

DYNAMIC LITTLEWOOD-TYPE INEQUALITIES

RAVI AGARWAL, MARTIN BOHNER, AND SAMIR SAKER

(Communicated by Sergei K. Suslov)

ABSTRACT. In this paper, we present some dynamic inequalities on time scales. As special cases, these results contain and improve some integral inequalities and some discrete inequalities formulated by Littlewood in connection with some work on the general theory of orthogonal series.

1. INTRODUCTION

One of the inequalities formulated by Littlewood [11, page 152] in connection with some work on the general theory of orthogonal series is given by

$$(1.1) \quad \sum_{n=1}^{\infty} a_n A_n^2 \left(\sum_{k=n}^{\infty} a_k^{3/2} \right)^2 \leq K \sum_{n=1}^{\infty} a_n^2 A_n^4, \quad \text{where } A_n := \sum_{k=1}^n a_k.$$

Littlewood posed the problem to decide whether an absolute constant K exists so that the inequality (1.1) holds for any nonnegative sequence $\{a_n\}$. Bennett in [4] solved this problem and proved that (1.1) holds with $K = 4$. In particular, Bennett [4] proved the inequality

$$(1.2) \quad \sum_{n=1}^{\infty} a_n^p A_n^q \left(\sum_{k=n}^{\infty} a_k^{1+p/q} \right)^q \leq \left(\frac{2pq - q}{p} \right)^q \sum_{n=1}^{\infty} a_n^{2p} A_n^{2q}, \quad \text{where } p, q \geq 1,$$

which contains (1.1) as a special case when $p = 1$ and $q = 2$. Gao [8] improved the constant $K = 4$ for the case $p = 1$ and $q = 2$ to $K = \sqrt{6}$. For the case $p = 2$ and $q = 1$, Bennett's constant $K = 3/2$ was improved by Gong [9] to $K = \sqrt[3]{2}$. In this paper, we prove four new dynamic inequalities of type (1.2). The results are given on a general time scale \mathbb{T} , i.e., a nonempty closed subset of the real numbers. If $\mathbb{T} = \mathbb{N}_0$, we get as a special case (1.2) and improvements of (1.2); e.g., one result allows for an improved constant $K = 4/9$ for $p = 1$ and $q = 2$ and for an improved constant $K = 1$ for $p = 2$ and $q = 1$. If $\mathbb{T} = \mathbb{R}$, we get as a special case two new integral inequalities. Any other time scale \mathbb{T} such as $\mathbb{T} = h\mathbb{N}_0$ for $h > 0$ or $\mathbb{T} = w^{\mathbb{N}_0}$ for $\omega > 1$ also leads to new dynamic inequalities, and some examples are given in this paper. For related dynamic inequalities on time scales, we refer the reader to the papers [1–3, 5, 12–15]. For the general theory of time scales, we refer the reader to [6, 7, 10].

The setup of this paper is as follows. In Section 2, we present the four main results. Section 3 gives examples, applications and remarks. Sections 4–6 are

Received by the editors April 28, 2013.

2010 *Mathematics Subject Classification*. Primary 26D10, 26D15, 34A40, 34N05, 39A12, 39A13.

Key words and phrases. Littlewood's inequality, time scales, dynamic inequality.

devoted to the development of auxiliary inequalities that are proved using the time scales Hölder inequality, the time scales chain rule, and the time scales integration by parts formulas, respectively. Section 7 presents the proofs of the main results. Finally, we give two further Littlewood type inequalities in Section 8.

2. MAIN RESULTS

The main results in this paper are the following four theorems. Here and throughout we assume that \mathbb{T} is a time scale which is unbounded above. Throughout, all occurring improper integrals are assumed to be finite.

Theorem 2.1. *Let $t_0 \in \mathbb{T}$ and $p, q \geq 1$. Assume $a \in C_{rd}(\mathbb{T}, [0, \infty))$ and define*

$$(2.1) \quad A(t) := \int_{t_0}^t a(\tau)\Delta\tau \quad \text{and} \quad \tilde{B}(t) := \int_t^\infty a^{1+p/q}(\tau)\Delta\tau.$$

Then

$$(2.2) \quad \int_{t_0}^\infty a^p(t)A^q(\sigma(t))\tilde{B}^q(t)\Delta t \leq \left(\frac{2pq - q}{p}\right)^q \int_{t_0}^\infty [a^p(t)A^q(\sigma(t))]^2 \Delta t.$$

Theorem 2.2. *Let $t_0 \in \mathbb{T}$ and $p, q \geq 1$. Assume $a \in C_{rd}(\mathbb{T}, [0, \infty))$ and define*

$$(2.3) \quad A(t) := \int_{t_0}^t a(\tau)\Delta\tau \quad \text{and} \quad \tilde{C}(t) := \int_t^\infty \left(\frac{A(\tau)}{A(\sigma(\tau))}\right)^{1+q/p} a^{1+p/q}(\tau)\Delta\tau.$$

