

THE POLAR CONE OF THE SET OF MONOTONE MAPS

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(Communicated by Walter Craig)

ABSTRACT. We prove that every element of the polar cone to the closed convex cone of monotone transport maps can be represented as the divergence of a measure field taking values in the positive definite matrices.

1. INTRODUCTION

The one-dimensional pressureless gas dynamics equations

$$(1.1) \quad \left. \begin{aligned} \partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) &= 0 \\ \partial_t (\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) &= 0 \end{aligned} \right\} \text{ in } [0, \infty) \times \mathbb{R}$$

has recently been shown equivalent (in the regime of sticky particles) to a first-order differential inclusion on the space of monotone transport maps from the reference measure space $([0, 1], \mathcal{L}^1|_{[0,1]}) =: (\Omega, \mathfrak{m})$ (where \mathcal{L}^1 is the one-dimensional Lebesgue measure) to \mathbb{R} ; see [7]. More precisely, to every density/velocity (ϱ, \mathbf{u}) solving (1.1) one can associate a unique map $X \in \mathcal{L}^2(\Omega, \mathfrak{m})$ with X monotone such that

$$(1.2) \quad \varrho(t, \cdot) = X(t, \cdot) \# \mathfrak{m} \quad \text{for all } t \in [0, \infty).$$

Here $\#$ indicates the push-forward of measures. Then X satisfies

$$(1.3) \quad \dot{X} + \partial I_{\mathcal{K}}(X) \ni \bar{V} \quad \text{for all } t \in [0, \infty),$$

where \mathcal{K} denotes the closed convex cone of all transport maps $X \in \mathcal{L}^2(\Omega, \mathfrak{m})$ that are monotone, and where $\partial I_{\mathcal{K}}$ is the subdifferential of the indicator function of \mathcal{K} . If X satisfies (1.3) and is related to ϱ through (1.2), then the Eulerian velocity \mathbf{u} can be recovered from the Lagrangian velocity $V := \dot{X}$ through

$$(1.4) \quad V(t, \cdot) = \mathbf{u}(t, X(t, \cdot)) \quad \text{for all } t \in [0, \infty).$$

Assuming finite kinetic energy, it is natural to require that

$$V(t, \cdot) \in \mathcal{L}^2(\Omega, \mathfrak{m}), \quad \mathbf{u}(t, \cdot) \in \mathcal{L}^2(\mathbb{R}, \varrho(t, \cdot)).$$

The relation (1.4) in particular determines the initial Lagrangian velocity \bar{V} in (1.3) in terms of the initial data $(\varrho, \mathbf{u})(0, \cdot) =: (\bar{\varrho}, \bar{\mathbf{u}})$ of the system (1.1).

It is shown in [7] that the solution of (1.3) can be written explicitly as

$$(1.5) \quad X(t, \cdot) = P_{\mathcal{K}}(\bar{X} + t\bar{V}) \quad \text{for all } t \in [0, \infty),$$

with $\bar{X} := X(0, \cdot) \in \mathcal{K}$ given by (1.2). Here $P_{\mathcal{K}}$ denotes the metric projection onto the cone \mathcal{K} . The connection between (1.1) and (1.3) makes it possible to apply classical results from the theory of first-order differential inclusions in Hilbert spaces

Received by the editors May 10, 2013 and, in revised form, June 3, 2013.

2010 *Mathematics Subject Classification*. Primary 49Q20.

Key words and phrases. Optimal transport, polar cone.

to study the pressureless gas dynamics equations, which form a system of hyperbolic conservation laws. We refer the reader to [3, 7] for further information.

