

## ON THE GAUSSIAN BEHAVIOR OF MARGINALS AND THE MEAN WIDTH OF RANDOM POLYTOPES

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ABSTRACT. We show that the expected value of the mean width of a random polytope generated by  $N$  random vectors ( $n \leq N \leq e\sqrt{n}$ ) uniformly distributed in an isotropic convex body in  $\mathbb{R}^n$  is of the order  $\sqrt{\log N}L_K$ . This completes a result of Dafnis, Giannopoulos and Tsolomitis. We also prove some results in connection with the 1-dimensional marginals of the uniform probability measure on an isotropic convex body, extending the interval in which the average of the distribution functions of those marginals behaves in a sub- or supergaussian way.

### 1. INTRODUCTION

In asymptotic convex geometry, the hyperplane conjecture is a very well-known problem that first appeared explicitly in [9]. This conjecture says that there exists an absolute constant  $c$  such that every convex body of volume 1 has a hyperplane section of volume greater than  $c$ . A result by Hensley [16] yields an equivalent formulation, saying that there exists an absolute constant  $c$  such that every convex body has isotropic constant  $L_K$  less than  $c$ .

The study of random polytopes began with Sylvester and the famous four-point problem nearly 150 years ago. Since then, a tremendous effort has been made to study expectations, variances, and distributions of several functionals on a random polytope. This turned out to be very useful and many applications have been found (see [3], [32] and references therein). Random polytopes also provided counterexamples to several conjectures (see, for instance, [14], [34], [15] or [23]).

In 1989, Milman and Pajor [22] showed a deep connection between the hyperplane conjecture and the study of random polytopes, proving that the expected volume of a random simplex in an isotropic convex body is closely related to the value of its isotropic constant.

In [12], the authors studied the expected value of the quermassintegrals of a random polytope generated by  $N$  random vectors uniformly distributed in an isotropic convex body in  $\mathbb{R}^n$ . They showed that if  $n \leq N \leq e\sqrt{n}$ , then the expected value of the smallest quermassintegral, which is the volume radius, is greater than  $c\sqrt{\log \frac{N}{n}}L_K$  and the expected value of the biggest one, which is the mean width, is smaller than  $C\sqrt{\log N}L_K$ . Actually, the upper estimate had been proved in [11]. A similar lower bound without  $L_K$  had also been proved there. This yields a sharp

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estimate for the expected value of any quermassintegral when  $n^2 \leq N \leq e\sqrt{n}$ , but leaves a gap for the range  $n \leq N \leq n^2$ . Our first purpose is to fill this gap for the expected value of the mean width. Denoting by  $a \sim b$  the fact that there exist positive absolute constants  $c, C$  such that  $ca \leq b \leq Ca$ , in Section 3 we will prove the following:

**Theorem 1.1.** *Let  $K \subseteq \mathbb{R}^n$  be a symmetric isotropic convex body and  $n \leq N \leq e\sqrt{n}$ . Let  $K_N = \text{conv}\{\pm X_1, \dots, \pm X_N\}$  be a random polytope, where  $X_1, \dots, X_N$  are independent random vectors uniformly distributed in  $K$ . Then,*

$$\mathbb{E}w(K_N) \sim \sqrt{\log N} L_K.$$

This estimate in the range  $n \leq N \leq n^2$  will be a consequence of the central limit theorem for convex bodies proved by Klartag [19] and the results proved by Sodin [33]. The central limit theorem for convex bodies was first considered in [2] and says that most of the 1-dimensional marginals  $\langle X, \theta \rangle$  of a random variable  $X$  uniformly distributed in an isotropic convex body  $K$  are, in a certain sense, approximately Gaussian. To be more precise, Klartag showed that the distribution function  $F_\theta(t)$  of most of these marginals is “almost” Gaussian whenever  $|t|$  is smaller than some power of  $n$ . It turns out that the Gaussian behavior for a particular value of  $t$  in this range will be enough to prove Theorem 1.1.

Along these lines a great deal of research was devoted in connection with the marginals of the uniform probability measure on an isotropic convex body and important results were obtained. For instance, in [10] Bourgain verified the hyperplane conjecture for the class of  $\psi_2$  bodies, *i.e.*, the class of convex bodies such that every direction (or 1-dimensional marginal) is subgaussian. Given an isotropic convex body  $K$ , we say that a direction  $\theta \in S^{n-1}$  is subgaussian with constant  $r > 0$  if

$$|\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| \leq e^{-\frac{t^2}{r^2}}$$

for all  $1 \leq t \leq r\sqrt{n}$ .