Then

$$(2.4) \quad \int_{t_0}^\infty a^p(t)A^q(t)\tilde{C}^q(t)\Delta t \leq \left(\frac{2pq - q}{p + q}\right)^q \int_{t_0}^\infty [a^p(t)A^q(t)]^2 \Delta t.$$

Theorem 2.3. *Let $t_0 \in \mathbb{T}$ and $p, q \geq 1$. Assume $a \in C_{rd}(\mathbb{T}, [0, \infty))$ and define*

$$(2.5) \quad \tilde{A}(t) := \int_t^\infty a(\tau)\Delta\tau \quad \text{and} \quad B(t) := \int_{t_0}^t a^{1+p/q}(\tau)\Delta\tau.$$

Then

$$(2.6) \quad \int_{t_0}^\infty a^p(t)\tilde{A}^q(t)B^q(\sigma(t))\Delta t \leq \left(\frac{2pq - q}{p}\right)^q \int_{t_0}^\infty [a^p(t)\tilde{A}^q(t)]^2 \Delta t.$$

Theorem 2.4. *Let $t_0 \in \mathbb{T}$ and $p, q \geq 1$. Assume $a \in C_{rd}(\mathbb{T}, [0, \infty))$ and define*

$$(2.7) \quad \tilde{A}(t) := \int_t^\infty a(\tau)\Delta\tau \quad \text{and} \quad C(t) := \int_{t_0}^t \left(\frac{\tilde{A}(\sigma(\tau))}{\tilde{A}(\tau)}\right)^{1+q/p} a^{1+p/q}(\tau)\Delta\tau.$$

Then

$$(2.8) \quad \int_{t_0}^\infty a^p(t)\tilde{A}^q(\sigma(t))C^q(\sigma(t))\Delta t \leq \left(\frac{2pq - q}{p + q}\right)^q \int_{t_0}^\infty [a^p(t)\tilde{A}^q(\sigma(t))]^2 \Delta t.$$

3. EXAMPLES

Since the cases $p = 1, q = 2$ and $p = 2, q = 1$ have been discussed in the literature, we collect the results of Theorems 2.1–2.4 for these two cases in the following two corollaries.

Corollary 3.1 (Results for $p = 1$ and $q = 2$). *Let $t_0 \in \mathbb{T}$, $a \in C_{rd}(\mathbb{T}, [0, \infty))$ and*

$$\begin{aligned}
 A(t) &:= \int_{t_0}^t a(\tau) \Delta\tau, & \tilde{A}(t) &:= \int_t^\infty a(\tau) \Delta\tau, \\
 B(t) &:= \int_{t_0}^t a^{3/2}(\tau) \Delta\tau, & \tilde{B}(t) &:= \int_t^\infty a^{3/2}(\tau) \Delta\tau, \\
 C(t) &:= \int_{t_0}^t \left(\frac{\tilde{A}(\sigma(\tau))}{\tilde{A}(\tau)} \right)^3 a^{3/2}(\tau) \Delta\tau, & \tilde{C}(t) &:= \int_t^\infty \left(\frac{A(\tau)}{A(\sigma(\tau))} \right)^3 a^{3/2}(\tau) \Delta\tau.
 \end{aligned}$$

Then we have

$$\begin{aligned}
 \int_{t_0}^\infty a(t) A^2(\sigma(t)) \tilde{B}^2(t) \Delta t &\leq 4 \int_{t_0}^\infty a^2(t) A^4(\sigma(t)) \Delta t, \\
 \int_{t_0}^\infty a(t) A^2(t) \tilde{C}^2(t) \Delta t &\leq \frac{4}{9} \int_{t_0}^\infty a^2(t) A^4(t) \Delta t, \\
 \int_{t_0}^\infty a(t) \tilde{A}^2(t) B^2(\sigma(t)) \Delta t &\leq 4 \int_{t_0}^\infty a^2(t) \tilde{A}^4(t) \Delta t, \\
 \int_{t_0}^\infty a(t) \tilde{A}^2(\sigma(t)) C^2(\sigma(t)) \Delta t &\leq \frac{4}{9} \int_{t_0}^\infty a^2(t) \tilde{A}^4(\sigma(t)) \Delta t.
 \end{aligned}$$