It is known that if X satisfies (1.5), then the difference $(\bar{X} + t\bar{V}) - X(t, \cdot)$ must be an element of the *polar cone* $N_{\mathcal{K}}(X(t, \cdot))$ of \mathcal{K} , which is defined as

$$(1.6) \quad N_{\mathcal{K}}(X) := \left\{ Y \in \mathcal{L}^2(\Omega, \mathfrak{m}) : \int_{\Omega} Y(X' - X) \leq 0 \quad \text{for all } X' \in \mathcal{K} \right\}$$

for all $X \in \mathcal{K}$. We observe that $N_{\mathcal{K}}(X)$ coincides with the subdifferential $\partial I_{\mathcal{K}}(X)$. Since \mathcal{K} is a cone, one can choose $X' = 2X$, $X' = 0$ in (1.6) to obtain that

$$(1.7) \quad Y \in N_{\mathcal{K}}(X) \iff \int_{\Omega} YX = 0, \int_{\Omega} YX' \leq 0 \quad \text{for all } X' \in \mathcal{K}.$$

One is therefore naturally led to the problem of characterizing the polar cone of the set of monotone transport maps, beyond the basic definition (1.6). It is shown in [7] that if $Y \in \mathcal{L}^2(\Omega, \mathfrak{m})$ is an element of the polar cone $N_{\mathcal{K}}(X)$, then Y coincides with the *derivative of a nonnegative function*. We refer the reader to [7] for more details, and to [4, 6, 9] for similar results.

In this paper, we will give a generalization of this result to the multi-dimensional case. We are interested in the following setting: We assume that a Borel probability measure ϱ on \mathbb{R}^d is given with finite second moments, so that $\int_{\mathbb{R}^d} |x|^2 \varrho(dx) < \infty$. We consider the closed convex cone of monotone transport maps

$$\mathcal{K}_{\varrho} := \left\{ \mathbf{f} \in \mathcal{L}^2(\mathbb{R}^d, \varrho) : \mathbf{f} \text{ is monotone} \right\}.$$

Here we call any Borel map $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ monotone if the support of the induced transport plan $\gamma_{\mathbf{f}} := (\text{id}, \mathbf{f})\# \varrho$, which is a Borel probability measure on the product space $\mathbb{R}^d \times \mathbb{R}^d$, is a monotone set. Recall that $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ is monotone if

$$(y_1 - y_2) \cdot (x_1 - x_2) \geq 0 \quad \text{for all } (x_i, y_i) \in \Gamma \text{ with } i = 1..2,$$

where \cdot denotes the Euclidean inner product on \mathbb{R}^d . Our goal is to find a representation of elements of the polar cone $\mathcal{K}_{\varrho}^{\perp}$ (at the zero map), defined as

$$\mathcal{K}_{\varrho}^{\perp} := \left\{ \mathbf{g} \in \mathcal{L}^2(\mathbb{R}^d, \varrho) : \int_{\mathbb{R}^d} \mathbf{g}(x) \cdot \mathbf{f}'(x) \varrho(dx) \leq 0 \quad \text{for all } \mathbf{f}' \in \mathcal{K}_{\varrho} \right\}.$$

Notice that since ϱ has finite second moments, any smooth monotone function with at most linear growth at infinity (see details below) is an element of \mathcal{K}_{ϱ} . Moreover, whenever $\mathbf{g} \in \mathcal{K}_{\varrho}^{\perp}$ is given, then the product $\mathbf{g}\varrho$ is in fact an \mathbb{R}^d -valued finite Borel measure, because of Cauchy-Schwarz inequality. We will show below in Theorem 2.1 that for any $\mathbf{g} \in \mathcal{K}_{\varrho}^{\perp}$ the measure $\mathbf{g}\varrho$ can be written as the divergence of a finite Borel measure taking values in the symmetric, positive semidefinite matrices. In the one-dimensional case, we therefore obtain the derivative of a nonnegative function (measure) as in [7]. Our proof relies on an application of the Hahn-Banach theorem and is inspired by a similar argument in [2] for the construction of Michell trusses. It is possible to prove a representation of the polar cone $\mathcal{K}_{\varrho}^{\perp}$ similar to ours by using a characterization of polar cones from [10] and subharmonic functions; see [6, 9] for instance. Compared to these presentations, our proof is shorter and simpler.

