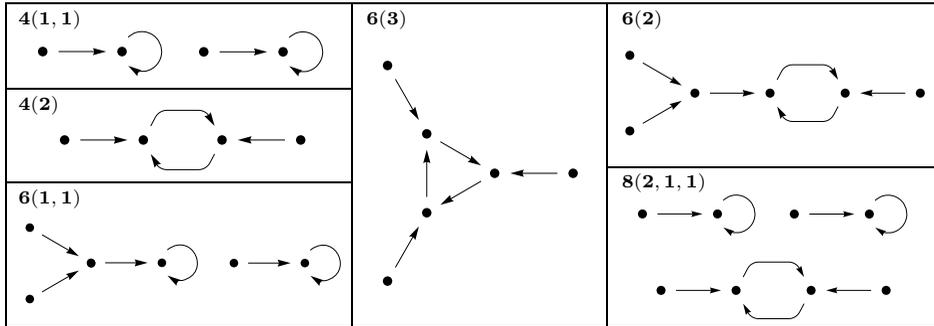


FIGURE 1

correspond to data $(x_1, \dots, x_n, c) \in \bar{\mathbb{Q}}^{n+1}$ for which x_1, \dots, x_n are preperiodic for f_c and generate the graph G under the action of f_c . In particular, if $(x_1, \dots, x_n, c) \in X_G(\mathbb{Q})$, then $G \subset \text{PrePer}(f_c, \mathbb{Q})$. For each graph G as in the theorem, the curve X_G is *rational*, and hence admits infinitely many parameters $c \in \mathbb{Q}$ for which $\text{PrePer}(f_c, \mathbb{Q})$ contains the graph G . While this is strong evidence for the theorem, the obstruction to making it a rigorous proof is that there are infinitely many admissible graphs H properly containing G , and so it is possible that all but finitely many points of $X_G(\mathbb{Q})$ actually correspond to a larger graph H .

Rather than taking an approach via rational points on curves, we start by constructing a sequence of parameters (c_n) for which there exist preperiodic points in the desired configuration, and for which we can control the prime factors occurring in the denominator. (This step uses the Prime Number Theorem in one case and van der Corput’s theorem on 3-term arithmetic progressions of primes in another.) We then apply a trick of Benedetto for bounding the number of rational preperiodic points of a polynomial. In essence, his technique focuses on a place p of genuinely bad reduction for f_c and captures the filled Julia set inside a collection of at most 2 p -adic disks of moderate radius, or 4 disks of very small radius. By the product formula, each such disk will contain at most one rational preperiodic point. This argument is highly reminiscent of the line of attack initiated by Call and Goldstone [3] (and then refined and extended by Benedetto). The bound on the number of rational preperiodic points so produced is sharp enough to obtain the desired result for the graphs with 4 or 8 vertices. For the graphs G with 6 vertices, we must also invoke nonexistence results of Poonen [10] to exclude the occurrence of graphs that properly contain G .

Benedetto’s Trick for bounding the number of rational preperiodic points of a polynomial is explained in §2. The proof of Theorem 1.1 is given in §3. We argue in §4 that the theorem is best possible in a precise sense.

2. BENEDETTO’S TRICK

The goal of this section is to explain (and extend) a technique of Benedetto for obtaining global bounds on $\#\text{PrePer}(f, \mathbb{Q})$ using local data and the product formula.

Let $p \leq \infty$ be a prime of \mathbb{Q} . We write \mathbb{Q}_p for the completion of \mathbb{Q} at p and $|\cdot|_p$ for its standard absolute value; we write \mathbb{C}_p for the completion of an algebraic

closure of \mathbb{Q}_p . In particular, $\mathbb{Q}_\infty = \mathbb{R}$, and $\mathbb{C}_\infty = \mathbb{C}$. For a polynomial f with rational coefficients, we define the *filled Julia set* for f at p to be

$$\mathcal{K}_{f,p} = \{x \in \mathbb{C}_p : f^n(x) \not\rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

The key observation is that preperiodic points for f lie in the filled Julia set $\mathcal{K}_{f,p}$ for every prime p .

