PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 2, February 2015, Pages 749–754
S 0002-9939(2014)12426-9
Article electronically published on October 10, 2014

ENTROPY CRITERIA AND STABILITY OF EXTREME SHOCKS: A REMARK ON A PAPER OF LEGER AND VASSEUR

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(Communicated by Walter Craig)

Abstract. We show that a relative entropy condition recently shown by Leger and Vasseur to imply uniqueness and stable L^2 dependence on initial data of Lax 1- or n-shock solutions of an $n \times n$ system of hyperbolic conservation laws with convex entropy implies Lopatinski stability in the sense of Majda. This means in particular that Leger and Vasseur's relative entropy condition represents a considerable improvement over the standard entropy condition of decreasing shock strength and increasing entropy along forward Hugoniot curves, which, in a recent example exhibited by Barker, Freistühler and Zumbrun, was shown to fail to imply Lopatinski stability, even for systems with convex entropy. This observation bears also on the parallel question of existence, at least for small BV or H^s perturbations.

1. Introduction

In this brief note, we examine for extreme Lax shock solutions of a system of conservation laws

$$u_t + f(u)_x = 0,\ u \in \mathbb{R}^n, \tag{1.1}$$

possessing a convex entropy

$$\eta, \quad P := \nabla^2 \eta > 0, \quad \nabla_u \eta \nabla_u f = \nabla_u q, \tag{1.2}$$

the relation between Lopatinski stability in the sense of Erpenbeck and Majda [Er, M1, M2, M3] and a relative entropy criterion introduced recently by Leger and Vasseur [LV].

A number of entropy criteria have been proposed over the years to distinguish physically admissible or stable shock waves. Some of the oldest [B, W] are decrease of characteristic speed (compressivity) or increase in entropy across the shock, or their instantaneous equivalents: monotone decrease of shock speed or increase of entropy along the forward 1-Hugoniot curve from a fixed left state. All of these conditions coincide for small-amplitude waves [Sm], agreeing also with the property of stability, or well-posedness of the shock solution with respect to nearby initial data, as determined by nonvanishing of a certain Lopatinski determinant [La, Er,

Received by the editors May 20, 2013.

2010 *Mathematics Subject Classification.* Primary 35L65, 35L67, 35B35.

The first author's research was partially supported by the Project "Instabilities in Hydrodynamics" funded by the Mairie de Paris (under the "Emergences" program) and the Fondation Sciences Mathématiques de Paris.

The second author was partially supported under NSF grants no. DMS-0300487 and DMS-0801745.

M1, M2, M3, Da]. However, for large-amplitude waves, the relations between these different criteria are unclear.

The related concept of entropy dissipation $\eta_t - q_x \leq 0$ in the sense of measures at the shock, or $[\eta] - \sigma[q] \leq 0$ for shock speed σ, is justified as a necessary condition for the physicality criterion of *admissibility*. Here we are thinking of admissibility in the sense of Lax, for small-amplitude waves, under the assumption of genuine nonlinearity and strictly convex entropy ([La, Theorem 5.7]), and also admissibility in the sense of existence of a nearby viscous shock profile, for arbitrary-amplitude waves, under the assumption of an associated entropy-compatible viscosity [K, KS].

The property of entropy dissipation may be seen to follow from monotone decrease of shock speed along the Hugoniot curve ([LV, Lemma 3]; see also [La]) making a link (if only one way) between large-amplitude admissibility and the more classical monotonicity condition defined originally in a small-amplitude context.

Most recently, Leger and Vasseur have introduced a *relative entropy condition* pertaining to arbitrary-amplitude waves.[1] Specifically, assuming L^∞ boundedness and a strong trace property on solutions,[2] these authors establish, for systems of conservation laws possessing a convex entropy, uniqueness and stable dependence in L^2 on initial data for perturbations of extreme Lax shock waves (without loss of generality 1-shocks) of arbitrary amplitude, under the conditions that

(i) shock speed is nonincreasing along the forward Hugoniot curve from a fixed left state; explicitly $\sigma'(s) \leq 0$ for $s \geq 0$, with notation introduced in Section 2 below, and

(ii) relative entropy (defined in (2.1)) is nondecreasing with respect to the right state along the forward Hugoniot curve; explicitly $d_s\eta(u|S_u(s)) \geq 0$ for $s \geq 0$.

The result of Leger and Vasseur is nominally an a priori short-time stability estimate and leaves open the question of existence, even for more standard perturbations that are small in BV or H^s.

The purpose of the present note is to verify that conditions (i)-(ii) imply satisfaction of the Lopatinski condition of Majda [M1, M2, M3, Le1]. Indeed, we show that satisfaction of the Lopatinski condition follows from the much weaker conditions that (i) and (ii) hold only for the single value $s = s_+$ corresponding to the right endstate; see (i′)-(ii′) below.