Corollary 3.2 (Results for $p = 2$ and $q = 1$). *Let $t_0 \in \mathbb{T}$, $a \in C_{rd}(\mathbb{T}, [0, \infty))$ and*

$$\begin{aligned}
 A(t) &:= \int_{t_0}^t a(\tau) \Delta\tau, & \tilde{A}(t) &:= \int_t^\infty a(\tau) \Delta\tau, \\
 B(t) &:= \int_{t_0}^t a^3(\tau) \Delta\tau, & \tilde{B}(t) &:= \int_t^\infty a^3(\tau) \Delta\tau, \\
 C(t) &:= \int_{t_0}^t \left(\frac{\tilde{A}(\sigma(\tau))}{\tilde{A}(\tau)} \right)^{3/2} a^3(\tau) \Delta\tau, & \tilde{C}(t) &:= \int_t^\infty \left(\frac{A(\tau)}{A(\sigma(\tau))} \right)^{3/2} a^3(\tau) \Delta\tau.
 \end{aligned}$$

Then we have

$$(3.1) \quad \int_{t_0}^\infty a^2(t) A(\sigma(t)) \tilde{B}(t) \Delta t \leq \frac{3}{2} \int_{t_0}^\infty a^4(t) A^2(\sigma(t)) \Delta t,$$

$$(3.2) \quad \int_{t_0}^\infty a^2(t) A(t) \tilde{C}(t) \Delta t \leq \int_{t_0}^\infty a^4(t) A^2(t) \Delta t,$$

$$(3.3) \quad \int_{t_0}^\infty a^2(t) \tilde{A}(t) B(\sigma(t)) \Delta t \leq \frac{3}{2} \int_{t_0}^\infty a^4(t) \tilde{A}^2(t) \Delta t,$$

$$(3.4) \quad \int_{t_0}^\infty a^2(t) \tilde{A}(\sigma(t)) C(\sigma(t)) \Delta t \leq \int_{t_0}^\infty a^4(t) \tilde{A}^2(\sigma(t)) \Delta t.$$

Now we summarize the main results for various time scales, starting with $\mathbb{T} = \mathbb{N}_0$.

Corollary 3.3 (Results for $\mathbb{T} = \mathbb{N}_0$). *Assume $\{a_n\}_{n \in \mathbb{N}}$ is a real-valued nonnegative sequence and define*

$$\begin{aligned}
 A_n &:= \sum_{k=0}^n a_k, & \tilde{A}_n &:= \sum_{k=n}^\infty a_k, & B_n &:= \sum_{k=0}^n a_k^{1+p/q}, & \tilde{B}_n &:= \sum_{k=n}^\infty a_k^{1+p/q}, \\
 C_n &:= \sum_{k=0}^n \left(\frac{\tilde{A}_{k+1}}{\tilde{A}_k} \right)^{1+q/p} a_k^{1+p/q}, & \tilde{C}_n &:= \sum_{k=n}^\infty \left(\frac{A_{k-1}}{A_k} \right)^{1+q/p} a_k^{1+p/q}.
 \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{k=0}^{\infty} a_k^p A_k^q \tilde{B}_k^q &\leq \left(\frac{2pq - q}{p}\right)^q \sum_{k=0}^{\infty} a_k^{2p} A_k^{2q}, \\ \sum_{k=1}^{\infty} a_k^p A_{k-1}^q \tilde{C}_k^q &\leq \left(\frac{2pq - q}{p + q}\right)^q \sum_{k=1}^{\infty} a_k^{2p} A_{k-1}^{2q}, \\ \sum_{k=0}^{\infty} a_k^p \tilde{A}_k^q B_k^q &\leq \left(\frac{2pq - q}{p}\right)^q \sum_{k=0}^{\infty} a_k^{2p} \tilde{A}_k^{2q}, \\ \sum_{k=0}^{\infty} a_k^p \tilde{A}_{k+1}^q C_k^q &\leq \left(\frac{2pq - q}{p + q}\right)^q \sum_{k=0}^{\infty} a_k^{2p} \tilde{A}_{k+1}^{2q}. \end{aligned}$$

Note that for $\mathbb{T} = \mathbb{R}$, we have $A = A^\sigma$ and hence $B = C$ and $\tilde{B} = \tilde{C}$. So the first two inequalities and the last two inequalities are the same, up to the constant. Hence we only summarize below the inequalities with the “better” constant.

Corollary 3.4 (Results for $\mathbb{T} = \mathbb{R}$). *Let $t_0 \in \mathbb{R}$. Assume $a : \mathbb{R} \rightarrow [0, \infty)$ is continuous and define*

$$\begin{aligned} A(t) &:= \int_{t_0}^t a(\tau) d\tau, & \tilde{A}(t) &:= \int_t^\infty a(\tau) d\tau, \\ B(t) &:= \int_{t_0}^t a^{1+p/q}(\tau) d\tau, & \tilde{B}(t) &:= \int_t^\infty a^{1+p/q}(\tau) d\tau. \end{aligned}$$

Then we have

$$\begin{aligned} \int_{t_0}^\infty a^p(t) A^q(t) \tilde{B}^q(t) dt &\leq \left(\frac{2pq - q}{p + q}\right)^q \int_{t_0}^\infty a^{2p}(t) A^{2q}(t) dt, \\ \int_{t_0}^\infty a^p(t) \tilde{A}^q(t) B^q(t) dt &\leq \left(\frac{2pq - q}{p + q}\right)^q \int_{t_0}^\infty a^{2p}(t) \tilde{A}^{2q}(t) dt. \end{aligned}$$

Finally, we let $\omega > 1$ and give Theorem 2.1 and Theorem 2.3 for the case $\mathbb{T} = \omega^{\mathbb{N}_0}$. This application yields Littlewood type inequalities for weighted sums.

