Caccioppoli Estimates Through an Anisotropic Picone’s Identity

JAROSLAV JAROŠ

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Abstract. Caccioppoli-type estimates for a class of nonlinear differential operators which include the $p$-Laplacian and the pseudo $p$-Laplacian as special cases are obtained by means of the differential identity involving an arbitrary norm in $\mathbb{R}^n$ which generalizes the well-known multidimensional Picone’s formula.

1. Introduction

One of the alternative ways to establish the basic Caccioppoli inequality in which the $L^2$ norm of the gradient of a harmonic (or subharmonic) function $v$ is estimated in terms of the $L^2$ norm of $v$ itself is to derive it from the Picone identity (see [15] or [17])

\begin{equation}
\|\nabla u\|^2 - \langle \nabla \left( \frac{u^2}{v} \right), \nabla v \rangle = \|\nabla u - \frac{u}{v} \nabla v\|^2,
\end{equation}

where $u$ and $v$ are differentiable functions on a given domain $\Omega \subset \mathbb{R}^n$ with $v(x) \neq 0$ in $\Omega$ and $\|\cdot\|_2$, $\nabla$ and $\langle \cdot, \cdot \rangle$ stand for the Euclidean norm, the usual gradient and the inner product in $\mathbb{R}^n$, respectively.

Indeed, if $v > 0$ is a weak (continuous) solution of $\text{div} (\nabla v) = 0$ in $\Omega$ and $\eta \in C^\infty_c(\Omega)$ is a nonnegative test function, then from the integrated form of (1.1) with $u = \eta v$, using Young’s and the Cauchy-Schwartz inequality, we easily obtain

\begin{equation}
0 = \int_\Omega \|\eta \nabla v\|^2 dx - 2 \int_\Omega \langle \eta \nabla v, \nabla (\eta v) \rangle dx \leq -\frac{1}{2} \int_\Omega \|\eta \nabla v\|^2 dx + 2 \int_\Omega \|v \nabla \eta\|^2 dx,
\end{equation}

which yields the desired estimate

\begin{equation}
\int_\Omega \|\eta \nabla v\|^2 dx \leq 4 \int_\Omega \|v \nabla \eta\|^2 dx.
\end{equation}

More generally, if $p > 1$ is fixed and $v > 0$ is a weak continuous solution (or subsolution) of the $p$-harmonic equation $\text{div}(\|\nabla v\|^{p-2}_2 \nabla v) = 0$, then the $L^p$-version of the Caccioppoli estimate ([11]; see also the inequality (5.27) in [16] and Corollary A.6 in [13])

\begin{equation}
\int_\Omega \|\eta \nabla v\|^p dx \leq p \int_\Omega \|v \nabla \eta\|^p dx
\end{equation}

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can easily be obtained from the $p$-Laplacian generalization of Picone’s identity (see [2] and [9])

\begin{equation}
\|\nabla u\|^p_2 - \left\langle \nabla \left( \frac{|u|^p}{vp-1} \right), \|\nabla v\|^p_2 - \frac{2}{p-1} \langle \nabla u, \|\nabla v\|^p_2 \rangle \right\rangle = \Phi_p(u, v),
\end{equation}

where

\begin{equation}
\Phi_p(u, v) := \|\nabla u\|^p_2 + (p - 1)\frac{|u|^p}{vp-1} \|\nabla v\|^p_2 - p\frac{|u|^{p-2}u}{vp-1} \langle \nabla u, \|\nabla v\|^{p-2} \rangle \geq 0,
\end{equation}

by setting $u = \eta v$ and making use of Young’s and the Hölder inequality.

The purpose of this paper is to extend the Caccioppoli inequality to a class of differential operators of the form

\begin{equation}
\Delta_{H,p} v := \text{div} \left( H(\nabla v)^{p-1} \nabla \xi H(\nabla v) \right)
\end{equation}

where $p > 1$, $H : \mathbb{R}^n \to [0, +\infty)$, $n \geq 2$, is a convex function of the class $C^1(\mathbb{R}^n \backslash \{0\})$ which is positively homogeneous of degree 1 and $\nabla$ and $\nabla \xi$ stand for usual gradient operators with respect to the variables $x$ and $\xi$, respectively. We refer to the operator $\Delta_{H,p}$ as the Finsler $p$-Laplacian (or the anisotropic $p$-Laplacian). A prototype of $H$ satisfying the above conditions is the $l_r$-norm

\begin{equation}
H(\xi) = \|\xi\|_r = \left( \sum_{i=1}^{n} |\xi_i|^{r} \right)^{1/r}, \quad r > 1,
\end{equation}

for which the operator defined by (1.6) has the form

\begin{equation}
\Delta_{r,p} v := \text{div} \left( \|\nabla v\|^{p-r} \nabla^r v \right)
\end{equation}

where

\begin{equation}
\nabla^r v := \left( \left| \frac{\partial v}{\partial x_1} \right|^{r-2} \frac{\partial v}{\partial x_1}, \ldots, \left| \frac{\partial v}{\partial x_n} \right|^{r-2} \frac{\partial v}{\partial x_n} \right).
\end{equation}