In this setting, the following question was posed by Milman: is it true that every convex body has at least one subgaussian direction? This question has been answered in the affirmative for the class of 1-unconditional convex bodies [8], for the class of zonoids [26], and for the class of isotropic convex bodies with small diameter [27]. (In fact, it was shown that the measure of subgaussian directions is greater than  $1 - e^{-\sqrt{n}}$  for this last class of convex bodies.) In [18], Klartag established the existence of a subgaussian direction up to a logarithmic factor in the dimension. See also [13].

In [30], Pivovarov considered the dual question of finding supergaussian directions. We say that a direction  $\theta \in S^{n-1}$  is supergaussian with constant  $r > 0$  if for all  $1 \leq t \leq \frac{\sqrt{n}}{r}$  we have

$$|\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| \geq e^{-r^2 t^2}.$$

He gave an affirmative answer up to a logarithmic factor for the class of 1-unconditional convex bodies.

In [28], Paouris showed that every isotropic convex body with bounded isotropic constant has “many” supergaussian directions. This includes several classes of convex bodies such as 1-unconditional convex bodies, zonoids, duals of zonoids, and the unit balls of the Schatten classes. He also proved that if for every isotropic

convex body a random direction is supergaussian with high probability, then the hyperplane conjecture is true.

Going in the same direction, Klartag proved in [17] that every non-degenerate  $n$ -dimensional measure has one direction that behaves in a “supergaussian way” for  $t$  in the interval  $1 \leq t \leq c(\log n)^{\frac{1}{4}}$ . As a consequence of the aforementioned central limit theorem this interval was extended to  $1 \leq t \leq n^\kappa$  for some constant  $\kappa$  (see [33]).

The approach used to prove the central limit theorem, as well as some previous weaker results (see [2], [5] or [33]), involved the study of the average of the distribution functions of the 1-dimensional marginals together with a concentration of measure phenomenon. More precisely, Sodin proved that if the Euclidean norm verifies a concentration hypothesis, then the average of the distribution function of the 1-dimensional marginals of an isotropic random vector is approximately Gaussian for  $|t| \leq n^\kappa$  for some absolute constant  $\kappa$ , which is smaller than  $\frac{1}{4}$  (due to “spherical approximation”). A similar result was obtained for  $k$ -dimensional marginals in [4]. In [19], the concentration hypothesis was shown to be true for every convex body.

In [1], a new approach to study the expected value of the support function of a random polytope generated by  $N$  vertices in an isotropic convex body was introduced. Using this approach, we will prove in Section 4 that a supergaussian estimate for the average of the distribution function of the 1-dimensional marginals of the uniform probability measure on an isotropic body holds for the whole range  $1 \leq t \leq \frac{\sqrt{n}}{c}$ , and not only if  $t \leq n^\kappa$ . Namely, if we define

$$F(t) = \int_{S^{n-1}} |\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| d\sigma(\theta),$$

where  $d\sigma$  denotes the uniform probability measure on  $S^{n-1}$ , we have the following:

**Theorem 1.2.** *There exists an absolute constant  $c$  such that for every symmetric isotropic convex body  $K \subseteq \mathbb{R}^n$  and every  $1 \leq t \leq \frac{\sqrt{n}}{c}$  we have*

$$F(t) \geq e^{-c^2t^2}.$$

Using the same idea, in Section 5 we will also show a subgaussian estimate for the average distribution function. However, in this case the estimate does not cover the whole range  $1 \leq t \leq c\sqrt{n}$  except for convex bodies with small diameter, *i.e.*,  $R(K) \leq C\sqrt{n}L_K$ .

**Theorem 1.3.** *There exists an absolute constant  $c$  such that for every symmetric isotropic convex body  $K \subseteq \mathbb{R}^n$  and any  $1 \leq t \leq n^{\frac{1}{4}}$*

$$F(t) \leq e^{-c^2t^2}.$$

*Furthermore, if  $K$  has small diameter, then this estimate is true for  $1 \leq t \leq C\sqrt{n}$ .*

As a consequence, we will find an interval in which a random direction  $\theta \in S^{n-1}$  verifies a subgaussian estimate with high probability. A restriction of the interval leads to better estimates for the measure of the set of “subgaussian directions” in the case of convex bodies with small diameter.