The following lemma provides stronger bounds for the filled Julia set in the case of finite p . It is an extension of [1, Lem. 4.3]; the proof strategy is essentially the same. For notation, if $p \leq \infty$ is a prime of \mathbb{Q} , $a \in \mathbb{C}_p$, and $s > 0$, we write $D(a, s) = \{x \in \mathbb{C}_p : |x - a|_p \leq s\}$ for the closed disk of radius s .

Lemma 2.1. *Let p be a finite prime. Let $f \in \mathbb{Q}[z]$ be a monic polynomial of degree $d \geq 2$, and let r be the radius of the minimal closed disk containing the filled Julia set $\mathcal{K}_{f,p}$. For any point $x_0 \in \mathcal{K}_{f,p}$, the following containments hold:*

$$\mathcal{K}_{f,p} \subset \bigcup_{f(x)=x_0} D(x, 1) \quad \text{and} \quad \mathcal{K}_{f,p} \subset \bigcup_{f^2(x)=x_0} D(x, r^{-1/d}).$$

Remark 2.2. In [1, Lem. 4.3], Benedetto proves that the filled Julia set $\mathcal{K}_{f,p}$ is contained in a union of at most d^2 open disks $D(x, 1)^- = \{y \in \mathbb{C}_p : |y - x|_p < 1\}$ as x varies over the solutions to $f^2(z) = x_0$.

Proof. Write $U_0 \subset \mathbb{C}_p$ for the minimal closed disk containing the filled Julia set of f ; it has radius r by assumption. Then $r \geq 1$, with equality if and only if f has potential good reduction [2, Lem. 2.5]. If $r = 1$, then U_0 is totally invariant (i.e., $f^{-1}(U_0) = f(U_0) = U_0$), and the result is trivial. Let us assume in what remains that $r > 1$.

Write x_1, \dots, x_d for the elements of $f^{-1}(x_0)$, repeated according to multiplicity. For each $i = 1, \dots, d$, denote by $x_{i,1}, \dots, x_{i,d}$ the elements of $f^{-1}(x_i)$, again with multiplicity.

Let V_1, \dots, V_m be the distinct pre-images of U_0 . Since f has genuinely bad reduction, we know $m > 1$ and each V_i maps surjectively onto U_0 . There must be two distinct disks V_s and V_t at distance exactly r , else the entire filled Julia set would fit into a disk of radius strictly less than r . Also, for each $i = 1, \dots, d$, each of V_s and V_t must contain at least one of $x_{i,1}, \dots, x_{i,d}$ since each of these disks maps surjectively onto U_0 . Finally, for every $i = 1, \dots, d$ and $w \in \mathbb{C}_p$, there is some $j = 1, \dots, d$ such that $|w - x_{i,j}|_p \geq r$; otherwise, all of the $x_{i,j}$ would be at distance less than r by the ultrametric inequality, contradicting the fact that V_s and V_t are at distance r .

For the first containment in the statement, let $w \in \mathbb{C}_p \setminus \bigcup_{f(x)=x_0} D(x, 1)$. For any $i = 1, \dots, d$, we see that

$$|f(w) - x_0|_p = \prod_{1 \leq j \leq d} |w - x_j|_p > 1^{d-1} \cdot r = r.$$

This means $f(w) \notin U_0$. By the total invariance of the filled Julia set, we conclude that $w \notin \mathcal{K}_{f,p}$.

A similar argument will prove the second containment in the statement. Let

$$w \in \mathbb{C}_p \setminus \bigcup_{f^2(x)=x_0} D(x, r^{-1/d}).$$

For any $i = 1, \dots, d$, we see that

$$|f(w) - x_i|_p = \prod_{1 \leq j \leq d} |w - x_{i,j}|_p > \left(r^{-1/d}\right)^{d-1} \cdot r = r^{1/d}.$$

Therefore

$$|f^2(w) - x_0|_p = \prod_{1 \leq i \leq d} |f(w) - x_i|_p > r.$$

As before, we conclude that $w \notin \mathcal{K}_{f,p}$. □

For the next statement, write $M_{\mathbb{Q}}$ for the set of primes of \mathbb{Q} , including $p = \infty$.