The class of operators of the form (1.8) includes the usual $p$-Laplacian and the so-called pseudo $p$-Laplace operator as the special cases corresponding to $r = 2$ and $p \in (1, \infty)$, and $r = p > 1$, respectively. Clearly, if $p = r = 2$, then (1.8) reduces to the standard Laplacian $\Delta$.

Anisotropic elliptic problems involving this kind of operator and related questions have recently been studied in several papers including [1], [3]–[8], [10] and [18]–[20].

Our main result in this note is Theorem 3.1 below which states that if $v > 0$ is a weak subsolution in $\Omega$ of the equation

$$-\Delta_{H,p} v = g(x)|v|^{p-2}v,$$

where $0 \leq g \in L_\text{loc}^\infty(\Omega)$, then, for any fixed $q > p - 1$ and $w = v^{q/p}$, the inequality

$$\int_{\Omega} H(\eta \nabla w)^p dx \leq \left( \frac{q}{q - p + 1} \right)^p \int_{\Omega} H(w \nabla \eta)^p dx + \frac{q^{p-1}}{q - p + 1} \int_{\Omega} g(x)w^p \eta^p dx$$

holds for any nonnegative function $\eta \in C^\infty_0(\Omega)$. Analogous result is valid also for positive supersolutions of the above equation and $q < p - 1$.

The paper is organized as follows. Section 2 contains a brief review of the basic properties of general norms in $\mathbb{R}^n$ that will be used in the sequel. Here we also present an extension of Picone’s identity to the Finsler $p$-Laplacian. In Section 3 we establish Caccioppoli-type estimates for positive sub- and supersolutions of nonlinear equations involving anisotropic elliptic operators.
2. Background

We begin by recalling basic properties of general norms in \( \mathbb{R}^n \) and their duals. For the proofs see [5] or [6].

Let \( H \) be an arbitrary norm in \( \mathbb{R}^n \), i.e., a convex function \( H : \mathbb{R}^n \to [0, \infty) \) satisfying \( H(\xi) > 0 \) for all \( \xi \neq 0 \) which is positively homogeneous of degree 1, so that

\[
H(t\xi) = |t|H(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.
\]

If we assume that \( H \in C^1(\mathbb{R}^n \setminus \{0\}) \), then from (2.1) it follows that

\[
\nabla_\xi H(t\xi) = \text{sgn } t \nabla_\xi H(\xi) \quad \text{for all } \xi \neq 0 \text{ and } t \neq 0
\]

and

\[
\langle \xi, \nabla_\xi H(\xi) \rangle = H(\xi) \quad \text{for all } \xi \in \mathbb{R}^n
\]

where the left-hand side is defined to be 0 if \( \xi = 0 \). Let \( \langle \cdot, \cdot \rangle \) denote the usual inner product in \( \mathbb{R}^n \) and define the dual norm \( H_0 \) of \( H \) by

\[
H_0(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H(\xi)} \quad \text{for } x \in \mathbb{R}^n.
\]

The \( H_0 \)-unit-ball, i.e., the set \( \{ x \in \mathbb{R}^n : H_0(x) \leq 1 \} \) is sometimes called the Wulff shape (or equilibrium crystal shape) of \( H \).

Any norm \( H \) of the class \( C^1 \) for \( \xi \neq 0 \) and its dual \( H_0 \) satisfy

\[
H_0(\nabla_\xi H(\xi)) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.
\]

Similarly, if \( H_0 \) is continuously differentiable for \( x \neq 0 \), then

\[
H(\nabla H_0(x)) = 1 \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.
\]

Also, the identities

\[
H\left[H_0(x)\nabla H_0(x)\right] \nabla_\xi H\left[H_0(x)\nabla H_0(x)\right] = x
\]

and

\[
H_0\left[H(\xi)\nabla_\xi H(\xi)\right] \nabla H_0\left[H(\xi)\nabla_\xi H(\xi)\right] = \xi
\]

hold for all \( x, \xi \in \mathbb{R}^n \), where \( H(0)\nabla_\xi H(0) \) and \( H_0(0)\nabla H_0(0) \) are defined to be 0.

