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ON QUADRATIC RATIONAL MAPS WITH PRESCRIBED GOOD REDUCTION

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Abstract. Given a number field K and a finite set S of places of K, the first main result of this paper shows that the quadratic rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over K which have good reduction at all places outside S form a Zariski-dense subset of the moduli space $\mathcal{M}_2$ parametrizing all isomorphism classes of quadratic rational maps. We then consider quadratic rational maps with double unramified fixed-point structure, and our second main result establishes a Zariski nondensity result for the set of such maps with good reduction outside S. We also prove a variation of this result for quadratic rational maps with unramified 2-cycle structure.

1. Introduction

Let K be a number field, and let S be a finite set of places of K which includes all of the Archimedean places. In 1963, Shafarevich proved that, up to K-isomorphism, there exist only finitely many elliptic curves over K having good reduction at all places $v \notin S$ (see [8], §IX.6). He conjectured a generalization of the result to abelian varieties, and this was proved in 1983 by Faltings [2] as part of his proof of Mordell's conjecture.

Motivated by an analogy between elliptic curves and dynamical systems on the projective line, Szpiro and Tucker [12] have asked whether there is a similar finiteness result for the set $\mathrm{Rat}_d(K)$ of rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of a given degree d defined over K. As they point out, however, if one uses the standard notions of isomorphism and good reduction for rational maps on $\mathbb{P}^1$, then simple counterexamples preclude a finiteness result of this type. For example, a rational map defined by a monic integral polynomial has everywhere good reduction, and for each fixed degree $d \geq 2$ one can easily find infinite families of such maps which are pairwise nonisomorphic.

By using a weaker notion of isomorphism defined by separate pre-composition and post-composition actions of PGL_2 on the set of rational maps of degree d, and by suitably altering the notion of good reduction, Szpiro-Tucker obtained a finiteness result of Shafarevich type for a certain class of rational maps. More recently, Petsche [6] has proved a different Shafarevich-type finiteness theorem along

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certain families of critically separable rational maps, using a notion of isomorphism defined via PGL_2-conjugation.

In the present paper we consider the following similar but somewhat more geometric question. Rather than a finiteness statement for the set of isomorphism classes of rational maps of degree d having prescribed good reduction, we ask instead whether or not this set is Zariski-dense in the moduli space $\mathcal{M}_d$ parametrizing $\bar{K}$-isomorphism classes of rational maps of degree d. Introduced in the complex-analytic setting by Milnor [5], and further developed geometrically by Silverman [9], $\mathcal{M}_d$ is an affine variety whose definition is given, via geometric invariant theory, as the quotient

$$\mathcal{M}_d = \mathrm{Rat}_d/\mathrm{PGL}_2$$

of the space Rat_d of all rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d, modulo the conjugation action of PGL_2, the automorphism group of $\mathbb{P}^1$. Let $\langle\cdot\rangle : \mathrm{Rat}_d \to \mathcal{M}_d$ denote the quotient map associated to this action.

Question. Let K be a number field and let S be a finite set of places of K including the Archimedean places. Is the set

$$\mathcal{G}_d(K,S) = \left\{ \langle\phi\rangle \in \mathcal{M}_d(K) \;\middle|\; \begin{array}{l} \phi \in \mathrm{Rat}_d(K) \text{ has good} \\ \text{reduction at all } v \in M_K \setminus S \end{array} \right\}$$

Zariski-dense in $\mathcal{M}_d$?

The first main result of this paper is the affirmative answer to this question in the simplest nontrivial setting of quadratic rational maps (the case $d = 2$). The definitions of the required terms are stated precisely in §3, and this Zariski-density result is stated as Theorem 1.

The remainder of this paper is spent showing that, despite our Zariski-density result in the case $d = 2$, if we replace arbitrary quadratic rational maps with objects possessing slightly more dynamical structure, then it is possible to obtain Zariski-nondensity results of Shafarevich type for the moduli space $\mathcal{M}_2$.

To explain our motivation in trying to obtain such results, we again consider the analogy with elliptic curves. It is an interesting fact that Shafarevich's finiteness theorem for elliptic curves may fail, in general, for genus-one curves. For example, if S is a finite set of places of $\mathbb{Q}$ containing the Archimedean place as well as all of the places of bad reduction for some rank-zero elliptic curve $E/\mathbb{Q}$, then there are infinitely many non-isomorphic genus-one curves over $\mathbb{Q}$ with good reduction outside S; the argument is explained by Mazur in [4], p. 241. (On the other hand, as he points out, it would follow from the Shafarevich-Tate conjecture that the set of K-isomorphism classes of genus-one curves over K with *everywhere* good reduction is finite.)

Motivated by the elliptic curve analogy, we consider quadratic rational maps with double unramified fixed-point structure. More precisely, we consider the space of triples $\Phi = (\phi, P_1, P_2)$, where $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a quadratic rational map defined over K, and where $P_1, P_2 \in \mathbb{P}^1(K)$ are distinct K-rational unramified fixed points of ϕ. The conjugation action of PGL_2 gives rise to a notion of K-isomorphism between two such triples, and we formulate a natural definition of good reduction for such a triple $\Phi = (\phi, P_1, P_2)$ at a non-Archimedean place v of K which, roughly speaking, requires that (up to K-isomorphism) the reduction $\tilde{\Phi}_v = (\tilde{\phi}_v, \tilde{P}_{1,v}, \tilde{P}_{2,v})$ constitute a quadratic rational map with double unramified fixed-point structure over the residue field $\mathbb{F}_v$ at v. In §4 we give the precise definitions of these terms and we prove the second main result of this paper, Theorem 2, which shows that among all quadratic rational maps with double unramified fixed-point structure, those having good reduction at all places v outside S form a Zariski-nondense subset of the moduli space $\mathcal{M}_2$. We regard this result as a geometric Shafarevich-type theorem for rational maps.

Finally, with very little extra effort we can also establish a variation on the Zariski-nondensity result of Theorem 2, in which maps with unramified 2-cycle structure take the place of maps with unramified double fixed-point structure. This result is stated as Theorem 6.

2. Preliminaries

2.1. **Review of quadratic rational maps.** We now fix notation and review basic facts about quadratic rational maps on the projective line; for further details see [10], §2.4, §4.3, §4.6.

An arbitrary quadratic rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\bar{K}$ is given in homogeneous coordinates as

$$\phi(X:Y) = (A(X,Y) : B(X,Y)),$$

where

$$A(X,Y) = a_0X^2 + a_1XY + a_2Y^2,$$
$$B(X,Y) = b_0X^2 + b_1XY + b_2Y^2$$

are binary quadratic forms in $\bar{K}[X,Y]$ having no common zeros in $\bar{K}^2 \setminus \{(0,0)\}$. The requirement that $A(X,Y)$ and $B(X,Y)$ share no common zeros in $\bar{K}^2 \setminus \{(0,0)\}$ is equivalent to the nonvanishing of the resultant

$$\mathrm{Res}(A,B) = \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix}$$

associated to the pair (A,B).