Benedetto’s Trick. *Let $d \geq 2$ be an integer, and let $(g_n)_{n \geq 1}$ be a sequence of monic polynomials of degree d with \mathbb{Q} -coefficients. Suppose that for each n there exists a partition*

$$M_{\mathbb{Q}} = A_n \cup B_n \cup C_n$$

for which the following hypotheses hold:

- (Good reduction) For each prime $\ell \in A_n$, the minimum radius of a closed disk that contains $\mathcal{K}_{g_n, \ell}$ is 1.
- (Controlled bad reduction) The set B_n is stable for $n \geq 1$ and contains ∞ . For each $\ell \in B_n$, there exists $R_\ell \geq 1$ independent of n such that the ℓ -adic filled Julia set $\mathcal{K}_{g_n, \ell} \cap \mathbb{Q}_\ell$ is contained inside the disk $D(0, R_\ell)$.
- (Uncontrolled bad reduction) The cardinality of C_n is stable. For each $\ell \in C_n$, if $r_{n, \ell}$ is the minimum radius of a closed disk containing the filled Julia set $\mathcal{K}_{g_n, \ell}$, then $\min\{r_{n, \ell} : \ell \in C_n\}$ tends to infinity with n .

If $\#C_n \geq 1$, then $\#\text{PrePer}(g_n, \mathbb{Q}) \leq d^{\#C_n+1}$ for all n sufficiently large.

Remark 2.3. Let s be the number of odd primes of bad reduction. For our application in the next section, we have $d = 2$ and $\#C_n = s$, so that our bound will be 2^{s+1} . The proof of [3, Thm. 6.9] produces the bound $2^{s+1} \cdot R_\infty$, which is generally weaker.

Proof. A preperiodic point for g_n must lie in the filled Julia set $\mathcal{K}_{g_n, \ell}$ for each prime ℓ of \mathbb{Q} . If α and β are distinct rational preperiodic points for g_n and $\ell \in A_n \cup B_n$, this means

$$(2.1) \quad |\alpha - \beta|_\ell \leq \begin{cases} 1 & \text{if } \ell \in A_n, \\ 2^{\epsilon(\ell)} R_\ell & \text{if } \ell \in B_n, \end{cases}$$

where $\epsilon(\ell) = 1$ if $\ell = \infty$ and 0 otherwise. We now use Lemma 2.1 to provide a bound for the primes in C_n .

Assign an ordering to the elements of C_n , say $C_n = \{\ell_0, \dots, \ell_t\}$. Write r_{n, ℓ_0} for the minimum radius of a disk that contains the filled Julia set $\mathcal{K}_{g_n, \ell_0}$. We apply Lemma 2.1 to obtain d^2 disks $D_{0,1}, \dots, D_{0,d^2}$ of radius $r_{n, \ell_0}^{-1/d}$ whose union contains the filled Julia set. (These disks depend implicitly on the index n .) For each prime ℓ_j with $j = 1, \dots, t$, we apply Lemma 2.1 to obtain d disks $D_{j,1}, \dots, D_{j,d}$ of radius 1 whose union contains the filled Julia set $\mathcal{K}_{g_n, \ell_j}$.