From the definition (2.4) we easily obtain the Hölder-type inequality

\[
\langle x, \xi \rangle \leq H(\xi)H_0(x) \quad \text{for all } x, \xi \in \mathbb{R}^n
\]

with equality holding if and only if

\[
x = H(\xi)\nabla_\xi H(\xi) \quad \text{(or, equivalently, } H_0(x) = H(\xi)).
\]

It is well known that a continuously differentiable function \( F \) defined in an open convex subset of \( \mathbb{R}^n \) is strictly convex if and only if

\[
F(y) - F(x) - \langle \nabla F(x), y - x \rangle > 0
\]

for all \( x \neq y \) (see, for example, [14]). As an immediate consequence of this result we have the following simple lemma.

**Lemma 2.1.** Let \( H \) be a norm in \( \mathbb{R}^n \) such that \( H \in C^1(\mathbb{R}^n \setminus \{0\}) \) and \( H^p, 1 < p < \infty \), is strictly convex. If

\[
H(\xi)^p + (p-1)H(\eta)^p - p\langle \xi, H(\eta)^{p-1}\nabla H(\eta) \rangle = 0
\]

for some \( \xi, \eta \in \mathbb{R}^n, \eta \neq 0, \) and \( H(\xi) = H(\eta), \) then \( \xi = \eta. \)
Proof. If $\xi, \eta \in \mathbb{R}^n$ with $\eta \neq 0$ satisfy $H(\xi) = H(\eta)$ and (2.12), then
\[
(2.13) \quad 0 = pH(\eta)^p - p\langle \eta, H(\eta)^{p-1}\nabla H(\eta) \rangle + p\langle \eta - \xi, H(\eta)^{p-1}\nabla H(\eta) \rangle
= p\langle \eta - \xi, H(\eta)^{p-1}\nabla H(\eta) \rangle.
\]
We claim that $pH(\eta)^{p-1}\nabla H(\eta) = \nabla(H(\eta))^p \neq 0$. Indeed, if $\nabla(H(\eta))^p$ were the zero vector for some $\eta \in \mathbb{R}^n$, i.e., even strictly convex function $H(\eta)^p$ attained its global minimum at $\eta$, then $\eta$ would necessarily be equal to 0, a contradiction. Therefore, by strict convexity of $H^p$, $\xi = \eta$, and the proof is complete. \quad \Box

Next, we present a generalization of Picone’s identity for the Finsler $p$-Laplacian $\Delta_{H,p}$.

Lemma 2.2. Let $H$ be an arbitrary norm in $\mathbb{R}^n$ which is of class $C^1$ for $x \neq 0$. Assume that $u, v \in W^{1,p}_p(\Omega) \cap C(\Omega)$ with $v(x) \neq 0$ in $\Omega$ and denote
\[
\Phi(u, v) := H(\nabla u)^p + (p-1) \frac{|u|^p}{|v|^p} H(\nabla v)^p - p \frac{|u|^{p-2}u}{|v|^{p-2}v} \langle \nabla u, H(\nabla v)^{p-1}\nabla \xi H(\nabla v) \rangle.
\]
Then
\[
(2.14) \quad H(\nabla u)^p - \langle \nabla \left( \frac{|u|^p}{|v|^{p-2}v} \right), H(\nabla v)^{p-1}\nabla \xi H(\nabla v) \rangle = \Phi(u, v)
\]
and $\Phi(u, v) \geq 0$ a.e. in $\Omega$. If, in addition, $H(\xi)^p$ is strictly convex in $\mathbb{R}^n$, then $\Phi(u, v) = 0$ a.e. in $\Omega$ if and only if $u$ is a constant multiple of $v$ in $\Omega$.

Proof. First, verify (2.14) directly by expanding the left-hand side of (2.14). Next, notice that $\Phi(u, v) = \Phi_1(u, v) + \Phi_2(u, v)$, where
\[
\Phi_1(u, v) := H(\nabla u)^p - pH(\nabla u)H\left(\frac{u}{v}\nabla v\right)^{p-1} + (p-1)H\left(\frac{u}{v}\nabla v\right)^p
\]
and
\[
\Phi_2(u, v) := p[H(\nabla u)H\left(\frac{u}{v}\nabla v\right)^{p-1} - \langle \nabla u, H\left(\frac{u}{v}\nabla v\right)^{p-1}\nabla \xi H\left(\frac{u}{v}\nabla v\right) \rangle],
\]
and apply the Young inequality to $\Phi_1$ and the generalized Hölder inequality (2.9) to $\Phi_2$ to conclude that both $\Phi_1(u, v) \geq 0$ and $\Phi_2(u, v) \geq 0$.