The group variety of automorphisms of $\mathbb{P}^1$ is denoted by PGL_2, and each $f \in \mathrm{PGL}_2$ is given in homogeneous coordinates by

$$f(X:Y) = (\alpha X + \beta Y : \gamma X + \delta Y)$$

for some nonsingular matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ with coefficients in $\bar{K}$. Given a quadratic rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, we denote by $\phi^f : \mathbb{P}^1 \to \mathbb{P}^1$ the rational map $\phi^f = f^{-1} \circ \phi \circ f$ defined via conjugation of ϕ by f. Explicitly, if we denote by $(A,B) : \bar{K}^2 \to \bar{K}^2$ the map defined by $(X,Y) \mapsto (A(X,Y), B(X,Y))$, and if we define binary quadratic forms $C(X,Y)$ and $D(X,Y)$ in $\bar{K}[X,Y]$ by the formula

$$(C,D) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \circ (A,B) \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then $\phi^f(X:Y) = (C(X,Y) : D(X,Y))$. The formula

$$\mathrm{Res}(C,D) = (\alpha\delta - \beta\gamma)^2 \mathrm{Res}(A,B), \tag{1}$$

which shows the effect of GL_2-conjugation on the resultant, can be verified from direct calculation.

2.2. **Review of the moduli space $\mathcal{M}_2$.** The moduli space $\mathcal{M}_2$ parametrizing isomorphism classes of quadratic rational maps was first studied complex analytically by Milnor [5], who showed that it is isomorphic to the affine plane $\mathbb{A}^2$. A bit later, Silverman [9] used geometric invariant theory to construct $\mathcal{M}_2$ (and more generally the moduli space $\mathcal{M}_d$ of rational maps of degree d) as a scheme over $\mathrm{Spec}(\mathbb{Z})$, and established the isomorphism $\mathcal{M}_2 \simeq \mathbb{A}^2$ in this more geometric context. Since that time, variations and generalizations have been studied by Petsche-Szpiro-Tepper [7], Levy [3], and others. Further references include [10], §4.4, and [11].

We now review the definition and basic properties of the space $\mathcal{M}_2$. The first step is to observe that the set of all quadratic rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\bar{K}$ is parametrized by an affine variety which is commonly denoted by Rat_2. To obtain this variety, note that since the map ϕ is unchanged by scaling its coefficients, one may identify ϕ with the point $(\mathbf{a} : \mathbf{b}) = (a_0 : a_1 : a_2 : b_0 : b_1 : b_2)$ of $\mathbb{P}^5$ defined by its coefficients. In this way, the space Rat_2 of all quadratic rational maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is identified with the open affine subvariety $\{(\mathbf{a} : \mathbf{b}) \mid \mathrm{Res}(A, B) \neq 0\}$ of $\mathbb{P}^5$.

Let

$$\begin{aligned} \mathrm{PGL}_2 \times \mathrm{Rat}_2 &\to \mathrm{Rat}_2, \\ (f, \phi) &\mapsto \phi^f \end{aligned} \tag{2}$$

be the conjugation action of PGL_2 on Rat_2, and let $\mathcal{A} = \Gamma(\mathrm{Rat}_2, \mathcal{O}_{\mathrm{Rat}_2})$ be the coordinate ring of Rat_2. The moduli space $\mathcal{M}_2$ is defined to be the affine variety $\mathrm{Spec}(\mathcal{A}^{\mathrm{PGL}_2})$, where $\mathcal{A}^{\mathrm{PGL}_2}$ is the subring of PGL_2-invariants in $\mathcal{A}$. Using standard facts from geometric invariant theory it can be shown that $\mathcal{M}_2$ is a geometric quotient for the action (2), which means roughly that the map

$$\langle \cdot \rangle : \mathrm{Rat}_2 \to \mathcal{M}_2 \tag{3}$$

induced by inclusion $\mathcal{A}^{\mathrm{PGL}_2} \subset \mathcal{A}$ possesses many of the nice properties one would expect from the quotient map of a group action in the classical sense. For example, the (geometric) fibers of the map (3) are closed, and they are precisely the orbits in Rat_2 with respect to the conjugation action of PGL_2.

To describe Milnor's isomorphism $\mathcal{M}_2 \simeq \mathbb{A}^2$ in detail, recall that each quadratic rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ has three (counting with multiplicity) fixed points $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{P}^1$, and for each fixed point, the multiplier λ_j associated to α_j is the leading coefficient $\lambda_j = \phi'(\alpha_j)$ of the power series expansion of $\phi(z)$ at $z = \alpha_j$ (with respect to some choice of affine coordinate z on $\mathbb{P}^1$). A standard calculation shows that the multiplier of a fixed point is invariant under PGL_2-conjugation. Since the fixed-point set $\mathrm{Fix}(\phi)$ is naturally an unordered triple, we obtain three scalar-valued PGL_2-invariant functions σ_1 and σ_2 on Rat_2, defined by the first two symmetric functions $\sigma_1(\phi) = \lambda_1 + \lambda_2 + \lambda_3$ and $\sigma_2(\phi) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ in the multipliers of the three fixed points of ϕ. (The third symmetric function, $\sigma_3(\phi) = \lambda_1\lambda_2\lambda_3$, gives only redundant information because of the identity $\sigma_3 - \sigma_1 + 2 = 0$.) Milnor's isomorphism $\mathcal{M}_2 \simeq \mathbb{A}^2$ is defined by

$$\begin{aligned} \sigma : \mathcal{M}_2 &\xrightarrow{\sim} \mathbb{A}^2, \\ \langle \phi \rangle &\mapsto (\sigma_1(\phi), \sigma_2(\phi)). \end{aligned} \tag{4}$$

See [5].

2.3. **Number-theoretic preliminaries.** We denote by M_K the set of places of the number field K. For each $v \in M_K$, the notation $|\cdot|_v$ refers to any absolute value on K associated to v. If v is non-Archimedean, $\mathcal{O}_v$ is the subring of v-integral elements of K, and $\mathcal{O}_v^\times$ is the group of units in $\mathcal{O}_v$. The notation $x \mapsto \tilde{x}_v$ will denote the reduction map $\mathcal{O}_v \to \mathbb{F}_v$ onto the residue field $\mathbb{F}_v$ of $\mathcal{O}_v$.

The letter S denotes a finite subset of M_K which includes all of the Archimedean places. The ring of S-integers in K is written $\mathcal{O}_S$, and $\mathcal{O}_S^\times$ denotes the group of units in this ring.