To each element $x \in \text{PrePer}(g_n, \mathbb{Q})$, we may assign an address

$$(a_0, \dots, a_t) \in \{1, \dots, d^2\} \times \{1, \dots, d\}^t$$

so that $x \in D_{j,a_j}$ for each $j = 0, \dots, t$. We claim that for n sufficiently large, no two rational preperiodic points share the same address. Evidently, this implies the desired inequality:

$$\#\text{PrePer}(g_n, \mathbb{Q}) \leq d^2 \cdot d^t = d^{\#C_n+1}.$$

To prove the claim, suppose that α and β are distinct rational preperiodic points for g_n with the same address. Then

$$(2.2) \quad |\alpha - \beta|_{\ell_j} \leq \begin{cases} r_{n,\ell_0}^{-1/d} & \text{if } j = 0, \\ 1 & \text{if } j = 1, \dots, t. \end{cases}$$

Combining (2.1) with (2.2) and applying the product formula gives

$$1 = \prod_{\ell \leq \infty} |\alpha - \beta|_{\ell} \leq r_{n,\ell_0}^{-1/d} \prod_{\ell \in B_n} 2^{\epsilon(\ell)} R_{\ell}.$$

Since $r_{n,\ell_0} \rightarrow \infty$ with n , we obtain a contradiction. □

3. PROOF OF THE THEOREM

In order to apply Benedetto's Trick, we make use of the following well-known result that gives bounds for the various filled Julia sets for the polynomial $f_c(z) = z^2 + c$. For example, see [4] for proofs.

Proposition 3.1. *Let $c \in \mathbb{Q}$.*

- *If $c \leq 1/4$, then $\mathcal{K}_{f_c,\infty} \cap \mathbb{R} \subset D\left(0, \frac{1}{2} + \sqrt{\frac{1}{4} - c}\right) \cap \mathbb{R}$.*
- *If $p < \infty$ is prime and $|c|_p \leq 1$, then $\mathcal{K}_{f_c,p} = D(0, 1)$.*
- *If $p < \infty$ is prime and $|c|_p > 1$, then $\mathcal{K}_{f_c,p} \subset \{x \in \mathbb{C}_p : |x|_p = |c|_p^{1/2}\}$.*

Each of the six graphs in the theorem requires a slightly different argument, so we will group them accordingly.

3.1. The graph 4(1, 1). For each odd rational prime p , we define

$$c_p = \frac{1}{4} - \frac{1}{p^2}.$$

One checks easily that the four points $\pm \frac{1}{2} \pm \frac{1}{p}$ are preperiodic for $f = f_{c_p}$ and that they yield the graph 4(1, 1). In particular,

$$\#\text{PrePer}(f, \mathbb{Q}) \geq 4.$$

We show the opposing inequality using Benedetto's Trick.

We use the partition of $M_{\mathbb{Q}}$ determined by

$$A_p = M_{\mathbb{Q}} \setminus \{2, p, \infty\} \quad B_p = \{2, \infty\} \quad C_p = \{p\}.$$

For $\ell \neq 2, p, \infty$, the minimum radius of a closed disk containing the filled Julia set $\mathcal{K}_{f,\ell}$ is 1 (Proposition 3.1). For $\ell = \infty$, Proposition 3.1 shows that the real filled Julia set $\mathcal{K}_{f,\infty} \cap \mathbb{R}$ is contained in the closed disk about the origin with radius

$$\frac{1}{2} + \sqrt{\frac{1}{4} - c_p} = \frac{1}{2} + \frac{1}{p} \leq \frac{5}{6}.$$

Set $R_{\infty} = 5/6$. If $\ell = 2$, then $|c_p|_2 = 4$, so that

$$\mathcal{K}_{f,2} \subset \{x \in \mathbb{C}_2 : |x|_2 = 2\} \subset D(0, 2).$$

Set $R_2 = 2$. Finally, if $\ell = p$, then $|c_p|_p = p^2$, so that

$$\mathcal{K}_{f,p} \subset \{x \in \mathbb{C}_p : |x|_p = p\}.$$

In particular, since $\frac{1}{2} \pm \frac{1}{p}$ are both preperiodic for f , and since they lie at distance p from each other, the smallest disk that contains $\mathcal{K}_{f,p}$ has radius $r_p = p$. Evidently the hypotheses of Benedetto's Trick are met, so that for p sufficiently large we obtain

$$\#\text{PrePer}(f, \mathbb{Q}) \leq 2^{\#C_p+1} = 4.$$

3.2. The graph 4(2). For each odd prime p , set

$$c_p = -\frac{3}{4} - \frac{1}{p^2}.$$

The four points $\pm\frac{1}{2} \pm \frac{1}{p}$ are preperiodic for $f = f_{c_p}$, and they yield the graph 4(2). The rest of the proof is identical to the one used for the graph 4(1,1), with the exception that we take $R_\infty = \frac{1}{2} + \frac{1}{3}\sqrt{10}$.