This observation immediately yields, by the existing theory of [Le1, M1, M2, M3], existence and stability for the classes of small BV or H^s perturbations under conditions (i′)-(ii′). One might hope that it could eventually be also of use in constructing approximate solutions and ultimately the demonstration of existence in the much more delicate L^2/strong trace setting considered in [LV].

We remark that Lopatinski stability is necessary for the physicality condition of *viscous stability*, or stability of an associated viscous profile [ZS].

Remark 1.1. From the above discussion, (i) is sufficient for entropy dissipation, which is necessary for the physicality condition of admissibility, and (i′)-(ii′) are sufficient for Lopatinski stability, which is necessary for the physicality condition of viscous stability. As strict entropy dissipation and Lopatinski stability are open conditions, whereas (i) and (i′)-(ii′), as nonstrict monotonicity conditions, are closed,

[1] See also an earlier analysis by Leger [Leg], proving L^2-contractivity of entropy solutions in the scalar case.

[2] Satisfied by BV solutions.

it is evident that (i) *and* (i′)-(ii′), *are sufficient but not necessary* for strict entropy dissipation and Lopatinski stability, respectively. For, a closed condition holds at the boundary of its region of satisfaction, whereupon an implied open condition must therefore hold at some point outside.

Remark 1.2. An examination of the argument of [LV] reveals that their hypothesis (ii) may be weakened to[3]

$$\text{(ii}^*\text{)} \quad (s-s_+)\big(\eta(u|S_u(s)) - \eta(u|S_u(s_+))\big) \geq 0, \quad \text{for } s \geq 0,$$

with notation introduced in Section 2, where $(u, S_u(s_+), \sigma(s_+))$ is the fixed 1-shock under consideration. Evidently, (ii*) implies our condition (ii′) stated in Assumption II below.

2. Definitions and result

Let η, q be a convex entropy/entropy flux pair. Then [La], for $P = \nabla^2\eta$, $A := \nabla f$, P is symmetric positive definite and PA is symmetric. Thus, A is self-adjoint with respect to the inner product induced by P, and so the eigenvectors of A corresponding to distinct eigenvalues are P-orthogonal. The *relative entropy* $\eta(u|v)$ is defined following [D, LV] as

$$\eta(u|v) := \eta(u) - \eta(v) - \nabla\eta(v)(u-v). \tag{2.1}$$

We assume that A is strictly hyperbolic,[4] with eigenvalues $a_1 < a_2 < \cdots < a_n$, and associated eigenvectors $r_1, r_2, \ldots, r_n$.

Assumptions I. *For a given left state u, suppose that there is a well-defined C^1 1-Hugoniot curve of states $S_u(s)$ and associated speeds $\sigma(s)$, $s \geq 0$, satisfying*

$$\sigma(s)(S_u(s) - u) = f(S_u(s)) - f(u), \tag{2.2}$$

with $S_u(0) = u$, $\sigma(0) = a_1(u)$. More precisely, assume that condition (2.2) *is everywhere full rank, with the linearized equations*

$$\sigma'(s)(S_u(s) - u) = \big(A(S_u(s)) - \sigma(s)\big)S_u'(s) \tag{2.3}$$

(hence, by the Implicit Function Theorem, also the nonlinear equations (2.2)*) uniquely solvable (up to a constant multiplier) for $(\sigma'(s), S_u'(s))$. Moreover, assume that the resulting discontinuity is a Lax* 1*-shock, in the sense that*

$$a_1(u) > \sigma(s) \quad \text{and} \quad a_1(S_u(s)) < \sigma(s) < a_2(S_u(s)) < a_3(S_u(s)) < \cdots < a_n(S_u(s)). \tag{2.4}$$

The *Lopatinski* (stability) *condition* for the shock $(u, S_u(s), \sigma(s))$, with notation introduced in Assumptions I above, is

$$\det\big((S_u(s) - u) \quad r_2(S_u(s)) \quad \ldots \quad r_n(S_u(s))\big) \neq 0. \tag{2.5}$$

This may be recognized as the condition that the Riemann problem be well-posed for data near $(u, S_u(s))$, more precisely, that the Jacobian of the associated Lax wave-map be full rank [La, Sm].

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[3]Specifically, in the course of the proof of Theorem 3 in [LV], assumption (ii) is used only in the key Lemma 4, which is invoked only in Lemma 8, in 1-3 page 291, where the weakened form (ii*) is used, not for all $s \geq 0$ but only in an interval $0 \leq s \leq s_+ + C$ (in their notation, $0 \leq s \leq s_u$), where $C > 0$ is determined by the L^∞ bound assumed on solutions.

[4]For Theorem 2.2 to hold, we only need strict hyperbolicity to hold at the right endstate $S_u(s_+)$ introduced in Assumptions I.

Condition (2.5) is a crucial building block both (through resulting a priori stability estimates) for the small H^s-perturbation existence/stability theory of Majda [M1, M2, M3, Me] and (through direct construction based on Riemann solutions) in the small-BV perturbation existence/stability theory of Lewicka and others [Le1, Le2, C] in the vicinity of a single large-amplitude shock.