2. THE MAIN RESULT

We will denote by $x \cdot y$ the Euclidean inner product of $x, y \in \mathbb{R}^k$, and by $|x|$ the induced norm. We write $\mathbb{R}^{l \times l}$ for the space of real matrices. For any $A, B \in \mathbb{R}^{l \times l}$ with components $A = (a_{ij})$ and $B = (b_{ij})$ we define an inner product

$$\langle A, B \rangle := \operatorname{tr}(AB^T) = \sum_{i,j=1}^l a_{i,j} b_{i,j}$$

(with B^T the transpose of B), which induces the Frobenius norm

$$\|A\| := \sqrt{\operatorname{tr}(AA^T)} = \sum_{i,j=1}^l a_{i,j}^2.$$

We denote by \mathcal{S}^l the space of symmetric real matrices and by \mathcal{S}_+^l the subset of positive semidefinite symmetric matrices. The space of all positive definite, but not necessarily symmetric matrices will be denoted by $\mathbb{R}_+^{l \times l}$. Recall that

$$A \in \mathbb{R}_+^{l \times l} \iff v \cdot (Av) \geq 0 \quad \text{for all } v \in \mathbb{R}^l.$$

Equivalently, we have $A \in \mathbb{R}_+^{l \times l}$ if and only if $A^{\operatorname{sym}} := (A + A^T)/2 \in \mathcal{S}_+^l$.

Let $\mathcal{C}_*(\mathbb{R}^d; \mathbb{R}^{l \times l})$ be the space of all continuous functions $w: \mathbb{R}^d \rightarrow \mathbb{R}^{l \times l}$ with the property that $\lim_{|x| \rightarrow \infty} w(x) \in \mathbb{R}_+^{l \times l}$ exists. Note that we can write

$$\mathcal{C}_*(\mathbb{R}^d; \mathbb{R}^{l \times l}) = \mathbb{R}^{l \times l} + \mathcal{C}_0(\mathbb{R}^d; \mathbb{R}^{l \times l}),$$

where $\mathcal{C}_0(\mathbb{R}^d; \mathbb{R}^{l \times l})$ is the closure of the space of all compactly supported continuous $\mathbb{R}^{l \times l}$ -valued maps, w.r.t. the sup-norm. In an analogous way, we define $\mathcal{C}_*(\mathbb{R}^d; \mathcal{S}^l)$ and $\mathcal{C}_*(\mathbb{R}^d; \mathcal{S}_+^l)$. For any map $u \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d)$ we denote by

$$e(u(x)) := Du(x)^{\operatorname{sym}} \quad \text{for all } x \in \mathbb{R}^d$$

its deformation tensor, which is an element of $\mathcal{C}(\mathbb{R}^d; \mathcal{S}^d)$. Let

$$\mathcal{C}_*^1(\mathbb{R}^d; \mathbb{R}^d) := \{u \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d) : Du \in \mathcal{C}_*(\mathbb{R}^d; \mathbb{R}^{d \times d})\},$$

$$\operatorname{MON}(\mathbb{R}^d) := \{u \in \mathcal{C}_*^1(\mathbb{R}^d; \mathbb{R}^d) : u \text{ is monotone}\},$$

so that $e(u) \in \mathcal{C}_*(\mathbb{R}^d; \mathcal{S}_+^d)$ if $u \in \operatorname{MON}(\mathbb{R}^d)$. The cone $\operatorname{MON}(\mathbb{R}^d)$ contains all linear maps $u(x) := Ax$ for $x \in \mathbb{R}^d$, where $A \in \mathbb{R}_+^{d \times d}$. See [1] for more details.