At this point we would also like to mention the paper [6] by Bobkov and Koldobsky, where they studied the rate of convergence of the average volume of sections of an isotropic convex body  $K$  to the gaussian density on the line with variance  $L_K^2$ , instead of the average of the distribution functions of the 1-dimensional marginals of the uniform probability measure on an isotropic body.

2. PRELIMINARIES AND NOTATION

Before we go into more detail we start with some basic definitions. A convex body  $K \subset \mathbb{R}^n$  is a compact convex set with non-empty interior. It is called symmetric if  $-x \in K$ , whenever  $x \in K$ . We will denote its volume (or Lebesgue measure) by  $|\cdot|$ . The volume of the Euclidean unit ball  $B_2^n$  will be denoted by  $w_n = |B_2^n|$ . We write  $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$  for the standard Euclidean sphere in  $\mathbb{R}^n$  and  $d\sigma$  for the uniform probability measure on  $S^{n-1}$ . A convex body is said to be isotropic if it has volume 1 and satisfies the following two conditions:

- $\int_K x \, dx = 0$  (center of mass at 0),
- $\int_K \langle x, \theta \rangle^2 \, dx = L_K^2 \quad \forall \theta \in S^{n-1}$ ,

where  $L_K$  is a constant independent of  $\theta$ , which is called the isotropic constant of  $K$ . Here,  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product in  $\mathbb{R}^n$ .

A probability measure  $\mu$  on  $\mathbb{R}^n$  is said to be isotropic if it is centered at 0 and its covariance matrix is the identity. Notice that a convex body is isotropic if and only if the uniform probability measure on  $\frac{K}{L_K}$  is isotropic.

Let  $K$  be a convex body and  $\theta \in S^{n-1}$  a unit vector. The support function of  $K$  in the direction  $\theta$  is defined by  $h_K(\theta) = \max\{\langle x, \theta \rangle : x \in K\}$ . The mean width of  $K$  is

$$w(K) = \int_{S^{n-1}} h_K(\theta) \, d\sigma(\theta).$$

Given a symmetric isotropic convex body  $K$ , we denote by  $K_N = \text{conv}\{\pm X_1, \dots, \pm X_N\}$  the random polytope, where  $X_1, \dots, X_N$  are independent random vectors uniformly distributed in  $K$ .

In the sequel, if  $\mu$  is an isotropic probability measure on  $\mathbb{R}^n$ ,  $f_\theta$  will denote the density of the random variable  $\langle X, \theta \rangle$  with  $X$  distributed according to  $\mu$ .  $\gamma$  will denote the density of a standard Gaussian, *i.e.*,

$$\gamma(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}.$$

The following lemma is very well known:

**Lemma 2.1.** *For every  $t \geq 1$*

$$\frac{\gamma(t)}{2t} \leq \int_t^\infty \gamma(s) \, ds \leq \frac{\gamma(t)}{t}.$$

The letters  $c, c', C, C', c_1, c_2, \dots$  will denote positive absolute constants, whose value may change from line to line.

Now, let us mention the central limit theorem in the form we will use it to prove Theorem 1.1 in Section 3.

Klartag’s central limit theorem for isotropic measures, combined with an argument by Sodin [33] gives the following:

**Theorem 2.2** ([19, Theorem 1.4]). *Let  $n \geq 1$  be an integer and let  $X$  be a random vector in  $\mathbb{R}^n$  with an isotropic, log-concave density. Then, there exists  $\Theta \subseteq S^{n-1}$*

with  $\sigma_{n-1}(\Theta) \geq 1 - Ce^{-\sqrt{n}}$  such that for all  $\theta \in \Theta$ , the real valued random variable  $\langle X, \theta \rangle$  has a density  $f_\theta : \mathbb{R}^n \rightarrow [0, \infty)$  with the following properties:

- (1)  $\int_{-\infty}^\infty |f_\theta(t) - \gamma(t)| dt \leq \frac{1}{n^\kappa}$ ,
- (2) For all  $|t| \leq n^\kappa$  we have  $\left| \frac{f_\theta(t)}{\gamma(t)} - 1 \right| \leq \frac{1}{n^\kappa}$ .