Given a quadratic rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over K and a non-Archimedean place v of K, we may scale the coefficients of $A(X,Y)$ and $B(X,Y)$ by a uniformizing parameter at v to obtain all coefficients in $\mathcal{O}_v$, with at least one coefficient in $\mathcal{O}_v^\times$. Reducing modulo the maximal ideal of $\mathcal{O}_v$, we obtain a reduced rational map $\tilde{\phi}_v : \mathbb{P}^1 \to \mathbb{P}^1$ over the residue field $\mathbb{F}_v$ of $\mathcal{O}_v$, defined by

$$\tilde{\phi}_v(X : Y) = (\tilde{A}(X,Y) : \tilde{B}(X,Y)).$$

Clearly $\deg(\tilde{\phi}_v) \le 2$, and it follows from properties of the resultant that $\deg(\tilde{\phi}_v) = 2$ if and only if $\mathrm{Res}(A,B) \in \mathcal{O}_v^\times$.

Given a point $P = (a : b) \in \mathbb{P}^1(K)$ and a non-Archimedean place v of K, we may scale the coordinates a and b by a uniformizing parameter at v to obtain both coordinates in $\mathcal{O}_v$, with at least one of the two in $\mathcal{O}_v^\times$; we obtain a reduced point $\tilde{P}_v = (\tilde{a}_v : \tilde{b}_v) \in \mathbb{P}^1(\mathbb{F}_v)$.

3. Prescribed good reduction for quadratic rational maps

In this section we show that the quadratic rational maps over K having good reduction at all places outside S comprise a Zariski-dense subset of the moduli space $\mathcal{M}_2$. We first recall the standard definitions of K-isomorphism and good reduction.

Definitions. Two quadratic rational maps $\phi, \psi \in \mathrm{Rat}_2(K)$ are *K-isomorphic* if $\psi = \phi^f$ for some automorphism $f \in \mathrm{PGL}_2(K)$. A quadratic rational map $\phi \in \mathrm{Rat}_2(K)$ has *good reduction* at a non-Archimedean place v of K if it is K-isomorphic to some $\psi \in \mathrm{Rat}_2(K)$ such that $\deg(\tilde{\psi}_v) = 2$.

We stress the contrast between the notion of *good reduction*, in which the two rational maps ϕ and ψ are required to be conjugate via a K-rational automorphism $f \in \mathrm{PGL}_2$, with the weaker notion of *potential good reduction*, in which the definition is relaxed to allow automorphisms f defined over $\bar{K}$. To illustrate the difference between the two, consider the elliptic curve setting: an elliptic curve E/K has potential good reduction at all places $v \in M_K \setminus S$ if and only if $j_E \in \mathcal{O}_S$, and so it is trivial that there are infinitely many ($\bar{K}$-isomorphism classes of) such curves. Similarly, in the case of rational maps, it would be straightforward to produce a Zariski-density result for $\bar{K}$-isomorphism classes of rational maps in $\mathcal{M}_2$ with prescribed *potential* good reduction using the fact that $\mathcal{M}_2(\mathbb{Z})$ is Zariski-dense in $\mathcal{M}_2 \simeq \mathbb{A}^2$.

Theorem 1. *Let K be a number field and let S be a finite set of places of K including the Archimedean places. Then the set*

$$\mathcal{G}_2(K,S) = \left\{ \langle\phi\rangle \in \mathcal{M}_2(K) \,\middle|\, \begin{array}{l} \phi \in \mathrm{Rat}_2(K) \text{ has good} \\ \text{reduction at all } v \in M_K \setminus S \end{array} \right\}$$

is Zariski-dense in $\mathcal{M}_2$.

Our primary proof of Theorem 1 uses the isomorphism (4), as well as a further result of Milnor [5] on quadratic rational maps in critical-point normal form. We also give an alternate proof which holds only when the group of S-units in K is infinite (thus, this alternate proof fails to apply only when K is either $\mathbb{Q}$ or a quadratic imaginary extension of $\mathbb{Q}$, and S consists of the sole Archimedean place of K). While it does not apply in full generality, this secondary proof is sufficiently different from the first proof that it may be of some interest. It is more self-contained, in that it does not rely on special properties of quadratic rational maps in critical-point normal form, and the ideas behind this secondary proof may find wider applicability toward possible generalizations to the higher degree case.

Proof of Theorem 1. We will consider maps $\phi \in \mathrm{Rat}_2$ given in critical-point normal form

$$\phi(X:Y) = (aX^2 + bY^2 : cX^2 + dY^2). \tag{5}$$

The critical points are $(1:0)$ and $(0:1)$, and

$$\mathrm{Res}(aX^2 + bY^2, cX^2 + dY^2) = (ad - bc)^2. \tag{6}$$

Let $\mathcal{F}$ be the image in the moduli space $\mathcal{M}_2$ of the set of all $\phi \in \mathrm{Rat}_2(\mathbb{Q})$ given in critical-point normal form (5) with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. It follows from (6) that each such map has good reduction at all non-Archimedean places $v \in M_K$; thus $\mathcal{F} \subset \mathcal{G}_2(K,S)$, and we are reduced to showing that $\mathcal{F}$ is Zariski-dense in $\mathcal{M}_2$.

A direct calculation shows that if ϕ is given in critical-point normal form (5) with $ad - bc = 1$, then

$$\begin{aligned}\sigma_1(\phi) &= 8ad - 6,\\ \sigma_2(\phi) &= 8a^2d^2 - 20ad + 4(a^3b + cd^3) + 12;\end{aligned}$$

see for example the explicit formula given by Silverman ([10, p. 189]) or the calculation for rational maps in critical-point normal form due to Milnor ([5, Corollary C.4]).

In view of the isomorphism (4), we may identify $\mathcal{M}_2$ with the affine plane $\mathbb{A}^2$ and we may use σ_1 and σ_2 as the two affine coordinates on $\mathcal{M}_2$. Arguing by contradiction, assume on the contrary that the Zariski closure $\overline{\mathcal{F}}$ of $\mathcal{F}$ is not all of $\mathcal{M}_2$. Then $\overline{\mathcal{F}}$ is a finite union of curves and points in $\mathcal{M}_2$. By Bezout's theorem, there exists a positive bound $B = B(\overline{\mathcal{F}}) > 0$ such that if L is any line in $\mathcal{M}_2$, then either $L \subseteq \overline{\mathcal{F}}$ or $|L \cap \overline{\mathcal{F}}| \leq B$. For each $\alpha \in \bar{K}$, let L_α be the vertical line $\{\sigma_1 = \alpha\}$ in $\mathcal{M}_2$. Then $\overline{\mathcal{F}}$ can contain at most finitely many of these lines; call them $L_{\alpha_1}, \dots, L_{\alpha_r}$.