Corollary 3.5 (Results for $\mathbb{T} = \omega^{\mathbb{N}_0}$). *Let $\omega > 1$. Assume $\{a_n\}_{n \in \mathbb{N}_0}$ is a real-valued nonnegative sequence and define*

$$\begin{aligned} A_n &:= \sum_{k=0}^n \omega^k a_k, & \tilde{A}_n &:= \sum_{k=n}^\infty \omega^k a_k, \\ B_n &:= \sum_{k=0}^n \omega^k a_k^{1+p/q}, & \tilde{B}_n &:= \sum_{k=n}^\infty \omega^k a_k^{1+p/q}. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{k=0}^{\infty} \omega^k a_k^p A_k^q \tilde{B}_k^q &\leq \left(\frac{2pq - q}{p}\right)^q \sum_{k=0}^{\infty} \omega^k a_k^{2p} A_k^{2q}, \\ \sum_{k=0}^{\infty} \omega^k a_k^p \tilde{A}_k^q B_k^q &\leq \left(\frac{2pq - q}{p}\right)^q \sum_{k=0}^{\infty} \omega^k a_k^{2p} \tilde{A}_k^{2q}. \end{aligned}$$

4. AN INEQUALITY BASED ON HÖLDER’S INEQUALITY

We start by giving the time scales Hölder inequality [6, Theorem 6.13].

Theorem 4.1 (Time scales Hölder inequality). *Let $r > 1$ and $r^* = r/(r - 1)$. Let $a, b \in \mathbb{T}$. If $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, then*

$$(4.1) \quad \int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^r \Delta t \right\}^{\frac{1}{r}} \left\{ \int_a^b |g(t)|^{r^*} \Delta t \right\}^{\frac{1}{r^*}}.$$

Now we prove an inequality based on a double application of Hölder’s inequality (4.1). This inequality will be used several times in the proofs of our main results. Although this inequality was found by trying to prove the main results of this paper, it is also an interesting result in its own right.

Theorem 4.2. *Let $p, q \geq 1$ and define*

$$(4.2) \quad r := \frac{2qp - q}{p}.$$

Assume $\varphi, \psi \in C_{rd}(\mathbb{T}, [0, \infty))$ and $I \subset \mathbb{T}$ is an interval. If there exists a constant $\theta > 0$ such that

$$(4.3) \quad \int_I \varphi(t)\psi^r(t)\Delta t \leq \theta \int_I \varphi^{1+p/q}(t)\psi^{r-1}(t)\Delta t,$$

then

$$(4.4) \quad \int_I \varphi^p(t)\psi^q(t)\Delta t \leq \theta^q \int_I \varphi^{2p}(t)\Delta t.$$

Proof. First note that our assumptions imply that $r \geq 1$. Since $r = 1$ iff $p = q = 1$ in which case (4.3) implies (4.4) trivially, we may assume $r > 1$. Defining

$$r^* := \frac{r}{r - 1} \quad \text{so that} \quad \frac{1}{r^*} + \frac{2p}{r} = 1 + \frac{p}{q},$$

we may use (4.3) and Hölder’s inequality (4.1) with conjugate exponents r and r^* to obtain

$$\begin{aligned} \int_I \varphi(t)\psi^r(t)\Delta t &\leq \theta \int_I \varphi^{1+p/q}(t)\psi^{r-1}(t)\Delta t \\ &= \theta \int_I \left[\varphi^{1/r^*}(t)\psi^{r-1}(t) \right] \varphi^{2p/r}(t)\Delta t \\ &\leq \theta \left\{ \int_I \left[\varphi^{1/r^*}(t)\psi^{r-1}(t) \right]^{r^*} \Delta t \right\}^{\frac{1}{r^*}} \left\{ \int_I \left[\varphi^{2p/r}(t) \right]^r \Delta t \right\}^{\frac{1}{r}} \\ &= \theta \left\{ \int_I \varphi(t)\psi^r(t)\Delta t \right\}^{\frac{1}{r^*}} \left\{ \int_I \varphi^{2p}(t)\Delta t \right\}^{\frac{1}{r}}, \end{aligned}$$

i.e.,

$$\left\{ \int_I \varphi(t)\psi^r(t)\Delta t \right\}^{1 - \frac{1}{r^*}} \leq \theta \left\{ \int_I \varphi^{2p}(t)\Delta t \right\}^{\frac{1}{r}},$$

i.e.,

$$(4.5) \quad \int_I \varphi(t)\psi^r(t)\Delta t \leq \theta^r \int_I \varphi^{2p}(t)\Delta t.$$