We will denote by $\mathcal{M}(\mathbb{R}^d; \mathbb{R}^k)$ the space of finite \mathbb{R}^k -valued Borel measures. In an analogous way, we define $\mathcal{M}(\mathbb{R}^d; \mathcal{S}^l)$ and $\mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^l)$. If $f_i, i = 1 \dots k$, are the components of $\mathbf{F} \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^k)$ and $u \in \mathcal{C}_b(\mathbb{R}^d; \mathbb{R}^k)$ we write

$$\int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) = \sum_{i=1}^k \int_{\mathbb{R}^d} u_i(x) f_i(dx).$$

We will say that \mathbf{F} has finite first moment if $\sum_{i=1}^k \int_{\mathbb{R}^d} |x| |f_i|(dx) < \infty$. If $\mu_{i,j} = \mu_{j,i}$, $i, j = 1 \dots l$, are the components of $\mathbf{M} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}^l)$ and $v \in \mathcal{C}_b(\mathbb{R}^d; \mathcal{S}^l)$, then

$$\int_{\mathbb{R}^d} \langle v(x), \mathbf{M}(dx) \rangle = \sum_{i,j=1}^l \int_{\mathbb{R}^d} v_{i,j}(x) \mu_{i,j}(dx).$$

For any $\mathbf{M} = (\mu_{i,j}) \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}^l)$ we have $\mathbf{M} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^l)$ if and only if

$$\sum_{i,j=1}^l \mu_{i,j} v_i v_j \quad \text{is a positive measure for all } v \in \mathbb{R}^l.$$

We can now state our representation result.

Theorem 2.1 (Stress tensor). *Assume that there exist a measure $\mathbf{F} \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ with finite first moment and a matrix-valued field $\mathbf{H} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^d)$ with*

$$(2.1) \quad G(u) := - \int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \langle e(u(x)), \mathbf{H}(dx) \rangle \geq 0$$

for all $u \in \text{MON}(\mathbb{R}^d)$. Then there exists $\mathbf{M} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^d)$ such that

$$(2.2) \quad G(u) = \int_{\mathbb{R}^d} \langle e(u(x)), \mathbf{M}(dx) \rangle \quad \text{for all } u \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^d),$$

$$(2.3) \quad \int_{\mathbb{R}^d} \text{tr}(\mathbf{M}(dx)) \leq - \int_{\mathbb{R}^d} x \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \text{tr}(\mathbf{H}(dx)).$$

Notice that the integrals in (2.1) are finite for any choice of $u \in \mathcal{C}_*^1(\mathbb{R}^d; \mathbb{R}^d)$, by our assumptions on \mathbf{F} and \mathbf{H} . Recall that the trace of a symmetric matrix is equal to the sum of its eigenvalues, which in the case of a positive semidefinite matrix are all nonnegative. Therefore (2.3) controls the size of the measure \mathbf{M} .

For $\mathbf{H} \equiv 0$ we obtain the representation announced in the introduction:

$$\int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) = - \int_{\mathbb{R}^d} \langle Du(x), \mathbf{M}(dx) \rangle$$

for all test functions u . Recall that \mathbf{M} takes values in the symmetric matrices. The more general form of (2.1) is motivated by a variational time discretization for the compressible Euler equations, for which a minimization problem of the form

$$(2.4) \quad \inf_{\mathbf{f} \in \mathcal{K}_\varrho} \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\mathbf{h}(x) - \mathbf{f}(x)|^2 \varrho(dx) + \int_{\mathbb{R}^d} e(x) \det(D\mathbf{f}(x)^{\text{sym}})^{1-\gamma} dx \right\}$$

for suitable $\mathbf{h} \in \mathcal{L}^2(\mathbb{R}^d, \varrho)$ and nonnegative $e \in \mathcal{L}^1(\mathbb{R}^d)$ must be solved, with $\gamma > 1$ some constant. Denoting by $\mathbf{f} \in \mathcal{K}_\varrho$ the minimizer of (2.4) and letting $\mathbf{g} := \mathbf{h} - \mathbf{f}$, we can write the corresponding first-order optimality condition (formally) as