3.3. The graph 6(1,1). The graph $\text{PrePer}(f_c, \mathbb{Q})$ contains 6(1,1) if and only if

$$c = -\frac{2(\eta^2 + 1)}{(\eta^2 - 1)^2}$$

for some $\eta \in \mathbb{Q} \setminus \{0, \pm 1\}$ [10, Thm. 3.2]. The known rational preperiodic points are

$$\pm \frac{2\eta}{\eta^2 - 1}, \quad \pm \frac{1}{2} \pm \frac{\eta^2 + 3}{2(\eta^2 - 1)}.$$

Make the change of variable $\eta = \frac{t+1}{t-1}$ to arrive at

$$c = -\frac{t^4 - 2t^3 + 2t^2 - 2t + 1}{4t^2}.$$

Define two sequences (p_n) and (q_n) of odd primes such that $p_n \neq q_n$ and $p_n/q_n \rightarrow 1$ as $n \rightarrow \infty$. (This is possible, for example, by the Prime Number Theorem, since the n th prime is of approximate size $n \log n$.) We define c_n by setting $t = p_n/q_n$ in the preceding formula:

$$c_n = -\frac{p_n^4 - 2p_n^3q_n + 2p_n^2q_n^2 - 2p_nq_n^3 + q_n^4}{(2p_nq_n)^2}.$$

For each such value c_n , the graph of preperiodic points for $f = f_{c_n}$ contains 6(1,1) by construction. Indeed, the six known preperiodic points are

$$\pm \frac{1}{2} \pm \frac{p_n^2 - p_nq_n + q_n^2}{2p_nq_n} \quad \text{and} \quad \pm \frac{p_n^2 - q_n^2}{2p_nq_n}.$$

Consider the partition of $M_{\mathbb{Q}}$ given by

$$A_n = M_{\mathbb{Q}} \setminus \{2, p_n, q_n, \infty\} \quad B_n = \{2, \infty\} \quad C_n = \{p_n, q_n\}.$$

For $\ell \in A_n$, we see that $|c_n|_\ell \leq 1$, so that $\mathcal{K}_{f,\ell} = D(0, 1)$ as desired. For $\ell = \infty$, the real filled Julia set $\mathcal{K}_{f,\infty} \cap \mathbb{R}$ is contained in the disk about the origin of radius

$$\frac{1}{2} + \sqrt{\frac{1}{4} - c} = \frac{1}{2} + \frac{p_n^2 - p_nq_n + q_n^2}{2p_nq_n} \rightarrow 1.$$

Hence, there is $R_\infty > 1$ so that $\mathcal{K}_{f,\infty} \cap \mathbb{R} \subset D(0, R_\infty)$ for all n . For $\ell = 2$, we may take $R_2 = 2$ (Proposition 3.1). Finally, for $\ell \mid p_nq_n$, we have $|c_n|_\ell = \ell^2$. There are distinct preperiodic points at distance ℓ from each other, so it follows that the

smallest closed disk containing $\mathcal{K}_{f,\ell}$ has radius $r_\ell = \ell$. Applying Benedetto's Trick shows that for n sufficiently large we have the bound