Assumptions II.

(i′) $\sigma'(s_+) \le 0$,

(ii′) $d_s\eta(u|S_u(s_+)) \ge 0$, *for shock* $(u, S_u(s_+), \sigma(s_+))$, $s_+ \ge 0$.

Lemma 2.1. *Condition* (ii′) *is equivalent to*

$$\langle S_u'(s_+), \nabla^2\eta(S_u(s_+))(S_u(s_+) - u)\rangle \ge 0. \tag{2.6}$$

Proof. Differentiating (2.1), we have

$$\begin{aligned} d_s\eta(u|S_u(s)) &= d_s\big[\eta(u) - \eta(S_u(s)) - \nabla\eta(S_u(s))(u - S_u(s))\big] \\ &= -\nabla\eta(S_u(s))S_u'(s) - \big[\big\langle S_u'(s), \nabla^2\eta(S_u(s))(u - S_u(s))\big\rangle - \nabla\eta(S_u(s))S_u'(s)\big] \\ &= -\big\langle S_u'(s), \nabla^2\eta(S_u(s))(u - S_u(s))\big\rangle, \end{aligned}$$

whence the assertion follows. □

Theorem 2.2. *Under Assumptions* I *and* II *Lopatinski condition* (2.5) *holds for* $(u, S_u(s_+), \sigma(s_+))$.

Proof. Failure of (2.5) implies that

$$S_u(s_+) - u = \sum_{2\le j\le n} \alpha_j r_j^+, \qquad r_j^+ := r_j(S_u(s_+)),$$

for some $\alpha_j \in \mathbb{R}$, which are not all equal to zero, since we may assume $s_+ > 0$, $S_u(s_+) \neq u$. Then, by Assumptions I,

$$(A_+ - \sigma(s_+))S_u'(s_+) = \sigma'(s_+) \sum_{2\le j\le n} \alpha_j r_j^+, \qquad A_+ := A(S_u(s_+)).$$

Hence, inverting $A_+ - \sigma(s_+)$ (as we may by (2.4)):

$$S_u'(s_+) = \sigma'(s_+) \sum_{2\le j\le n} \beta_j r_j^+, \qquad \beta_j := (a_j(S_u(s_+)) - \sigma(s_+))^{-1}\alpha_j.$$

We remark that, since $(u, S_u(s_+), \sigma(s_+))$ is a 1-shock (2.4), there holds $\beta_j\alpha_j \ge 0$, and, since the shock is nontrivial, $\beta_j\alpha_j > 0$ for at least one j. Hence, by $\sigma'(s_+) \neq 0$ (a consequence of Assumptions I), positive definiteness of P, and the fact that the eigenvectors of A are P-orthogonal, we deduce

$$\big\langle S_u'(s_+), P_+(A_+ - \sigma(s_+))S_u'(s_+)\big\rangle = \sigma'(s_+)^2 \sum_{2\le j\le n} \beta_j\alpha_j \big\langle r_j^+, P_+ r_j^+\big\rangle > 0, \tag{2.7}$$

with notation $P_+ := P(S_u(s_+)) = \nabla^2\eta(S_u(s_+))$. However, Assumptions II and Lemma 2.1 imply

$$\sigma'(s_+)\big\langle S_u'(s_+), P_+(S_u(s_+) - u)\big\rangle \le 0,$$

so that, by (2.3),

$$\big\langle S_u'(s_+), P_+(A_+ - \sigma(s_+))S_u'(s_+)\big\rangle \le 0,$$

in contradiction with (2.7). □

3. Discussion and open problems

The corresponding absolute entropy conditions that shock strength is decreasing and absolute entropy $\eta(S_u(s))$ is increasing along the forward Hugoniot curve have been shown [B] to hold *globally* for very general gas dynamical equations of state. However, recently, Barker, Freistühler, and Zumbrun [BFZ] have shown by explicit example that there exist systems satisfying these conditions and also possessing a convex entropy, but for which nonetheless *the Lopatinski condition can fail*. Thus, the relative entropy condition represents a considerable sharpening of the older absolute entropy condition.

An interesting open problem would be to find an analog of this condition for intermediate shocks; however, we see no obvious candidate for this. Certainly, the approach of Theorem 2.2 breaks down, since there is no relation between $P(u)$- and $P(S_u(s))$-orthogonality. An interesting, but more speculative, problem would be to make use of the Lopatinski condition to construct approximate solutions in the small L^2-perturbation class, towards an eventual possible small L^2/strong trace class existence theory. It would be extremely interesting, of course, to find some analog also for the corresponding viscous shock stability problem, whether directly as in [LV], or indirectly as here through the study of spectral stability and the linearized eigenvalue problem.

Acknowledgements

The second author thanks the Ecole Normale Supérieure, Ulm, and coordinating host Thomas Alazard for their hospitality during the visit in which this work was carried out. Both authors thank the anonymous referee for helpful comments regarding the presentation.

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