$$\begin{aligned} & - \int_{\mathbb{R}^d} \mathbf{g}(x) \cdot \mathbf{f}'(x) \varrho(dx) \\ & - (\gamma - 1) \int_{\mathbb{R}^d} e(x) \det(D\mathbf{f}(x)^{\text{sym}})^{-\gamma} \text{tr} \left(\text{cof}(D\mathbf{f}(x)^{\text{sym}})^{\text{T}} D\mathbf{f}'(x) \right) dx \geq 0 \end{aligned}$$

for all $\mathbf{f}' \in \mathcal{K}_\varrho$. From this, assumption (2.1) follows if we define

$$\mathbf{F} := \mathbf{g}\varrho \quad \text{and} \quad \mathbf{H} := (\gamma - 1)e \det(D\mathbf{f}^{\text{sym}})^{-\gamma} \text{cof}(D\mathbf{f}^{\text{sym}})^{\text{T}}.$$

One can then check that \mathbf{F} has finite first moments and that $\mathbf{H} \in \mathcal{M}(\mathbb{R}^d; \mathcal{S}_+^d)$. This application will be discussed in more detail in an upcoming publication.

2.1. Positive functionals. In this section, we will discuss a general result about extensions of positive functionals, which is due to Riedl [8]. Let us start with some notation: In the following, we denote by E a normed vector space. We call *positive cone* any subset $C \subset E$ with $C \neq E$ with the following properties:

$$(2.5) \quad C + C \subset C, \quad \lambda C \subset C \quad \text{for all } \lambda > 0, \quad C \cap (-C) = \{0\}.$$

The positive cone C induces a partial ordering \geq on the space E by

$$y \geq x \iff y - x \in C.$$

A linear map $F: L \rightarrow \mathbb{R}$ defined on a subspace $L \subset E$ is called *positive* if

$$(2.6) \quad F(x) \geq 0 \quad \text{for all } x \in L \cap C.$$

A linear map $F: E \rightarrow \mathbb{R}$ is called *functional* if it is continuous.

Proposition 2.2. *Let E be a Banach space, partially ordered by a positive cone C . If some subspace $L \subset E$ contains an interior point of C , then every positive linear map $F_0: L \rightarrow \mathbb{R}$ can be extended to a positive functional $F: E \rightarrow \mathbb{R}$.*

Proof. See Theorem 10.10 of [8]. We include a proof for the reader's convenience.

Step 1. We first observe that $E = L - C$. Indeed if $x_0 \in L$ is an inner point of C , then there exists a $\delta > 0$ with $B_\delta(x_0) \subset C$. Moreover, for all $x \in E$ there exists $\lambda > 0$ (choose $\lambda := 2\|x\|/\delta$, for example) with the property that

$$x/\lambda \subset B_\delta(0) = x_0 - B_\delta(x_0) \subset x_0 - C.$$

Since L is a subspace we obtain, using $\lambda C \subset C$ for all $\lambda > 0$, that

$$E \subset \bigcup_{\lambda > 0} \lambda(x_0 - C) \subset \mathbb{R}x_0 - C \subset L - C.$$

Step 2. Since $E = L - C$, for every $x \in E$ there exist $y_\pm \in L$ and $z_\pm \in C$ such that $\pm x = y_\pm - z_\pm$, which implies that $y_+ \geq x \geq -y_-$. We now define

$$(2.7) \quad p(x) := \inf \left\{ F_0(y) : y \in L, y \geq x \right\} \quad \text{for all } x \in E.$$