$$\#\text{PrePer}(f_{c_n}, \mathbb{Q}) \leq 2^{\#C_n+1} = 8.$$

To complete the proof, let us assume that n is sufficiently large so that $6 \leq \#\text{PrePer}(f_{c_n}, \mathbb{Q}) \leq 8$. If the lower bound is strict, then there exists a point $x \in \mathbb{A}^1(\mathbb{Q})$ that is preperiodic for $f = f_{c_n}$ and that is not represented in the graph $6(1, 1)$. If the orbit of x contains an m -cycle $\{x_1, \dots, x_m\}$ for some $m \geq 2$, then the points $\{-x_1, \dots, -x_m\}$ are also preperiodic and map into this m -cycle (because $f(z) = f(-z)$). Hence, $\#\text{PrePer}(f, \mathbb{Q}) \geq 6 + 2m > 8$, a contradiction. Therefore, the orbit of x must contain a fixed point. As f has at most 2 fixed points, it follows that $f(\pm x)$ is an element of the graph $6(1, 1)$.

Let us say that the point x is of type a_b if $f^b(x)$ lies in an a -cycle, but $f^{b-1}(x)$ does not. By the preceding paragraph, the point x is of type 1_2 or 1_3 . The latter case is immediately ruled out by [10, Thm. 3.6]. In the former case, f has four points of type 1_2 , which is ruled out by [10, Thm. 3.2]. We conclude that $\#\text{PrePer}(f_{c_n}, \mathbb{Q}) = 6$, as desired.

3.4. The graph 6(2). The strategy in this case is virtually identical to the one used in the preceding section, so we will simply sketch the differences.

The graph $\text{PrePer}(f_c, \mathbb{Q})$ contains 6(2) if and only if

$$c = -\frac{\nu^4 + 2\nu^3 + 2\nu^2 - 2\nu + 1}{(\nu^2 - 1)^2}$$

for some $\nu \in \mathbb{Q} \setminus \{0, \pm 1\}$ [10, Thm. 3.3]. Making the change of coordinate $\nu = \frac{t+1}{t-1}$ yields

$$c = -\frac{t^4 + 2t^3 + 2t^2 - 2t + 1}{4t^2}.$$

Choose two sequences of primes (p_n) and (q_n) as before and set $t = p_n/q_n$. Then

$$c_n = -\frac{p_n^4 + 2p_n^3q_n + 2p_n^2q_n^2 - 2p_nq_n^3 + q_n^4}{(2p_nq_n)^2},$$

and the six preperiodic points that arise in this construction are

$$\pm \frac{p_n^2 + q_n^2}{2p_nq_n} \quad \text{and} \quad \pm \frac{1}{2} \pm \frac{p_n^2 + p_nq_n - q_n^2}{2p_nq_n}.$$

At the infinite prime, we see that $c_n \rightarrow -1$ with n . Thus there is $R_\infty > \frac{1}{2}(1 + \sqrt{5})$ so that the real filled Julia set $\mathcal{K}_{f_{c_n}, \infty} \cap \mathbb{R}$ is contained in the closed disk about the origin of radius R_∞ . The remainder of the application of Benedetto's Trick is identical to the preceding section. We conclude that

$$6 \leq \#\text{PrePer}(f_{c_n}, \mathbb{Q}) \leq 8$$

for all sufficiently large n . If the lower inequality is sharp, then there is a rational preperiodic point x of type 2_2 or 2_3 that has not been accounted for. Both of these cases are outlawed by [10, Thm. 3.3, 3.6]. Hence, $\#\text{PrePer}(f_{c_n}, \mathbb{Q}) = 6$, as desired.

3.5. **The graph 6(3).** According to [10, Thm. 1.3], the graph $\text{PrePer}(f_c, \mathbb{Q})$ contains 6(3) if and only if

$$(3.1) \quad c = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau + 1)^2}$$

for some $\tau \in \mathbb{Q} \setminus \{0, -1\}$. The six preperiodic points that one obtains by taking such a c are

$$\pm \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau + 1)}, \quad \pm \frac{\tau^3 - \tau - 1}{2\tau(\tau + 1)}, \quad \pm \frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau + 1)}.$$