Here,  $C, \kappa > 0$  are universal constants.

In the case of symmetric  $X$ , those results give  $\kappa = \frac{1}{24}$ , obtaining that there exists  $\Theta \subseteq S^{n-1}$  with  $\sigma_{n-1}(\Theta) \geq 1 - Ce^{-\sqrt{n}}$  such that for any  $\theta \in \Theta$ ,

$$(2.1) \quad \left| \frac{f_\theta(t)}{\gamma(t)} - 1 \right| \leq \frac{C'}{n^{\frac{1}{24}}}$$

whenever  $|t| < cn^{\frac{1}{24}}$ .

Let us also introduce some notation and results we will need to prove the estimates for the average of the distribution functions of the 1-dimensional marginals.

A convex function  $M : [0, \infty) \rightarrow [0, \infty)$  with  $M(0) = 0$  and  $M(t) > 0$  for  $t > 0$  is called an Orlicz function (see for instance [21] or [31]).

Let  $X$  be a random vector in  $\mathbb{R}^n$ . For every  $\theta \in S^{n-1}$  we define an Orlicz function  $M_\theta$  by

$$M_\theta(s) = \int_0^s \int_{\{\frac{1}{t} \leq |\langle X, \theta \rangle\}} |\langle X, \theta \rangle| d\mathbb{P} dt.$$

This Orlicz function was used in [1] to study the expected value of the support function of a random polytope in the direction  $\theta$  since if  $M_\theta$  is the Orlicz function associated to a random vector uniformly distributed on an isotropic body  $K$  and  $K_N$  is a random polytope on  $K$ , then we have (see [1], Corollary 2.2 )

$$\mathbb{E}h_{K_N}(\theta) \sim \inf \left\{ s > 0 : M_\theta \left( \frac{1}{s} \right) \leq \frac{1}{N} \right\}.$$

The following proposition was obtained:

**Proposition 2.3** ([1, Proposition 4.3]). *Let  $K$  be a symmetric convex body in  $\mathbb{R}^n$  of volume 1. Let  $s > 0$ ,  $\theta \in S^{n-1}$  and  $M_\theta$  be the Orlicz function associated to the random variable  $\langle X, \theta \rangle$ , where  $X$  is uniformly distributed in  $K$ . Then,*

$$(2.2) \quad \int_{S^{n-1}} M_\theta \left( \frac{1}{s} \right) d\sigma(\theta) = \int_K M_{\langle \theta, e_1 \rangle} \left( \frac{\|x\|_2}{s} \right) dx,$$

where  $M_{\langle \theta, e_1 \rangle}$  is the Orlicz function associated to the random variable  $\langle \theta, e_1 \rangle$  with  $\theta$  uniformly distributed on  $S^{n-1}$ . For any  $s \leq \|x\|_2$

$$(2.3) \quad M_{\langle \theta, e_1 \rangle} \left( \frac{\|x\|_2}{s} \right) = \frac{2w_{n-1}}{nw_n} \int_0^{\cos^{-1}(\frac{s}{\|x\|_2})} \frac{\sin^n y}{\cos^2 y} dy,$$

and 0 otherwise.

With this representation the existence of some directions for which  $\mathbb{E}h_{K_N}(\theta) \geq C\sqrt{\log N}L_K$  holds was established. Using this very same approach we are going to prove Theorem 1.2 and Theorem 1.3 in Section 4.

3. EXPECTED VALUE OF THE MEAN WIDTH OF A RANDOM POLYTOPE

In this section we are going to prove Theorem 1.1. It is a direct consequence of the following theorem, which will fill the aforementioned “gap” in the range of  $n$  in the results already proved in [12].