Then $p(x) \leq F_0(y_+) < \infty$. On the other hand, for every $y \in L$ with $y \geq x$ we have $y \geq -y_-$. Since $y + y_- \in L \cap C$, we have $F_0(y + y_-) \geq 0$, by positivity of F_0 . This implies that $F_0(y) \geq -F_0(y_-) > -\infty$. We conclude that $p(x)$ is finite for all $x \in E$. It is easy to check that for all $x_1, x_2 \in E$ and for all $\lambda > 0$ we have

$$p(x_1 + x_2) \leq p(x_1) + p(x_2), \quad p(\lambda x_1) = \lambda p(x_1).$$

For every $x \in L$ and $z \in E$ with $z \geq x$, we have $F_0(x) \leq p(z)$ (in particular, we may choose $z = x$). Indeed for every $y \in L$ with $y \geq z$, we have $y \geq x$, thus $y - x \in L \cap C$. Hence $F_0(y - x) \geq 0$, by positivity, which yields $F_0(y) = F_0(x) + F_0(y - x) \geq F_0(x)$. Taking the inf over all $y \in L$ with $y \geq z$, we obtain the estimate.

Step 3. We can now apply the Hahn-Banach theorem and obtain a linear map $F: E \rightarrow \mathbb{R}$ with $F(x) \leq p(x)$ for all $x \in E$. In order to show that F is positive, let $x \in C$. Then $0 \geq -x$ and $0 \in L$, so we may choose $y = 0$ in the definition of $p(-x)$ (see (2.7)) to obtain $p(-x) \leq 0$. Therefore $F(-x) \leq p(-x) \leq 0$, and so $F(x) \geq 0$ for all $x \in C$. To prove that F is an extension of F_0 , let $x \in L$. Then we may choose $y = -x$ in (2.7) to obtain $p(-x) \leq F_0(-x)$ for all $x \in L$. Then

$$-F(x) = F(-x) \leq p(-x) \leq F_0(-x) = -F_0(x),$$

hence $F_0(x) \leq F(x)$. Applying the same argument to $-x \in L$, we get $F_0(x) \geq F(x)$. It follows that $F_0(x) = F(x)$ for all $x \in L$. Therefore F is an extension of F_0 .

Step 4. It remains to prove that F is continuous. Let x_0 be the interior point of C from Step 1, for which $B_\delta(x_0) \subset C$. Then $B_\delta(0) \subset \pm(x_0 - C)$. Let $\lambda := F(x_0) \geq 0$ (recall that $F(x) \geq 0$ for all $x \in C$). Then for all $x \in B_\delta(0)$ we have $x_0 - x \in C$, thus $F(x_0 - x) \geq 0$. It follows that $F(x_0) \geq F(x)$. Similarly, we obtain $F(x) \geq -F(x_0)$. Then either F vanishes (if $\lambda = 0$), or the preimage of the nonempty interval $(-\lambda, \lambda)$ contains a neighborhood of 0, and so F (being linear) is continuous. \square

2.2. Proof of Theorem 2.1. We apply Proposition 2.2 with

$$E := \mathcal{C}_*(\mathbb{R}^d; \mathcal{S}^d), \quad C := \mathcal{C}_*(\mathbb{R}^d; \mathcal{S}_+^d), \quad L := \{e(u) : u \in \mathcal{C}_*^1(\mathbb{R}^d; \mathbb{R}^d)\}.$$

Clearly C satisfies conditions (2.5). The identity map id is an element of $\text{MON}(\mathbb{R}^d)$, with constant deformation tensor $e(\text{id})$ equal to the identity matrix $\mathbf{1} \in \mathcal{S}_+^d$. Since the eigenvalues of a symmetric matrix depend continuously on the matrix entries, we have that $e(\text{id}) = \mathbf{1}$ is an interior point of C : For all $\|v - \text{id}\|_E$ sufficiently small, the eigenvalues of $v(x)$ are larger than $1/2$ for all $x \in \mathbb{R}^d$ and $v \in E$.