Now we define

$$\gamma := \frac{1}{2p-1} \quad \text{so that} \quad \frac{2p\gamma}{1+\gamma} = 1 \quad \text{and} \quad \frac{2q}{1+\gamma} = r.$$

Note that our assumptions imply $0 < \gamma \leq 1$. Since $\gamma = 1$ iff $p = 1$ in which case $r = q$ and (4.5) implies (4.4) trivially, we may assume $0 < \gamma < 1$. Thus we may use Hölder’s inequality (4.1) again, this time with conjugate exponents $2/(1-\gamma)$ and $2/(1+\gamma)$, to arrive at

$$\begin{aligned} \int_I \varphi^p(t)\psi^q(t)\Delta t &= \int_I \varphi^{p(1-\gamma)}(t) [\varphi^{p\gamma}(t)\psi^q(t)] \Delta t \\ &\leq \left\{ \int_I [\varphi^{p(1-\gamma)}(t)]^{\frac{2}{1-\gamma}} \Delta t \right\}^{\frac{1-\gamma}{2}} \left\{ \int_I [\varphi^{p\gamma}(t)\psi^q(t)]^{\frac{2}{1+\gamma}} \Delta t \right\}^{\frac{1+\gamma}{2}} \\ &= \left\{ \int_I \varphi^{2p}(t)\Delta t \right\}^{\frac{1-\gamma}{2}} \left\{ \int_I \varphi(t)\psi^r(t)\Delta t \right\}^{\frac{1+\gamma}{2}} \\ &\leq \theta^{r(1+\gamma)/2} \left\{ \int_I \varphi^{2p}(t)\Delta t \right\}^{\frac{1-\gamma}{2}} \left\{ \int_I \varphi^{2p}(t)\Delta t \right\}^{\frac{1+\gamma}{2}} \\ &= \theta^q \int_I \varphi^{2p}(t)\Delta t, \end{aligned}$$

where we have used (4.5) in the last inequality. This shows the validity of (4.4). \square

5. TWO INEQUALITIES BASED ON THE CHAIN RULE

We start by giving the time scales chain rule [6, Theorem 1.87].

Theorem 5.1 (Time scales chain rule). *Assume $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable, and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ such that*

$$(5.1) \quad (f \circ g)^\Delta(t) = f'(g(c))g^\Delta(t).$$

Now we use the time scales chain rule from Theorem 5.1 to establish two inequalities for the derivative of the power of an integral. These inequalities are used in the proofs of the main result, and they are also important results in their own right.

Theorem 5.2. *Let $\beta \geq 1$ and $t_0 \in \mathbb{T}$. Assume $d : \mathbb{T} \rightarrow [0, \infty)$ is integrable on $[t_0, \infty)$ and define*

$$D(t) := \int_{t_0}^t d(\tau)\Delta\tau \quad \text{and} \quad \tilde{D}(t) := \int_t^\infty d(\tau)\Delta\tau.$$

Then we have

$$(5.2) \quad \beta D^{\beta-1}(t)d(t) \leq (D^\beta)^\Delta(t) \leq \beta D^{\beta-1}(\sigma(t))d(t)$$

and

$$(5.3) \quad \beta \tilde{D}^{\beta-1}(t)d(\sigma(t)) \leq -(\tilde{D}^\beta)^\Delta(t) \leq \beta \tilde{D}^{\beta-1}(t)d(t).$$

Proof. By Theorem 5.1, there exists $c \in [t, \sigma(t)]$ such that

$$(D^\beta)^\Delta(t) = \beta D^{\beta-1}(c)D^\Delta(t) = \beta D^{\beta-1}(c)d(t),$$

which in view of $t \leq c \leq \sigma(t)$, the nondecreasing nature of D and thus $D^{\beta-1}$, and the nonnegativity of d yields (5.2). Again by Theorem 5.1, there exists $c \in [t, \sigma(t)]$ with

$$-(\tilde{D}^\beta)^\Delta(t) = -\beta\tilde{D}^{\beta-1}(c)\tilde{D}^\Delta(t) = \beta\tilde{D}^{\beta-1}(c)d(t),$$

which in view of $t \leq c \leq \sigma(t)$, the nonincreasing nature of \tilde{D} and thus $\tilde{D}^{\beta-1}$, and the nonnegativity of d yields (5.3). □

6. TWO INEQUALITIES BASED ON INTEGRATION BY PARTS

We start by giving the two time scales integration by parts formulas [6, Theorem 1.77 (v) and (vi)].