We will obtain a contradiction by showing that $\mathcal{F}$ can meet a vertical line L_α at an arbitrarily large number of points for lines $L_\alpha \notin \{L_{\alpha_1}, \dots, L_{\alpha_r}\}$. Let N be a positive integer, let p be an arbitrary prime number, and for each $0 \leq n \leq N-1$ define

$$\phi_{n,N}(X:Y) = (p^n X^2 + Y^2 : (p^{2N} - 1)X^2 + p^{2N-n}Y^2).$$

Since $(p^n)(p^{2N-n}) - (p^{2N}-1)(1) = 1$, we have $\phi_{n,N} \in \mathcal{F}$; denote it by $\mathcal{F}(N) = \{\langle \phi_{n,N} \rangle \mid 0 \leq n \leq N-1\}$. Using Milnor's calculation of σ_1 and σ_2 in terms of A and Σ, we have

$$\begin{aligned}\sigma_1(\langle \phi_{n,N} \rangle) &= 8p^{2N} - 6,\\ \sigma_2(\langle \phi_{n,N} \rangle) &= 8p^{4N} - 20p^{2N} + 4(p^{3n} + (p^{2N}-1)p^{6N-3n}) + 12.\end{aligned}$$

This shows that $\mathcal{F}(N)$ is contained in the line $L_{8p^{2N}-6}$. Further, note that for fixed N, the numbers $p^{3n} + (p^{2N}-1)p^{6N-3n}$ are distinct as n ranges from $0 \leq n \leq N-1$

(for example, because the p-adic absolute value of $p^{3n} + (p^{2N} - 1)p^{6N-3n}$ is p^{-3n}). Therefore, the σ_2-coordinates of the N points $\langle\phi_{n,N}\rangle$ are distinct for $0 \le n \le N-1$, whereby $|\mathcal{F}(N)| = N$. We have shown that $\mathcal{F}$ meets the line $L_{8p^{2N}-6}$ in at least N points; taking N large enough produces a contradiction, since there are only finitely many N for which $N \le B$ or $L_{8p^{2N}-6} \in \{L_{\alpha_1}, \dots, L_{\alpha_r}\}$. $\square$

Alternate proof of Theorem 1. (This proof holds only under the additional assumption that the S-unit group $\mathcal{O}_S^\times$ of K is infinite.)

We will consider maps $\phi \in \mathrm{Rat}_2$ given in fixed-point normal form

$$\phi(X:Y) = (X^2 + \lambda_1 XY : \lambda_2 XY + Y^2). \tag{7}$$

For rational maps in this form, the fixed points, their multipliers, and the resultant are particularly easy to calculate. The fixed points of ϕ are $(0:1)$, $(1:0)$, and $(1-\lambda_1 : 1-\lambda_2)$, with multipliers λ_1, λ_2, and $\lambda_3 = \frac{2-\lambda_1-\lambda_2}{1-\lambda_1\lambda_2}$, respectively (see [10], §4.6), and

$$\mathrm{Res}(X^2 + \lambda_1 XY, \lambda_2 XY + Y^2) = 1 - \lambda_1\lambda_2.$$

For each pair of nonzero elements $\alpha, \beta \in \bar{K}^\times$, define $\phi_{\alpha,\beta} \in \mathrm{Rat}_2$ to be the map given in fixed-point normal form (7) with $\lambda_1 = \alpha$ and $\lambda_2 = \frac{1-\beta}{\alpha}$. We obtain a map

$$\begin{aligned} u : \mathbb{G}_m \times \mathbb{G}_m &\to \mathcal{M}_2, \\ u(\alpha, \beta) &= \langle\phi_{\alpha,\beta}\rangle. \end{aligned}$$

We first show that u is dominant. For each $\alpha \in \bar{K}^\times$, define $u_\alpha : \mathbb{G}_m \to \mathcal{M}_2$ by $u_\alpha(z) = u(\alpha, z)$, and let $Z_\alpha = \overline{u_\alpha(\mathbb{G}_m)}$ be the Zariski closure of the image of u_α. In view of the isomorphism (4), we may identify $\mathcal{M}_2$ with the affine plane $\mathbb{A}^2$ and we may use σ_1 and σ_2 as the two affine coordinates on $\mathcal{M}_2$. Direct calculations show that $u_1(z) = (3 - z, 3 - 2z)$, and therefore Z_1 is the line $2\sigma_1 - \sigma_2 = 3$ in $\mathcal{M}_2$. Similarly, $u_{-1}(z) = (-3 + z + \frac{4}{z}, 7 - 2(z + \frac{4}{z}))$, and therefore Z_{-1} is the line $2\sigma_1 + \sigma_2 = 1$ in $\mathcal{M}_2$. Now let $Z = \overline{u(\mathcal{O}_S^\times \times \mathcal{O}_S^\times)}$ be the Zariski closure of the image of u. Then since the torus $\mathbb{G}_m \times \mathbb{G}_m$ is irreducible, Z is irreducible, and since Z contains the two distinct lines Z_1 and Z_{-1}, Z must have dimension 2. Therefore $Z = \mathcal{M}_2$ and u is dominant.

Define $\mathcal{V} = u(\mathcal{O}_S^\times \times \mathcal{O}_S^\times)$ to be the image in $\mathcal{M}_2$ under u of the set of all $\phi_{\alpha,\beta} \in \mathrm{Rat}_2(K)$ for which both α and β are in the S-unit group $\mathcal{O}_S^\times$. The calculation $\mathrm{Res}(X^2 + \alpha XY, (\frac{1-\beta}{\alpha})XY + Y^2) = \beta$ shows that each such map has good reduction at all places $v \in M_K \setminus S$, and therefore $\mathcal{V} \subset \mathcal{G}_2(K, S)$. To complete the proof of the theorem, we only need to show that $\mathcal{V}$ is Zariski-dense in $\mathcal{M}_2$.

Since $\mathcal{O}_S^\times$ is infinite, the subgroup $\mathcal{O}_S^\times \times \mathcal{O}_S^\times$ is Zariski-dense in $\mathbb{G}_m \times \mathbb{G}_m$. Therefore

$$u(\mathbb{G}_m \times \mathbb{G}_m) = u(\overline{\mathcal{O}_S^\times \times \mathcal{O}_S^\times}) \subseteq \overline{u(\mathcal{O}_S^\times \times \mathcal{O}_S^\times)} = \overline{\mathcal{V}}. \tag{8}$$

Here we have used the fact, which is true of all continuous maps on topological spaces, including morphisms of algebraic varieties, that $f(\overline{X}) \subseteq \overline{f(X)}$. Taking the Zariski closure of both sides of (8) we obtain $\overline{u(\mathbb{G}_m \times \mathbb{G}_m)} \subseteq \overline{\mathcal{V}}$. Since u is dominant, we have $\overline{u(\mathbb{G}_m \times \mathbb{G}_m)} = \mathcal{M}_2$, and therefore $\overline{\mathcal{V}} = \mathcal{M}_2$, completing the proof. $\square$

4. Prescribed good reduction for quadratic rational maps with double unramified fixed-point structure

One of the goals of the paper is to emphasize that dynamical analogues of theorems for elliptic curves over number fields may fail because general rational maps lack the richer structure of elliptic curves. As mentioned in the introduction, when one replaces elliptic curve in the statement of Shafarevich's theorem with genus-one curve, the theorem becomes false. In that setting, the extra structure provided by a marked K-rational point acting as the origin for the elliptic curve has a dramatic influence on the set of K-isomorphism classes of such objects. With the elliptic curve analogy in mind, in this section we consider rational maps equipped with some additional structure arising from fixed points.