On the subspace $L \subset E$, we define the functional F_0 as

$$F_0(v) := - \int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \langle v(x), \mathbf{H}(dx) \rangle \quad \text{where } v = e(u).$$

Note that F_0 is well defined: If there exists another map $\tilde{u} \in \mathcal{C}_*^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $e(\tilde{u}(x)) = v(x)$ for all $x \in \mathbb{R}^d$, then we have $e(u - \tilde{u}) \equiv 0$, by linearity. Consequently, there exist an antisymmetric matrix $B \in \mathbb{R}^{d \times d}$ and $c \in \mathbb{R}^d$ such that

$$\bar{u}(x) := u(x) - \tilde{u}(x) = Bx + c \quad \text{for all } x \in \mathbb{R}^d.$$

Indeed assume that $e(\bar{u}(x)) = 0$ and define

$$W\bar{u}(x) := \frac{D\bar{u}(x) - D\bar{u}(x)^T}{2} \quad \text{for all } x \in \mathbb{R}^d.$$

Then $\partial_k(W\bar{u})_{i,j} \equiv 0$ for all indices i, j, k . Since $D\bar{u} = e(\bar{u}) + W\bar{u}$ it follows that $D\bar{u}$ is a constant matrix with vanishing symmetric part, so \bar{u} is a rigid deformation. We now observe that both $\pm\bar{u} \in \text{MON}(\mathbb{R}^d)$, which implies $F_0(e(\bar{u})) = 0$ because of (2.1). As F_0 is linear, we conclude that F_0 is well defined. Similarly, one can check that $F_0(v) \geq 0$ for all $v \in L \cap C$, so the linear map $F_0 : L \rightarrow \mathbb{R}$ is positive.

Applying Proposition 2.2, we obtain that F_0 can be extended to a continuous linear map $F : E \rightarrow \mathbb{R}$. Notice that $\mathcal{C}_*(\mathbb{R}^d; \mathbb{R})$ is a separable and *closed subalgebra* of the space $\mathcal{C}_b(\mathbb{R}^d; \mathbb{R})$ of bounded continuous \mathcal{S}^d -functions. As is well known, to any closed subalgebra of a space of bounded continuous functions, there corresponds a compactification of the domain. In our case, we obtain the one-point (also called Alexandroff) compactification of \mathbb{R}^d , which we will denote by $\beta\mathbb{R}^d$. Then $\mathcal{C}_*(\mathbb{R}^d; \mathcal{S}^d)$ is isomorphic to $\mathcal{C}(\beta\mathbb{R}^d; \mathcal{S}^d)$. We refer the reader to [5] Section 4.8 for more details. By the Riesz representation theorem, there therefore exists a finite Radon measure $\mathbf{M} \in \mathcal{M}(\beta\mathbb{R}^d; \mathcal{S}^d)$ that represents the functional F in the sense that

$$F(v) = \int_{\beta\mathbb{R}^d} \langle v(x), \mathbf{M}(dx) \rangle \quad \text{for all } v \in \mathcal{C}_*(\mathbb{R}^d; \mathcal{S}^d).$$

Since $F(v) \geq 0$ for all $v \in \mathcal{C}_*(\mathbb{R}^d; \mathcal{S}_+^d)$ we obtain that \mathbf{M} takes in fact values in \mathcal{S}_+^d . Moreover, as F is an extension of F_0 , the following identity holds:

$$F_0(v) = - \int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \langle v(x), \mathbf{H}(dx) \rangle = \int_{\beta\mathbb{R}^d} \langle v(x), \mathbf{M}(dx) \rangle$$

for any $v = e(u)$ and $u \in \mathcal{C}_*^1(\mathbb{R}^d; \mathbb{R}^d)$; see (2.2). In particular, we may choose $u = \text{id}$ (with $e(\text{id}) = \mathbf{1}$) to obtain the control (recall that \mathbf{M} is \mathcal{S}_+^d -valued)

$$\int_{\beta\mathbb{R}^d} \text{tr}(\mathbf{M}(dx)) = - \int_{\mathbb{R}^d} x \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \text{tr}(\mathbf{H}(dx)).$$

Restricting the representation from $\beta\mathbb{R}^d$ to \mathbb{R}^d , we obtain the result.

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