As c has three poles as a rational function in τ , controlling the primes that occur in its denominator is a trickier endeavor than in the previous sections. We make use of the following result from analytic number theory:

Theorem 3.2 (van der Corput, 1939). *For all N sufficiently large, there exists a 3-term arithmetic progression $p, p + k, p + 2k$ of primes in the interval $(N, 2N)$.*

Proof. The proof that there are infinitely many 3-term arithmetic progressions of primes is given in [12]. In a private communication, Andrew Granville has informed the author that one may impose the added condition $N < p, p + k, p + 2k < 2N$ in the proof. \square

The theorem implies that there exist 3-term arithmetic progressions $p, p+k, p+2k$ with p arbitrarily large and $k < p$. For such an arithmetic progression, it follows that

$$\frac{1}{3} < \frac{p}{p + 2k} < 1.$$

Let (p_n) and (q_n) be sequences of primes such that $p_n \rightarrow \infty$ and $p_n < \frac{p_n + q_n}{2} < q_n$ is a 3-term arithmetic progression as above. Define $c_n \in \mathbb{Q}$ by setting $\tau = p_n/q_n$ in (3.1):

$$c_n = -\frac{p_n^6 + 2p_n^5q_n + 4p_n^4q_n^2 + 8p_n^3q_n^3 + 9p_n^2q_n^4 + 4p_nq_n^5 + q_n^6}{[2p_nq_n(p_n + q_n)]^2}.$$

Partition $M_{\mathbb{Q}}$ using the sets

$$A_n = M_{\mathbb{Q}} \setminus \{2, p_n, (p_n + q_n)/2, q_n, \infty\}, \quad B_n = \{2, \infty\},$$

$$C_n = \{p_n, (p_n + q_n)/2, q_n\}.$$

Set $f = f_{c_n}$. Since the ratio p_n/q_n is positive and bounded away from zero and infinity, it follows that $|c_n|_{\infty}$ is bounded in the Archimedean metric, and hence there is $R_{\infty} > 1$ such that $\mathcal{K}_{f, \infty} \cap \mathbb{R}$ is contained in the disk $D(0, R_{\infty})$ for all n (Proposition 3.1). The remainder of the application of Benedetto’s Trick is similar to the section on the graph 6(1, 1). We conclude that

$$6 \leq \#\text{PrePer}(f_{c_n}, \mathbb{Q}) \leq 2^{\#C_n+1} = 16$$

for all sufficiently large n . If the lower inequality is sharp, we let x be a rational preperiodic point that does not lie in the graph 6(3) (of known points). Then the orbit of x contains an m -cycle $\{x_1, \dots, x_m\}$ for some $m \geq 1$ as well as the points $\{-x_1, \dots, -x_m\}$. Now $m < 6$, since otherwise we have $\#\text{PrePer}(f_{c_n}, \mathbb{Q}) \geq 2m + 6 > 16$. By results of Morton [8] and Flynn/Poonen/Schaefer [5], there does not exist a rational point of period 4 or 5 for any parameter $c \in \mathbb{Q}$. Hence $m < 4$. The (unconditional) results in [10] now show that f_{c_n} has at most one periodic orbit of

period 3 and that it has no other periodic orbit. That is, $\#\text{PrePer}(f_{c_n}, \mathbb{Q}) = 6$ for n sufficiently large.

3.6. The graph $8(2, 1, 1)$. According to [10, Thm. 2.1], the graph $\text{PrePer}(f_c, \mathbb{Q})$ contains $8(2, 1, 1)$ if and only if

$$c = -\frac{3\mu^4 + 10\mu^2 + 3}{4(\mu^2 - 1)^2}$$

for some $\mu \in \mathbb{Q} \setminus \{0, \pm 1\}$. The eight preperiodic points that one obtains by taking such a c are

$$\pm \frac{1}{2} \pm \frac{\mu^2 + 1}{\mu^2 - 1}, \quad \pm \frac{1}{2} \pm \frac{2\mu}{\mu^2 - 1}.$$