Theorem 6.1 (Integration by parts). *If $a, b, c \in \mathbb{T}$ and $f, g \in C_{\text{rd}}(\mathbb{T}, \mathbb{R})$, then*

$$(6.1) \quad \int_a^b f(\sigma(t))g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(t)\Delta t$$

and

$$(6.2) \quad \int_a^b f(t)g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t.$$

Now we use the two integration by parts formulas from Theorem 6.1 together with Theorem 4.2 to obtain two Littlewood type inequalities.

Theorem 6.2. *Let $p, q \geq 1$ and define r by (4.2). Let $t_0 \in \mathbb{T}$. Assume $\varphi, d \in C_{\text{rd}}(\mathbb{T}, [0, \infty))$ and define*

$$\Phi(t) := \int_{t_0}^t \varphi(\tau)\Delta\tau \quad \text{and} \quad \tilde{D}(t) := \int_t^\infty d(\tau)\Delta\tau.$$

If there exists a constant $\kappa > 0$ such that

$$(6.3) \quad \Phi(\sigma(t))d(t) \leq \kappa\varphi^{1+p/q}(t) \quad \text{for all } t \geq t_0,$$

then

$$(6.4) \quad \int_{t_0}^\infty \varphi^p(t)\tilde{D}^q(t)\Delta t \leq (r\kappa)^q \int_{t_0}^\infty \varphi^{2p}(t)\Delta t.$$

Proof. We show that (4.3) is satisfied with $I = [t_0, \infty)$, $\psi = \tilde{D}$ and $\theta = r\kappa$. Then the claim follows from Theorem 4.2. First note that

$$\Phi(t_0) = \lim_{t \rightarrow \infty} \tilde{D}(t) = 0,$$

so that the integration by parts formula (6.2) yields

$$\begin{aligned} \int_{t_0}^\infty \varphi(t)\psi^r(t)\Delta t &= \int_{t_0}^\infty \Phi^\Delta(t)\tilde{D}^r(t)\Delta t \\ &= \int_{t_0}^\infty \Phi(\sigma(t)) \left[-(\tilde{D}^r)^\Delta(t) \right] \Delta t \\ &\leq \int_{t_0}^\infty \Phi(\sigma(t))r\tilde{D}^{r-1}(t)d(t)\Delta t \\ &\leq \theta \int_{t_0}^\infty \varphi^{1+p/q}(t)\psi^{r-1}(t)\Delta t, \end{aligned}$$

where we have used (5.3) and (6.3) in the two inequalities. □

Theorem 6.3. *Let $p, q \geq 1$ and define r by (4.2). Let $t_0 \in \mathbb{T}$. Assume $\varphi, d \in C_{rd}(\mathbb{T}, [0, \infty))$ and define*

$$\tilde{\Phi}(t) := \int_t^\infty \varphi(\tau)\Delta\tau \quad \text{and} \quad D(t) := \int_{t_0}^t d(\tau)\Delta\tau.$$

If there exists a constant $\kappa > 0$ such that

$$(6.5) \quad \tilde{\Phi}(t)d(t) \leq \kappa\varphi^{1+p/q}(t) \quad \text{for all } t \geq t_0,$$

then

$$(6.6) \quad \int_{t_0}^\infty \varphi^p(t)D^q(\sigma(t))\Delta t \leq (r\kappa)^q \int_{t_0}^\infty \varphi^{2p}(t)\Delta t.$$

Proof. We show that (4.3) is satisfied with $I = [t_0, \infty)$, $\psi = D^\sigma$ and $\theta = r\kappa$. Then the claim follows from Theorem 4.2. First note that

$$D(t_0) = \lim_{t \rightarrow \infty} \tilde{\Phi}(t) = 0,$$

so that the integration by parts formula (6.1) yields

$$\begin{aligned} \int_{t_0}^\infty \varphi(t)\psi^r(t)\Delta t &= \int_{t_0}^\infty \left[-\tilde{\Phi}^\Delta(t)\right] D^r(\sigma(t))\Delta t \\ &= \int_{t_0}^\infty \tilde{\Phi}(t)(D^r)^\Delta(t)\Delta t \\ &\leq \int_{t_0}^\infty \tilde{\Phi}(t)rD^{r-1}(\sigma(t))d(t)\Delta t \\ &\leq \theta \int_{t_0}^\infty \varphi^{1+p/q}(t)\psi^{r-1}(t)\Delta t, \end{aligned}$$

where we have used (5.2) and (6.5) in the two inequalities. □

7. PROOFS OF THE MAIN RESULTS

Now we use the results from the previous sections to prove the main results of this paper.