**Theorem 3.1.** *Let  $K$  be a symmetric isotropic convex body and  $K_N$  a random polytope in  $K$ , with  $n \leq N \leq n^\delta$ . There exist absolute constants  $c, C$  and a set  $\Theta \subseteq S^{n-1}$  with  $\sigma(\Theta) \geq 1 - Ce^{-\sqrt{n}}$  such that for every  $\theta \in \Theta$*

$$\mathbb{E}h_{K_N}(\theta) \geq \frac{c}{\sqrt{\delta}} \sqrt{\log N} L_K.$$

*Proof.* First of all, notice that for every  $\theta \in S^{n-1}$

$$\begin{aligned} M_\theta \left( \frac{1}{s} \right) &= \int_0^{\frac{1}{s}} \int_{K \cap \{|\langle x, \theta \rangle| \geq \frac{1}{t}\}} |\langle x, \theta \rangle| \, dx \, dt \\ &\geq \int_{\frac{1}{2s}}^{\frac{1}{s}} \frac{1}{t} |\{x \in K : |\langle x, \theta \rangle| \geq \frac{1}{t}\}| \, dt \\ &\geq \left( \frac{1}{s} - \frac{1}{2s} \right) s |\{x \in K : |\langle x, \theta \rangle| \geq 2s\}| \\ &= \frac{1}{2} |\{x \in K : |\langle x, \theta \rangle| \geq 2s\}|. \end{aligned}$$

Thus, if  $s_0 = t_0 L_K$  such that  $M_\theta \left( \frac{1}{s_0} \right) = \frac{1}{N}$ , we have that

$$\frac{2}{N} \geq |\{x \in K : |\langle x, \theta \rangle| \geq 2t_0 L_K\}| = \mathbb{P}\{|\langle Y, \theta \rangle| \geq 2t_0\},$$

where  $Y$  is a random variable distributed uniformly on  $\frac{K}{L_K}$ . Thus, if for some  $t$  we have  $\mathbb{P}\{|\langle Y, \theta \rangle| \geq t\} > \frac{2}{N}$ , then  $t_0 \geq \frac{t}{2}$ . At this point we would like to mention that one can obtain the same conclusion without using the Orlicz function  $M_\theta$ . However, we decided to do it this way, since we will also use this estimate in the proof of Theorem 1.3. From (2.1) we have that if  $Y$  is a log-concave isotropic (covariance matrix equals the identity) random vector in  $\mathbb{R}^n$ , then there exists a subset  $\Theta \subseteq S^{n-1}$  with measure greater than  $1 - Ce^{-\sqrt{n}}$  such that for any  $\theta \in \Theta$ ,

$$\left| \frac{f_\theta(t)}{\gamma(t)} - 1 \right| \leq \frac{C'}{n^{\frac{1}{24}}} \text{ when } |t| \leq cn^{\frac{1}{24}}.$$

Applying this result to the uniform probability measure on  $\frac{K}{L_K}$ , we have that there exists  $\Theta \subseteq S^{n-1}$  with measure greater than  $1 - Ce^{-\sqrt{n}}$  such that for any  $\theta \in \Theta$  and any  $0 \leq t \leq cn^{\frac{1}{24}}$  we have, using Lemma 2.1,

$$\begin{aligned} \mathbb{P}\{|\langle Y, \theta \rangle| < t\} &\leq \left( 1 + \frac{C'}{n^{\frac{1}{24}}} \right) \left( 1 - 2 \int_t^\infty \gamma(s) \, ds \right) \\ &\leq \left( 1 + \frac{C'}{n^{\frac{1}{24}}} \right) \left( 1 - \frac{\gamma(t)}{t} \right), \end{aligned}$$

and so

$$\mathbb{P}\{|\langle Y, \theta \rangle| \geq t\} \geq 1 - \left( 1 + \frac{C'}{n^{\frac{1}{24}}} \right) \left( 1 - \frac{\gamma(t)}{t} \right).$$

Taking  $t = \alpha\sqrt{\log N}$  we have that  $t \leq cn^{\frac{1}{24}}$ , since  $N \leq n^\delta$ . Thus,

$$\begin{aligned} \mathbb{P}\{|\langle Y, \theta \rangle| \geq \alpha\sqrt{\log N}\} &\geq \frac{e^{-\frac{\alpha^2}{2} \log N}}{\sqrt{2\pi}\alpha\sqrt{\log N}} - \frac{C'}{n^{\frac{1}{24}}} \left(1 - \frac{e^{-\frac{\alpha^2}{2} \log N}}{\sqrt{2\pi}\alpha\sqrt{\log N}}\right) \\ &\geq \frac{1}{\sqrt{2\pi}\alpha N^{\frac{\alpha^2}{2}}\sqrt{\log N}} - \frac{C'}{n^{\frac{1}{24}}} \\ &\geq \frac{1}{\sqrt{2\pi}\alpha N^{\frac{\alpha^2}{2}}\sqrt{\log N}} - \frac{C'}{N^{\frac{1}{24\delta}}} > \frac{2}{N} \end{aligned}$$