Definition. Let $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ be the set of all triples of the form $\Phi = (\phi, P_1, P_2)$, where $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a quadratic rational map defined over K, and where $P_1, P_2 \in \mathbb{P}^1(K)$ are distinct K-rational unramified fixed points of ϕ. We call such a triple Φ a *quadratic rational map with double unramified fixed-point structure over* K, or, when the context is clear, for brevity we may refer to Φ simply as a map.

Remark 1. We have defined $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ only as a set, but observe also that $\mathrm{Rat}_{2,2}^{\mathrm{uf}}$ may be naturally viewed as an open subset of a closed surface in $\mathrm{Rat}_2 \times \mathbb{P}^1 \times \mathbb{P}^1$; thus $\mathrm{Rat}_{2,2}^{\mathrm{uf}}$ is a quasiprojective variety. Further, the map $\mathrm{Rat}_{2,2}^{\mathrm{uf}} \to \mathrm{Rat}_2$ obtained by forgetting the fixed-point structure is dominant; indeed, the set of quadratic rational maps that fail to have three distinct unramified fixed points forms a proper Zariski-closed subset of Rat_2. Finally, given a map $\phi \in \mathrm{Rat}_2(K)$, the three fixed points of ϕ are K'-rational for some extension K'/K of degree at most six. So from the point of view of studying geometric questions concerning generic rational maps, the extra conditions required of a quadratic rational map with double unramified fixed-point structure over K are not terribly restrictive. (Later we will explain our choice of double, rather than single or triple, unramified fixed-point structure.)

Given a map $\Phi = (\phi, P_1, P_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$, and an automorphism $f \in \mathrm{PGL}_2(K)$, observe that $f^{-1}(P_1)$ and $f^{-1}(P_2)$ are distinct unramified fixed points of $\phi^f = f^{-1} \circ \phi \circ f$; we may therefore define $\Phi^f \in \mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$, the conjugate of Φ with respect to f, by

$$\Phi^f = (\phi^f, f^{-1}(P_1), f^{-1}(P_2)).$$

This notion of $\mathrm{PGL}_2(K)$-conjugation on $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ gives rise to the following definitions.

Definitions. Two maps Φ and Ψ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ are *K-isomorphic* if $\Psi = \Phi^f$ for some automorphism $f \in \mathrm{PGL}_2(K)$. A map Φ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ has *good reduction* at a non-Archimedean place v of K if it is K-isomorphic to some $\Psi = (\psi, Q_1, Q_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ such that $\deg(\tilde{\psi}_v) = 2$ and such that $\tilde{Q}_1$ and $\tilde{Q}_2$ are distinct unramified fixed points of $\tilde{\psi}_v$.

We emphasize that this notion of good reduction for a map $\Phi = (\phi, P_1, P_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ is stronger than the standard definition of good reduction for its underlying rational map $\phi \in \mathrm{Rat}_2(K)$; a natural additional condition has been added to ensure that reduction modulo the maximal ideal of $\mathcal{O}_v$ preserves the double unramified fixed-point structure of the triple $\tilde{\Phi} = (\tilde{\phi}_v, \tilde{P}_{1,v}, \tilde{P}_{2,v})$ over the residue field $\mathbb{F}_v$.

Abusing notation slightly, for each $\Phi = (\phi, P_1, P_2)$ in $\mathrm{Rat}^{\mathrm{uf}}_{2,2}(K)$, define $\langle\Phi\rangle = \langle\phi\rangle$. Thus, one may view $\langle\cdot\rangle : \mathrm{Rat}^{\mathrm{uf}}_{2,2}(K) \to \mathcal{M}_2(K)$ as the map which forgets the fixed-point structure of Φ and preserves only the PGL_2-conjugacy class $\langle\phi\rangle$ of its underlying rational map.

The main theorem of this section is the following, which shows that the set of all Φ in $\mathrm{Rat}^{\mathrm{uf}}_{2,2}(K)$ having good reduction outside S forms a Zariski-nondense subset of the moduli space $\mathcal{M}_2$.

Theorem 2. *Let K be a number field and let S be a finite set of places of K including the Archimedean places. Then the set*

$$\mathcal{G}^{\mathrm{uf}}_{2,2}(K,S) = \left\{ \langle\Phi\rangle \in \mathcal{M}_2(K) \;\middle|\; \begin{array}{l} \Phi \in \mathrm{Rat}^{\mathrm{uf}}_{2,2}(K) \textit{ has good} \\ \textit{reduction at all } v \in M_K \setminus S \end{array} \right\}$$

is not Zariski-dense in $\mathcal{M}_2$.

We need three preliminary propositions before we can give the proof of Theorem 2. The first states that maps in $\mathrm{Rat}^{\mathrm{uf}}_{2,2}(K)$ having good reduction at a non-Archimedean place v of K can be represented (up to K-isomorphism) in a certain simple form.

Proposition 3. *Suppose $\Phi \in \mathrm{Rat}^{\mathrm{uf}}_{2,2}(K)$ has good reduction at a non-Archimedean place v of K. Then Φ is K-isomorphic to $\Psi = (\psi, (1:0), (0:1))$ for some quadratic rational map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ given by*

$$\psi(X:Y) = (X^2 + aXY : bXY + cY^2) \tag{9}$$

for $a, b, c \in \mathcal{O}_v^\times$ such that

$$\mathrm{Res}(X^2 + aXY, bXY + cY^2) = c(c - ab) \in \mathcal{O}_v^\times.$$

Proof. According to the definition of good reduction, possibly replacing Φ with some member of its K-isomorphism class, we may assume without loss of generality that $\Phi = (\phi, P_1, P_2)$, where $\deg(\tilde{\phi}_v) = 2$ and where $\tilde{P}_1$ and $\tilde{P}_2$ are distinct unramified fixed points of the reduced map $\tilde{\phi}_v$.

The fact that $\deg(\tilde{\phi}_v) = 2$ means that we may write $\phi(X,Y) = (A(X,Y) : B(X,Y))$ for forms $A(X,Y), B(X,Y)$ in $\mathcal{O}_v[X,Y]$ with $\mathrm{Res}(A,B) \in \mathcal{O}_v^\times$. Set $P_1 = (\alpha_1 : \beta_1)$ for $\alpha_1, \beta_1 \in \mathcal{O}_v$ and at least one of the two α_1, β_1 in the unit group $\mathcal{O}_v^\times$, and set $P_2 = (\alpha_2 : \beta_2)$ subject to the same requirements. Since $\tilde{P}_1 \neq \tilde{P}_2$ in $\mathbb{P}^1(\mathbb{F}_v)$, we have $\tilde{\alpha}_2\tilde{\beta}_1 - \tilde{\alpha}_1\tilde{\beta}_2 \neq 0$ in $\mathbb{F}_v$. In other words $\alpha_2\beta_1 - \alpha_1\beta_2 \in \mathcal{O}_v^\times$, and therefore the matrix $\left(\begin{smallmatrix}\alpha_1 & \alpha_2 \\ \beta_1 & \beta_2\end{smallmatrix}\right)$ is an element of $\mathrm{GL}_2(\mathcal{O}_v)$.