Proof of Theorem 2.1. We show that (6.3) is satisfied with $\varphi = a(A^{q/p})^\sigma$, $d = -\tilde{B}^\Delta$ and $\kappa = 1$. Then the claim follows from Theorem 6.2. Indeed, the elementary estimate

$$\begin{aligned} A^{1+q/p}(\sigma(t)) &= \int_{t_0}^{\sigma(t)} a(s) \left\{ \int_{t_0}^{\sigma(s)} a(\tau)\Delta\tau \right\}^{\frac{q}{p}} \Delta s \\ &\geq \int_{t_0}^{\sigma(t)} a(s) \left\{ \int_{t_0}^{\sigma(s)} a(\tau)\Delta\tau \right\}^{\frac{q}{p}} \Delta s \\ &= \int_{t_0}^{\sigma(t)} a(s)A^{q/p}(\sigma(s))\Delta s = \int_{t_0}^{\sigma(t)} \varphi(s)\Delta s = \Phi(\sigma(t)) \end{aligned}$$

implies

$$\kappa\varphi^{1+p/q}(t) = d(t)A^{1+q/p}(\sigma(t)) \geq d(t)\Phi(\sigma(t))$$

and hence concludes the proof. □

Proof of Theorem 2.2. We show that (6.3) is satisfied with $\varphi = aA^{q/p}$, $d = -\tilde{C}^\Delta$ and $\kappa = p/(p+q)$. Then the claim follows from Theorem 6.2. Indeed, an application of the chain rule (5.2) yields

$$(A^{1+q/p})^\Delta(t) \geq \left(1 + \frac{q}{p}\right) A^{q/p}(t)a(t) = \frac{1}{\kappa}\varphi(t)$$

so that

$$\Phi(\sigma(t)) = \int_{t_0}^{\sigma(t)} \varphi(\tau)\Delta\tau \leq \kappa \int_{t_0}^{\sigma(t)} (A^{1+q/p})^\Delta(\tau)\Delta\tau = \kappa A^{1+q/p}(\sigma(t)),$$

and thus

$$d(t)\Phi(\sigma(t)) \leq \kappa a^{1+p/q}(t)A^{1+q/p}(t) = \kappa\varphi^{1+p/q}(t),$$

which concludes the proof. □

Proof of Theorem 2.3. We show that (6.5) is satisfied with $\varphi = a\tilde{A}^{q/p}$, $d = B^\Delta$ and $\kappa = 1$. Then the claim follows from Theorem 6.3. Indeed, the elementary estimate

$$\begin{aligned} \tilde{A}^{1+q/p}(t) &= \int_t^\infty a(s) \left\{ \int_t^\infty a(\tau)\Delta\tau \right\}^{\frac{q}{p}} \Delta s \\ &\geq \int_t^\infty a(s) \left\{ \int_s^\infty a(\tau)\Delta\tau \right\}^{\frac{q}{p}} \Delta s \\ &= \int_t^\infty a(s)\tilde{A}^{q/p}(s)\Delta s = \int_t^\infty \varphi(s)\Delta s = \tilde{\Phi}(t) \end{aligned}$$

implies

$$\kappa\varphi^{1+p/q}(t) = d(t)\tilde{A}^{1+q/p}(t) \geq d(t)\tilde{\Phi}(t)$$

and hence concludes the proof. □

Proof of Theorem 2.4. We show that (6.5) is satisfied with $\varphi = a(\tilde{A}^{q/p})^\sigma$, $d = C^\Delta$ and $\kappa = p/(p+q)$. Then the claim follows from Theorem 6.3. Indeed, an application of the chain rule (5.3) yields

$$-(\tilde{A}^{1+q/p})^\Delta(t) \geq \left(1 + \frac{q}{p}\right) \tilde{A}^{q/p}(\sigma(t))a(t) = \frac{1}{\kappa}\varphi(t)$$

so that

$$\tilde{\Phi}(t) = \int_{t_0}^\infty \varphi(\tau)\Delta\tau \leq -\kappa \int_{t_0}^{\sigma(t)} (\tilde{A}^{1+q/p})^\Delta(\tau)\Delta\tau = \kappa\tilde{A}^{1+q/p}(t),$$

and thus

$$d(t)\tilde{\Phi}(t) \leq \kappa a^{1+p/q}(t)\tilde{A}^{1+q/p}(\sigma(t)) = \kappa\varphi^{1+p/q}(t),$$

which concludes the proof. □

8. TWO FURTHER LITTLEWOOD INEQUALITIES

Finally, we state two Littlewood type inequalities that follow directly from Theorem 6.2 and Theorem 6.3 by an appropriate choice of d . These results may prove to be interesting in their own right, and they have further corollaries by using different time scales, which we will not explicitly give here.