whenever  $N \geq N_0$  if we take  $\alpha^2 = \frac{1}{24\delta}$ . Thus, for every  $\theta \in \Theta$ ,

$$\mathbb{E}h_{K_N}(\theta) \sim s_0 > \frac{1}{2\sqrt{24\delta}}\sqrt{\log N}L_K. \quad \square$$

#### 4. SUPERGAUSSIAN ESTIMATES

In this section we use the technique introduced in [1] to prove Theorem 1.2, extending the interval in which the average of the distribution function behaves in a supergaussian way.

*Proof of Theorem 1.2.* First of all, notice that for any  $s > 0$  and any isotropic convex body  $K$  we have

$$\begin{aligned} \int_{S^{n-1}} M_\theta \left(\frac{1}{s}\right) d\sigma(\theta) &= \int_{S^{n-1}} \int_0^{\frac{1}{s}} \int_{K \cap \{|\langle x, \theta \rangle| \geq \frac{1}{t}\}} |\langle x, \theta \rangle| dx dt d\sigma(\theta) \\ &\leq \int_{S^{n-1}} \frac{1}{s} \int_{K \cap \{|\langle x, \theta \rangle| \geq s\}} |\langle x, \theta \rangle| dx d\sigma(\theta) \\ &\leq \int_{S^{n-1}} \frac{1}{s} L_K |\{x \in K : |\langle x, \theta \rangle| \geq s\}|^{\frac{1}{2}} d\sigma(\theta) \\ &\leq \frac{L_K}{s} \left( \int_{S^{n-1}} |\{x \in K : |\langle x, \theta \rangle| \geq s\}| d\sigma(\theta) \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, taking  $s = tL_K$ , we have that for any  $t > 0$ ,

$$\left( \int_{S^{n-1}} |\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| d\sigma(\theta) \right)^{\frac{1}{2}} \geq t \int_{S^{n-1}} M_\theta \left(\frac{1}{tL_K}\right) d\sigma(\theta).$$

Using the representation of the average of  $M_\theta$  as an integral on  $K$  (Proposition 2.3) and the lower bound obtained in the proof of Theorem 4.2 in [1], we obtain that for every positive  $t$ ,

$$t \int_{S^{n-1}} M_\theta \left(\frac{1}{tL_K}\right) d\sigma(\theta) \geq t \int_{K \setminus 2tL_K B_2^n} \frac{c_{n-1} \|x\|_2}{nw_n tL_K} e^{-\frac{Cn t^2 L_K^2}{\|x\|_2^2}} dx.$$

Using the small ball probability estimate from [25], there exists an absolute constant  $c_1$  such that  $|K \setminus c_1\sqrt{n}L_K| \geq \frac{1}{2}$ . Thus, if  $0 < t \leq \frac{c_1}{2}\sqrt{n}$ ,

$$\begin{aligned} \left( \int_{S^{n-1}} |\{x \in K : |\langle x, \theta \rangle| > tL_K\}| d\sigma(\theta) \right)^{\frac{1}{2}} &\geq \int_{K \setminus c_1\sqrt{n}L_K B_2^n} \frac{c_{n-1} \|x\|_2}{nw_n L_K} e^{-\frac{Cn t^2 L_K^2}{\|x\|_2^2}} dx \\ &\geq c_2 e^{-Ct^2} \geq e^{-c_3 t^2}, \end{aligned}$$

if  $1 \leq t \leq \frac{c_1}{2} \sqrt{n}$ . Thus, taking  $c = \max\{\frac{2}{c_1}, 2c_3\}$ , we obtain that for every isotropic convex body  $K$  and every  $1 \leq t \leq \frac{\sqrt{n}}{c}$ ,

$$\int_{S^{n-1}} |\{x \in K : |\langle x, \theta \rangle| > tL_K\}| d\sigma(\theta) \geq e^{-ct^2}. \quad \square$$

As a consequence, we obtain the following estimate of the measure of the set of directions verifying a supergaussian estimate for a particular  $t$ . However, opposite to what will happen for a subgaussian estimate, the estimate of the measure of this set of directions will be very small for big values of  $t$ .