Define $\psi = \phi^f$, where $f \in \mathrm{PGL}_2(K)$ is given by $f(X:Y) = (\alpha_1 X + \alpha_2 Y : \beta_1 X + \beta_2 Y)$. This means that $\psi(X:Y) = (C(X,Y) : D(X,Y))$, where the forms $C(X,Y)$ and $D(X,Y)$ are defined by

$$(C,D) = \begin{pmatrix}\alpha_1 & \alpha_2 \\ \beta_1 & \beta_2\end{pmatrix}^{-1} \circ (A,B) \circ \begin{pmatrix}\alpha_1 & \alpha_2 \\ \beta_1 & \beta_2\end{pmatrix}.$$

Since $\left(\begin{smallmatrix}\alpha_1 & \alpha_2 \\ \beta_1 & \beta_2\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_v)$, formula (1) shows that the forms $C(X,Y)$ and $D(X,Y)$ have coefficients in $\mathcal{O}_v$ and resultant $\mathrm{Res}(C,D)$ in $\mathcal{O}_v^\times$. In particular, $\deg(\tilde{\psi}_v) = 2$.

Since $f(1:0) = P_1$ and $f(0:1) = P_2$, it follows that $(1:0)$ and $(0:1)$ are fixed points of ψ, and therefore we have $C(X,Y) = c_0X^2 + c_1XY$ and $D(X,Y) = d_1XY + d_2Y^2$ for elements $c_0, c_1, d_1, d_2 \in \mathcal{O}_v$, with $\mathrm{Res}(C,D) = c_0d_2(c_0d_2 - c_1d_1) \in \mathcal{O}_v^\times$.

This immediately forces $c_0, d_2 \in \mathcal{O}_v^\times$, since otherwise c_0 or d_2 would be an element of the maximal ideal of $\mathcal{O}_v$, making $c_0 d_2(c_0 d_2 - c_1 d_1) \in \mathcal{O}_v^\times$ impossible.

Since $\left(\begin{smallmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_v)$, the automorphism f reduces to an automorphism $\tilde{f} \in \mathrm{PGL}_2(\mathbb{F}_v)$, and $\tilde{\psi}_v = \tilde{\phi}_v^{\tilde{f}}$. Since $\tilde{P}_1$ and $\tilde{P}_2$ are unramified fixed points of $\tilde{\phi}_v$, it follows that $(\tilde{1} : \tilde{0})$ and $(\tilde{0} : \tilde{1})$ are unramified fixed points of $\tilde{\psi}_v$. Standard calculations show that since $(\tilde{1} : \tilde{0})$ is an unramified point of $\tilde{\psi}_v$, $\tilde{d}_1$ is nonzero in $\mathbb{F}_v$, and since $(\tilde{0} : \tilde{1})$ is an unramified point of $\tilde{\psi}_v$, $\tilde{c}_1$ is nonzero in $\mathbb{F}_v$. Consequently, both c_1 and d_1 are in $\mathcal{O}_v^\times$.

Finally, setting $a = \frac{c_1}{c_0}$, $b = \frac{d_1}{c_0}$, and $c = \frac{d_2}{c_0}$, we obtain a representation for the map ψ in the desired form (9), with $a, b, c \in \mathcal{O}_v^\times$ and $c(c - ab) = \frac{d_2}{c_0}(\frac{d_2}{c_0} - \frac{c_1}{c_0}\frac{d_1}{c_0}) = c_0^{-3} d_2(c_0 d_2 - c_1 d_1) \in \mathcal{O}_v^\times$. □

Given a map Φ in $\mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ having good reduction at all places outside S, Proposition 3 shows that for each $v \in M_K \setminus S$, Φ is K-isomorphic to some map Ψ possessing a particularly simple form which realizes this good reduction at v. A priori, the map Ψ may vary from place to place, but the following proposition shows that if $\mathcal{O}_S$ is a principal ideal domain, then a global map Ψ can be found that satisfies the conclusion of Proposition 3 at every place $v \in M_K \setminus S$.

Proposition 4. *Assume that $\mathcal{O}_S$ is a principal ideal domain. Suppose that $\Phi \in \mathrm{Rat}_{2,2}^{\mathrm{uf}}(K)$ has good reduction at all places $v \in M_K \setminus S$. Then Φ is K-isomorphic to $\Psi = (\psi, (1:0), (0:1))$ for some quadratic rational map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ given by*

$$\psi(X : Y) = (X^2 + aXY : bXY + cY^2)$$

for $a, b, c \in \mathcal{O}_S^\times$ such that

$$\mathrm{Res}(X^2 + aXY, bXY + cY^2) = c(c - ab) \in \mathcal{O}_S^\times.$$

Proof. Replacing $\Phi = (\phi, P_1, P_2)$ with its conjugate by a suitable automorphism in $\mathrm{PGL}_2(K)$ which takes $(1 : 0)$ to P_1 and $(0 : 1)$ to P_2, we may assume without loss of generality that $\Phi = (\phi, (1 : 0), (0 : 1))$ for some map ϕ given by

$$\phi(X : Y) = (X^2 + a_0 XY : b_0 XY + c_0 Y^2)$$

for coefficients $a_0, b_0, c_0 \in K$.

Fix a place $v \in M_K \setminus S$. Since Φ has good reduction at v, it follows from Proposition 3 that Φ is K-isomorphic to some map $\Psi_v = (\psi_v, (1 : 0), (0 : 1))$, where

$$\psi_v(X : Y) = (X^2 + a_v XY : b_v XY + c_v Y^2) \tag{10}$$

for $a_v, b_v, c_v \in \mathcal{O}_v^\times$ such that

$$\mathrm{Res}(X^2 + a_v XY, b_v XY + c_v Y^2) = c_v(c_v - a_v b_v) \in \mathcal{O}_v^\times.$$

Let $f_v \in \mathrm{PGL}_2(K)$ be the automorphism such that $\Psi_v = \Phi^{f_v}$. Since both ψ_v and ϕ fix the points $(1 : 0)$ and $(0 : 1)$, the automorphism f_v must fix these points as well, so we may write $f_v(X, Y) = (\alpha_v X : Y)$ for some $\alpha_v \in K^\times$. Conjugating ϕ by f_v we obtain

$$\phi^{f_v}(X : Y) = (X^2 + \alpha_v^{-1} a_0 XY : b_0 XY + \alpha_v^{-1} c_0 Y^2). \tag{11}$$

Since $\psi_v = \phi^{f_v}$, comparing (10) and (11) we obtain the three identities

$$\begin{aligned} a_v &= \alpha_v^{-1} a_0, \\ b_v &= b_0, \\ c_v &= \alpha_v^{-1} c_0. \end{aligned}$$