Corollary 8.1. *Let $t_0 \in \mathbb{T}$ and $p, q \geq 1$. If $a \in C_{\text{rd}}(\mathbb{T}, [0, \infty))$, then*

$$\int_{t_0}^{\infty} a^p(t) \left(\int_t^{\infty} \frac{a^{1+p/q}(\tau) \Delta\tau}{\int_{t_0}^{\sigma(\tau)} a(s) \Delta s} \Delta\tau \right)^q \Delta t \leq \left(\frac{2pq - q}{p} \right)^q \int_{t_0}^{\infty} a^{2p}(t) \Delta t$$

and

$$\int_{t_0}^{\infty} a^p(t) \left(\int_{t_0}^{\sigma(t)} \frac{a^{1+p/q}(\tau) \Delta\tau}{\int_{\tau}^{\infty} a(s) \Delta s} \Delta\tau \right)^q \Delta t \leq \left(\frac{2pq - q}{p} \right)^q \int_{t_0}^{\infty} a^{2p}(t) \Delta t.$$

REFERENCES

- [1] Ravi P. Agarwal, Donal O'Regan, Martin Bohner, and Samir H. Saker, *Some dynamic Wirtinger-type inequalities and their applications*, Pacific J. Math. **252** (2011), no. 1, 1–18, DOI 10.2140/pjm.2011.252.1. MR2862138 (2012j:26015)
- [2] Moulay Rchid Sidi Ammi and Delfim F. M. Torres, *Hölder's and Hardy's two dimensional diamond-alpha inequalities on time scales*, An. Univ. Craiova Ser. Mat. Inform. **37** (2010), no. 1, 1–11. MR2609350 (2011i:26034)
- [3] Matloob Anwar, Rabia Bibi, Martin Bohner, and Josip Pečarić, *Integral inequalities on time scales via the theory of isotonic linear functionals*, Abstr. Appl. Anal. 2011, Art. ID 483595, 16 pp., DOI 10.1155/2011/483595. MR2819769 (2012f:26024)
- [4] Grahame Bennett, *An inequality suggested by Littlewood*, Proc. Amer. Math. Soc. **100** (1987), no. 3, 474–476, DOI 10.2307/2046432. MR891148 (88g:26022)
- [5] Martin Bohner and Billür Kaymakçalan, *Opial inequalities on time scales*, Ann. Polon. Math. **77** (2001), no. 1, 11–20, DOI 10.4064/ap77-1-2. MR1867033 (2003e:26022)
- [6] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, An introduction with applications, Birkhäuser Boston Inc., Boston, MA, 2001. MR1843232 (2002c:34002)
- [7] Martin Bohner and Allan Peterson (eds.), *Advances in dynamic equations on time scales*, Birkhäuser Boston, Inc., Boston, MA, 2003. MR1962542 (2004d:34003)
- [8] Peng Gao, *On an inequality suggested by Littlewood*, J. Inequal. Appl. 2011, 2011:5, 10 pp., DOI 10.1186/1029-242X-2011-5. MR2819744 (2012f:26030)
- [9] Wei Ming Gong, *On a problem of Littlewood* (Chinese, with English and Chinese summaries), J. Yiyang Teachers College **14** (1997), no. 5, 15–16. MR1655812
- [10] Stefan Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), no. 1-2, 18–56, DOI 10.1007/BF03323153. MR1066641 (91m:26027)
- [11] J. E. Littlewood, *Some new inequalities and unsolved problems*, Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965), Academic Press, New York, 1967, pp. 151–162. MR0222231 (36 #5283)
- [12] Umut Mutlu Ozkan and Hüseyin Yildirim, *Hardy-Knopp-type inequalities on time scales*, Dynam. Systems Appl. **17** (2008), no. 3-4, 477–486. MR2569514 (2010i:26036)
- [13] Samir H. Saker, *Opial's type inequalities on time scales and some applications*, Ann. Polon. Math. **104** (2012), no. 3, 243–260, DOI 10.4064/ap104-3-3. MR2914534

- [14] Samir H. Saker, *Some Opial dynamic inequalities involving higher order derivatives on time scales*, Discrete Dyn. Nat. Soc. (2012), Art. ID 157301, 22. MR2984232
- [15] Adnan Tuna and Servet Kutukcu, *Some integral inequalities on time scales*, Appl. Math. Mech. (English Ed.) **29** (2008), no. 1, 23–29, DOI 10.1007/s10483-008-0104-y. MR2445568 (2009e:26047)

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY-KINGSVILLE, KINGSVILLE, TEXAS 78363

E-mail address: `agarwal@tamuk.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY, ROLLA, MISSOURI 65409-0020

E-mail address: `bohner@mst.edu`

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, MANSOURA UNIVERSITY, MANSOURA 35516, EGYPT

E-mail address: `shsaker@mans.edu.eg`