**Corollary 4.1.** *There exist absolute constants  $c, c'$  such that for every isotropic symmetric convex body  $K \subseteq \mathbb{R}^n$  and for every  $1 \leq t \leq \frac{\sqrt{n}}{c}$ , the set of directions verifying the supergaussian estimate*

$$|\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| \geq e^{-c^2t^2}$$

has measure greater than  $e^{-c't^2}$ .

*Proof.* By Theorem 1.2

$$\int_{S^{n-1}} |\{x \in K : |\langle x, \theta \rangle| < tL_K\}| d\sigma(\theta) \leq 1 - e^{-c^2t^2}.$$

Thus, by Markov's inequality, we have

$$\sigma\left\{\theta \in S^{n-1} : |\{x \in K : |\langle x, \theta \rangle| < tL_K\}| > 1 - e^{-c'^2t^2}\right\} \leq \frac{1 - e^{-c^2t^2}}{1 - e^{-c'^2t^2}}.$$

Consequently,

$$\sigma\left\{\theta \in S^{n-1} : |\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| < e^{-c'^2t^2}\right\} \leq \frac{1 - e^{-c^2t^2}}{1 - e^{-c'^2t^2}},$$

and so

$$\begin{aligned} \sigma\left\{\theta \in S^{n-1} : |\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| \geq e^{-c'^2t^2}\right\} &\geq 1 - \frac{1 - e^{-c^2t^2}}{1 - e^{-c'^2t^2}} \\ &= \frac{e^{-c^2t^2}(1 - e^{-(c'^2 - c^2)t^2})}{1 - e^{-c'^2t^2}}. \end{aligned}$$

Choosing  $c'^2 = 2c^2$  we obtain the result. □

### 5. SUBGAUSSIAN ESTIMATES

In this section we will extend the interval in which the average of the distribution function verifies a subgaussian estimate. As a consequence, we will obtain an estimate of the measure of the directions verifying a subgaussian estimate in some interval.

*Proof of Theorem 1.3.* As we have seen in the proof of Theorem 1.1 for every  $\theta \in S^{n-1}$ , we have

$$M_\theta\left(\frac{1}{s}\right) \geq \frac{1}{2}|\{x \in K : |\langle x, \theta \rangle| \geq 2s\}|.$$

Thus, using the upper bound shown in [1]

$$F(t) \leq 2 \int_{S^{n-1}} M_\theta\left(\frac{2}{tL_K}\right) d\sigma(\theta) \leq \frac{4\omega_{n-1}}{n\omega_n tL_K} \int_K |x| e^{-\frac{(n-1)t^2 L_K^2}{2|x|^2}} dx.$$



By Paouris' concentration of measure result, there exist constants such that for every  $\gamma > c_1$ ,  $|K \setminus \gamma\sqrt{n}L_K B_2^n| \leq e^{-c_2\gamma\sqrt{n}}$ . Hence,

$$\begin{aligned} \int_K |x| e^{-\frac{(n-1)t^2 L_K^2}{2|x|^2}} dx &= \int_{K \cap \gamma\sqrt{n}L_K B_2^n} |x| e^{-\frac{(n-1)t^2 L_K^2}{2|x|^2}} dx + \int_{K \setminus \gamma\sqrt{n}L_K B_2^n} |x| e^{-\frac{(n-1)t^2 L_K^2}{2|x|^2}} dx \\ &\leq \gamma\sqrt{n}L_K e^{-\frac{ct^2}{\gamma^2}} + (n+1)L_K |K \setminus \gamma\sqrt{n}L_K B_2^n| \\ &\leq \gamma\sqrt{n}L_K e^{-\frac{ct^2}{\gamma^2}} + (n+1)L_K e^{-c_2\gamma\sqrt{n}}. \end{aligned}$$