Since $\mathcal{O}_S$ is a principal ideal domain, there exists $\alpha \in K^\times$ such that $|\alpha|_v = |\alpha_v|_v$ for each $v \in M_K \setminus S$. Define $\Psi = (\psi, (1:0), (0:1))$ for

$$\psi(X:Y) = (X^2 + aXY : bXY + cY^2)$$

with coefficients given by

$$\begin{aligned} a &= \alpha^{-1} a_0, \\ b &= b_0, \\ c &= \alpha^{-1} c_0. \end{aligned}$$

Then $\Psi = \Phi^f$ for the automorphism $f \in \mathrm{PGL}_2(K)$ defined by $f(X:Y) = (\alpha X : Y)$. Furthermore, for each place $v \in M_K \setminus S$ we have $|a|_v = |\alpha^{-1}a_0|_v = |\alpha_v^{-1}a_0|_v = |a_v|_v = 1$, whereby $a \in \mathcal{O}_v^\times$. Similar calculations show that b, c, and $c(c-ab)$ are all elements of the unit group $\mathcal{O}_v^\times$ as well. Since these elements are in $\mathcal{O}_v^\times$ for all $v \in M_K \setminus S$, we have $a, b, c, c(c-ab) \in \mathcal{O}_S^\times$. □

Lemma 5. *Given $u \in \bar{K}$, define V_u to be the set of all $\langle\phi\rangle$ in $\mathcal{M}_2$ for $\phi \in \mathrm{Rat}_2$ of the form*

$$\phi(X:Y) = (X^2 + aXY : bXY + cY^2) \tag{12}$$

with $\frac{ab}{c} = u$. Then V_u is a proper Zariski-closed subset of $\mathcal{M}_2$.

Proof. Define W_u to be the set of maps $\phi \in \mathrm{Rat}_2$ of the form (12) with $\frac{ab}{c} = u$. In the notation of §2.1 and §2.2, W_u is the intersection of the three hypersurfaces $\{a_2 = 0\}$, $\{b_0 = 0\}$, and $\{a_1 b_1 = u a_0 b_2\}$ in Rat_2, and thus W_u is Zariski-closed in Rat_2. Since V_u is the image of W_u under the closed quotient map $\langle\cdot\rangle : \mathrm{Rat}_2 \to \mathcal{M}_2$, it follows that V_u is Zariski-closed in $\mathcal{M}_2$.

It remains to show that V_u is a proper subset of $\mathcal{M}_2$. We will prove the stronger statement that V_u and $V_{u'}$ are disjoint whenever $u \neq u'$, for if $V_u \cap V_{u'}$ is nonempty, then there exist maps $\phi \in W_u$ and $\phi' \in W_{u'}$ which are PGL_2-conjugate to one another, say $\phi' = \phi^f$ for $f \in \mathrm{PGL}_2$. In the obvious notation we therefore have $\frac{ab}{c} = u$ and $\frac{a'b'}{c'} = u'$. Since both ϕ and ϕ' fix both $(0:1)$ and $(1:0)$, the automorphism f must fix these points as well, whereby $f(X:Y) = (\alpha X : Y)$ for some $\alpha \in \bar{K}$. But for maps of the form (12), the quantity $\frac{ab}{c}$ is invariant under the action of automorphisms of this form, because conjugating ϕ by f we have

$$\phi^f(X:Y) = (X^2 + \alpha^{-1}aXY : bXY + \alpha^{-1}cY^2)$$

and $\frac{(\alpha^{-1}a)b}{\alpha^{-1}c} = \frac{ab}{c}$. It follows that $u = u'$. This completes the verification that $V_u \cap V_{u'} = \emptyset$ whenever $u \neq u'$, and therefore each V_u is a proper subset of $\mathcal{M}_2$. □

Proof of Theorem 2. As enlarging S proves a stronger statement, first let us increase the size of S so that $\mathcal{O}_S$ is a principal ideal domain.

Each point in $\mathcal{G}^{\mathrm{uf}}_{2,2}(K,S)$ is $\langle\Phi\rangle$ for some $\Phi\in\mathrm{Rat}^{\mathrm{uf}}_{2,2}(K)$ with good reduction at all places $v\in M_K\setminus S$, and according to Proposition 4 we may assume without loss of generality that each such map takes the form $\Phi=(\phi,(1:0),(0:1))$ where

$$\phi(X:Y)=(X^2+aXY:bXY+cY^2)$$

for $a,b,c,c(c-ab)\in\mathcal{O}_S^\times$. It follows that $(\frac{c-ab}{c},\frac{ab}{c})$ is a solution in $(\mathcal{O}_S^\times)^2$ to the unit equation $x+y=1$. Since there are only finitely many such solutions([1],§5.1), there exists a finite list of units $u_1,\dots,u_r\in\mathcal{O}_S^\times$ such that

$$\mathcal{G}^{\mathrm{uf}}_{2,2}(K,S)\subseteq V_{u_1}\cup\cdots\cup V_{u_r}, \tag{13}$$

where V_u denotes the the set of all $\langle\phi\rangle$ in $\mathcal{M}_2$ for $\phi\in\mathrm{Rat}_2$ of the form $\phi(X:Y)=(X^2+aXY:bXY+cY^2)$ with $\frac{ab}{c}=u$.

By Lemma 5, each V_u is a proper Zariski-closed subset of $\mathcal{M}_2$, and therefore (13) shows that $\mathcal{G}^{\mathrm{uf}}_{2,2}(K,S)$ is not Zariski-dense in $\mathcal{M}_2$. □

Remark 2. Theorem 2 cannot, in general, be improved to a finiteness theorem. For if $\mathcal{O}_S^\times$ is infinite and both u and $1-u$ are S-units, then considering the maps

$$\phi(X:Y)=(X^2+aXY:a^{-1}uXY+Y^2)$$

as a varies in $\mathcal{O}_S^\times$ produces an infinite subset of $\mathcal{G}^{\mathrm{uf}}_{2,2}(K,S)\cap V_u$.

On the other hand, $\mathcal{G}^{\mathrm{uf}}_{2,2}(\mathbb{Q},\{\infty\})$ and $\mathcal{G}^{\mathrm{uf}}_{2,2}(\mathbb{Q}(i),\{\infty\})$ are both empty, where $i=\sqrt{-1}$ and where ∞ denotes the sole Archimedean place of these fields, because the rings $\mathbb{Z}$ and $\mathbb{Z}[i]$ have no solutions in units to the unit equation $x+y=1$. Denoting by ρ a primitive sixth root of unity, the set $\mathcal{G}^{\mathrm{uf}}_{2,2}(\mathbb{Q}(\rho),\{\infty\})$ is finite (by Proposition 4 along with the fact that the ring of integers in $\mathbb{Q}(\rho)$ has a finite unit group) and nonempty (consider $\phi(X:Y)=(X^2+\rho XY:XY+Y^2)$).

Remark 3. Since a generic quadratic rational map in Rat_2 has three distinct unramified fixed points over $\bar{K}$, it would be reasonable to ask why we consider rational maps with double (rather than single or triple) unramified fixed-point structure.