Make the change of variable $\mu = \frac{1+t}{1-t}$ to get

$$c = -\frac{t^4 + t^2 + 1}{4t^2}.$$

Define two sequences (p_n) and (q_n) of odd primes such that $p_n \neq q_n$ and $p_n/q_n \rightarrow 1$ as $n \rightarrow \infty$. We define c_n by setting $t = p_n/q_n$ in the preceding formula:

$$c_n = -\frac{p_n^4 + p_n^2 q_n^2 + q_n^4}{4p_n^2 q_n^2}.$$

For each such value c_n , the graph of preperiodic points for $f = f_{c_n}$ contains $8(2, 1, 1)$ by construction. Indeed, the eight known preperiodic points are $\pm \frac{1}{2} \pm \frac{p_n^2 \pm q_n^2}{2p_n q_n}$.

Consider the partition of $M_{\mathbb{Q}}$ given by

$$A_n = M_{\mathbb{Q}} \setminus \{2, p_n, q_n, \infty\} \quad B_n = \{2, \infty\} \quad C_n = \{p_n, q_n\}.$$

For $\ell \in A_n$, we see that $|c_n|_{\ell} \leq 1$, so that $\mathcal{K}_{f, \ell} = D(0, 1)$ as desired. For $\ell = \infty$, the real filled Julia set $\mathcal{K}_{f, \infty} \cap \mathbb{R}$ is contained in the disk about the origin of radius

$$\frac{1}{2} + \sqrt{\frac{1}{4} - c} = \frac{1}{2} + \frac{p_n^2 + q_n^2}{2p_n q_n} \rightarrow \frac{3}{2}.$$

Hence, there is $R_{\infty} > 3/2$ so that $\mathcal{K}_{f, \infty} \cap \mathbb{R} \subset D(0, R_{\infty})$ for all n . For $\ell = 2$, we may take $R_2 = 2$ (Proposition 3.1). Finally, for $\ell \mid p_n q_n$, we have $|c_n|_{\ell} = \ell^2$. There are distinct preperiodic points at distance ℓ from each other, so it follows that the smallest closed disk containing $\mathcal{K}_{f, \ell}$ has radius $r_{\ell} = \ell$. Applying Benedetto's Trick shows that for n sufficiently large we have the bound

$$\#\text{PrePer}(f_{c_n}, \mathbb{Q}) \leq 2^{\#C_n + 1} = 8.$$

4. OPTIMALITY

Recall from the introduction that a finite directed graph G is **admissible** if there is a parameter $c \in \bar{\mathbb{Q}}$ such that $\text{PrePer}(f_c, \bar{\mathbb{Q}}) \supset G$. Theorem 1.1 is best possible in the following sense.

Proposition 4.1. *Let G be an admissible graph that is isomorphic to $\text{PrePer}(f_c, \mathbb{Q})$ for infinitely many parameters $c \in \mathbb{Q}$. Then G is either empty or it is among the six listed in Theorem 1.1.*

Proof. The empty graph occurs for infinitely many parameters $c \in \mathbb{Q}$. For example, set $c = 1/p$ for p a prime and observe that any preperiodic point must be of the form $x = u/\sqrt{p}$ for some $u \in \mathbb{Z} \setminus \{0\}$ (Proposition 3.1). Hence $\text{PrePer}(f_c, \mathbb{Q}) = \emptyset$. In what remains, we may assume that G is nonempty.

The graph G may not contain a directed cycle of length at least 4, for otherwise Morton's genus formula [7, Thm. C] shows that the corresponding algebraic curve X_G has genus at least 2, and hence only finitely many rational points. If f_c admits no rational periodic point of period greater than 3, then Poonen has shown that the graph $\text{PrePer}(f_c, \mathbb{Q})$ is one of 12 types [10, Thm. 3, Fig. 1]. One of his graphs is empty, six correspond to the ones appearing in Theorem 1.1, and he proves that the remaining five types may occur for only finitely many rational parameters $c \in \mathbb{Q}$. \square

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