Consequently,

$$\begin{aligned} F(t) &\leq \frac{C}{t} \left( \gamma e^{-\frac{ct^2}{\gamma^2}} + \sqrt{n} e^{-c_2\gamma\sqrt{n}} \right) \\ &\leq \frac{C}{t} \left( \gamma e^{-\frac{ct^2}{\gamma^2}} + e^{\sqrt{n}} e^{-c_2\gamma\sqrt{n}} \right) \\ &\leq \frac{C}{t} \left( \gamma e^{-\frac{ct^2}{\gamma^2}} + e^{(1-c_2\gamma)t^2} \right) \end{aligned}$$

if  $1 \leq t \leq n^{\frac{1}{4}}$  and  $\gamma$  is a constant big enough. Thus, we obtain that for  $t$  in the aforementioned interval

$$F(t) \leq \frac{C}{t} e^{-ct^2} \leq e^{-c't^2}.$$

Notice that if  $R(K) \leq C\sqrt{n}L_K$ , then we do not need to split the integral as the sum of two integrals and we obtain that

$$F(t) \leq \frac{C}{t} e^{-ct^2} \leq e^{-c't^2}$$

for every  $1 \leq t \leq C\sqrt{n}$ . □

As a consequence we will obtain that if we consider an interval  $t_0 \leq t \leq n^{\frac{1}{4}}$  with  $t_0$  big, then the measure of the directions that are subgaussian in such an interval will be big. It is explicitly stated in the following corollary.

**Corollary 5.1.** *Let  $K \subseteq \mathbb{R}^n$  be an isotropic symmetric convex body and let  $1 \leq t_0 \leq n^{\frac{1}{4}}$ . Then, the set of directions  $\theta \in S^{n-1}$  that verify*

$$|\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| \leq e^{-c^2 t^2} \text{ for every } t_0 \leq t \leq n^{\frac{1}{4}}$$

*has measure greater than  $1 - e^{-c'^2 t_0^2}$ .*

*Furthermore, if  $K$  has small diameter, we can take  $t_0 \leq C\sqrt{n}$  and  $t_0 \leq t \leq C\sqrt{n}$ .*

*Proof.* Applying Markov's inequality in Theorem 1.3, we have that for every  $1 \leq t \leq n^{\frac{1}{4}}$ ,

$$\sigma\{\theta \in S^{n-1} : |\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| \geq e^{-c'^2 t^2}\} \leq e^{-(c^2 - c'^2)t^2}.$$

Taking  $c'^2 = \frac{1}{2}c^2$  we have that there exists a constant  $c_0$  such that for every  $1 \leq t \leq n^{\frac{1}{4}}$ ,

$$\sigma\{\theta \in S^{n-1} : |\{x \in K : |\langle x, \theta \rangle| \geq tL_K\}| \geq e^{-c_0^2 t^2}\} \leq e^{-c_0^2 t^2}.$$

Now take  $t_0 < t_1 < \dots < t_I = n^{\frac{1}{4}}$ . Then,

$$\sigma\{\theta \in S^{n-1} : \exists 0 \leq i \leq I \text{ with } |\{x \in K : |\langle x, \theta \rangle| \geq t_i L_K\}| \geq e^{-c_0^2 t_i^2}\} \leq \sum_{i=0}^I e^{-c_0^2 t_i^2}.$$

Taking  $t_i^2 = t_0^2 + \lambda i$ , this probability is bounded by  $e^{-c_0^2 t_0^2 \frac{1}{1-e^{-c_0^2 \lambda}}}$ . If for every  $i$  we have

$$|\{x \in K : |\langle x, \theta \rangle| \geq t_i L_K\}| \leq e^{-c_0^2 t_i^2},$$

then for every  $t_i \leq t \leq t_{i+1}$  we have

$$\begin{aligned} |\{x \in K : |\langle x, \theta \rangle| \geq t L_K\}| &\leq |\{x \in K : |\langle x, \theta \rangle| \geq t_i L_K\}| \leq e^{-c_0^2 t_i^2} \\ &= e^{-c_0^2 t^2} e^{c_0^2 (t^2 - t_i^2)} \leq e^{-c_0^2 t^2} e^{c_0^2 (t_{i+1}^2 - t_i^2)} \\ &= e^{-c_0^2 t^2} e^{c_0^2 \lambda}. \end{aligned}$$

Choosing  $\lambda$ , a constant smaller than 1, we obtain the result.

If  $K$  has small diameter, the same proof works in the interval  $t_0 \leq t \leq C\sqrt{n}$  with  $1 \leq t_0 \leq C\sqrt{n}$ .  $\square$

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