First, Theorem 2 would be false in general if double unramified fixed-point structure were replaced by *single* unramified fixed-point structure, and a counterexample is given by the same family occurring in the second proof of Theorem 1. Recall that $\mathcal{V}$ is the set of all $\langle\phi\rangle$ in $\mathcal{M}_2$ for rational maps $\phi\in\mathrm{Rat}_2(K)$ taking the form $\phi(X:Y)=(X^2+\alpha XY:(\frac{1-\beta}{\alpha})XY+Y^2)$ for S-units $\alpha,\beta\in\mathcal{O}_S^\times$. At all places $v\in M_K\setminus S$, the point $(0:1)$ reduces to an unramified fixed point of $\tilde{\phi}_v$, but assuming that $\mathcal{O}_S^\times$ is infinite, $\mathcal{V}$ is Zariski-dense in $\mathcal{M}_2$.

One might define the space $\mathrm{Rat}^{\mathrm{uf}}_{2,3}(K)$ of quadratic rational maps with *triple* unramified fixed-point structure over K, but the Zariski-nondensity of the image in $\mathcal{M}_2$ of the set of all such maps having prescribed good reduction would follow at once from Theorem 2. Indeed, the map $\mathrm{Rat}^{\mathrm{uf}}_{2,3}(K)\to\mathcal{M}_2(K)$ factors through the map $\mathrm{Rat}^{\mathrm{uf}}_{2,3}(K)\to\mathrm{Rat}^{\mathrm{uf}}_{2,2}(K)$, which remembers the first two fixed points and forgets the third. In an obvious extension of the notation of Theorem 2, we therefore have $\mathcal{G}^{\mathrm{uf}}_{2,3}(K,S)\subseteq\mathcal{G}^{\mathrm{uf}}_{2,2}(K,S)$.

Remark 4. Although we have not attempted to carry out the necessary details, it may be possible to use geometric invariant theory to define a new moduli space $\mathcal{M}^{\mathrm{uf}}_{2,2}=\mathrm{Rat}^{\mathrm{uf}}_{2,2}/\mathrm{PGL}_2$ as the quotient of the quasiprojective variety $\mathrm{Rat}^{\mathrm{uf}}_{2,2}$ of quadratic rational maps with double unramified fixed-point structure modulo the conjugation action of PGL_2. While such a space may be of interest on its own merits, for our purposes there would be little to gain in such a construction, because a Zariski-nondensity result in $\mathcal{M}_{2,2}^{\mathrm{uf}}$ for maps with prescribed good reduction would follow trivially from Theorem 2, using the dominant map $\mathcal{M}_{2,2}^{\mathrm{uf}} \to \mathcal{M}_2$ obtained from forgetting the double unramified fixed-point structure.

5. Prescribed good reduction for quadratic rational maps with unramified 2-cycle structure

With nearly the same proof, a variation on Theorem 2 can be established in which 2-cycle structure is used in place of double fixed-point structure.

Definition. Let $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ be the set of all triples of the form $\Phi = (\phi, P_1, P_2)$, where $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a quadratic rational map defined over K, and where $P_1, P_2 \in \mathbb{P}^1(K)$ are distinct K-rational points which are not ramified points of ϕ and for which $\phi(P_1) = P_2$ and $\phi(P_2) = P_1$. We call such a triple Φ a *quadratic rational map with double unramified* 2*-cycle structure over* K.

Definitions. Two maps Φ and Ψ in $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ are *K-isomorphic* if $\Psi = \Phi^f$ for some automorphism $f \in \mathrm{PGL}_2(K)$, where

$$\Phi^f = (\phi^f, f^{-1}(P_1), f^{-1}(P_2)).$$

A map Φ in $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ has *good reduction* at a non-Archimedean place v of K if it is K-isomorphic to some $\Psi = (\psi, Q_1, Q_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ such that $\deg(\tilde{\psi}_v) = 2$ and such that $\tilde{Q}_1$ and $\tilde{Q}_2$ are distinct points in $\mathbb{P}^1(\mathbb{F}_v)$ which are not ramified points of $\tilde{\psi}_v$ and for which $\tilde{\psi}_v(\tilde{Q}_1) = \tilde{Q}_2$ and $\tilde{\psi}_v(\tilde{Q}_2) = \tilde{Q}_1$.

For each $\Phi = (\phi, P_1, P_2)$ in $\mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$, define $\langle\Phi\rangle = \langle\phi\rangle$. Thus, as before, the map $\langle\cdot\rangle : \mathrm{Rat}_{2,2}^{\mathrm{uc}}(K) \to \mathcal{M}_2(K)$ forgets the 2-cycle structure of Φ and preserves only the PGL_2-conjugacy class $\langle\phi\rangle$ of its underlying rational map.

Theorem 6. *Let K be a number field and let S be a finite set of places of K including the Archimedean places. Then the set*

$$\mathcal{G}_{2,2}^{\mathrm{uc}}(K,S) = \left\{ \langle\Phi\rangle \in \mathcal{M}_2(K) \,\middle|\, \begin{array}{l} \Phi \in \mathrm{Rat}_{2,2}^{\mathrm{uc}}(K) \textit{ has good} \\ \textit{reduction at all } v \in M_K \setminus S \end{array} \right\}$$

is not Zariski-dense in $\mathcal{M}_2$.

Proof. This proof follows precisely the same strategy as that of Theorem 2, and so we give only a sketch to highlight where this proof differs from the previous one.

Again, without loss of generality, we may enlarge S so that $\mathcal{O}_S$ is a principal ideal domain. Each point in $\mathcal{G}_{2,2}^{\mathrm{uc}}(K,S)$ is $\langle\Phi\rangle$ for some $\Phi \in \mathrm{Rat}_{2,2}^{\mathrm{uc}}(K)$ with good reduction at all places $v \in M_K \setminus S$, and in a similar fashion as in Proposition 4, it may be shown that each such map (up to K-isomorphism) takes the form $\Phi = (\phi, (1:0), (0:1))$, where

$$\phi(X:Y) = (aXY + bY^2 : X^2 + cXY)$$

for $a, b, c, b(b-ac) \in \mathcal{O}_S^\times$. It follows that $(\frac{b-ac}{b}, \frac{ac}{b})$ is a solution in $(\mathcal{O}_S^\times)^2$ to the unit equation $x+y=1$. Since there are only finitely many such solutions ([1], §5.1), there exists a finite list of units $u_1, \dots, u_r \in \mathcal{O}_S^\times$ such that

$$\mathcal{G}_{2,2}^{\mathrm{uf}}(K,S) \subseteq Z_{u_1} \cup \cdots \cup Z_{u_r},$$

where Z_u denotes the set of all $\langle\phi\rangle$ in $\mathcal{M}_2$ for $\phi \in \mathrm{Rat}_2$ of the form $\phi(X : Y) = (aXY + bY^2 : X^2 + cXY)$ with $\frac{ac}{b} = u$, and each Z_u is a proper Zariski-closed subset of $\mathcal{M}_2$. $\square$

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