Finally, $\Phi(u, v) = 0$ means that equality case simultaneously occurs in the Young inequality and the Hölder inequality (2.9) a.e. in $\Omega$. A necessary and sufficient condition for the first equality is that
\[
(2.15) \quad H(\nabla u) = H\left(\frac{u}{v}\nabla v\right) \quad \text{a.e. in } \Omega.
\]

If $(u\nabla v/v)(x_0) \neq 0$ for some $x_0 \in S := \{x \in \Omega : \Phi(u, v) = 0\}$, then by Lemma 2.1 we have $\nabla u = u \nabla v/v$ at $x_0$, or, equivalently, $\nabla(u/v)(x_0) = 0$. On the other hand, if $u\nabla v/v = 0$ on some subset $S_0$ of $S$, then $\nabla u = 0$ a.e. in $S_0$ which implies $\nabla(u/v) = 0$ a.e. in $S_0$. Summarizing the above facts we get $\nabla(u/v) = 0$ a.e. in $\Omega$ which forces $u/v$ to be constant in $\Omega$. \quad \Box

Remark 2.1. In the special case where $H$ is an $l_r$-norm, $r \in (1, \infty)$, the differential identity (2.14) reduces to
\[
(2.16) \quad \|\nabla u\|_r^p - \left\langle \nabla \left( \frac{|u|^p}{|v|^{p-2}v} \right), \|\nabla v\|_r^{p-r}\nabla v \right\rangle = \Phi_{p,r}(u, v),
\]
where $\nabla^r v$ is defined in (1.9) and

$$(2.17) \quad \Phi_{p,r}(u,v) := \|\nabla u\|_p^p + (p-1)\frac{|u|^p}{|v|^p} \|\nabla v\|_p^p - p \frac{|u|^{p-2}u}{|v|^{p-2}v} \langle \nabla u, \|\nabla v\|_p^{p-r} \nabla^r v \rangle \geq 0.$$ 

3. Caccioppoli-type estimates

Consider the equation

$$(3.1) \quad -\Delta_{H,p} v = g(x)|v|^{p-2}v,$$ 

where $0 \leq g \in L^\infty_{locc}(\Omega)$. Following the classical terminology, we will say that a continuous function $v \in W^{1,p}_{loc}(\Omega)$ is a weak solution of (3.1), if for any function $\eta \in W^{1,p}_{c}(\Omega) \cap C(\Omega)$ we have

$$(3.2) \quad \int_{\Omega} H(\nabla v)^{p-1} \langle \nabla_\xi H(\nabla v), \nabla \eta \rangle \, dx = \int_{\Omega} g|v|^{p-2}v\eta \, dx.$$ 

Here $W^{1,p}_{c}(\Omega)$ is used to denote the set of all functions $u \in W^{1,p}_{locc}(\Omega)$ which are compactly supported in $\Omega$.

Weak subsolutions and supersolutions of (3.1) are defined analogously with test functions $\eta \in W^{1,p}_{c}(\Omega) \cap C(\Omega)$ satisfying $\eta \geq 0$ and “=” in (3.2) replaced by “$\leq$” and “$\geq$”, respectively.

We associate with (3.1) the homogeneous (of degree $p$) functional

$$(3.3) \quad J_H(u;\Omega) := \int_{\Omega} [H(\nabla u)^p - g(x)|u|^p] \, dx, \quad u \in W^{1,p}_{c}(\Omega) \cap C(\Omega).$$

If $v(x) > 0$ is a (continuous) weak subsolution of (3.1) in $\Omega$ and $u \in W^{1,p}_{c}(\Omega) \cap C(\Omega)$, then we can choose

$$\eta = \frac{|u|^p}{v^{p-1}}$$

as a test function in (3.2) and conclude, by Picone’s identity (2.14), that

$$(3.4) \quad J_H(u;\Omega) \leq \int_{\Omega} \left[ H(\nabla u)^p - H(\nabla v)^{p-1} \langle \nabla_\xi H(\nabla v), \nabla \left( \frac{|u|^p}{v^{p-1}} \right) \rangle \right] \, dx$$

$$= \int_{\Omega} \Phi(u,v) \, dx.$$ 

Clearly, for positive supersolutions $v$ of (3.1) and any $u \in W^{1,p}_{c}(\Omega) \cap C(\Omega)$, the reversed inequality

$$(3.5) \quad J_H(u;\Omega) \geq \int_{\Omega} \Phi(u,v) \, dx$$

holds true. For solutions (3.4) becomes an equality.

The following estimate is the main result of this paper.

**Theorem 3.1** (Caccioppoli inequality). Let $v > 0$ be a weak subsolution of (3.1) in $\Omega$. Then, for any fixed $q > p - 1$ and $w = v^{q/p}$, the inequality

$$(3.6) \quad \int_{\Omega} H(\eta \nabla w)^p \, dx \leq \left( \frac{q}{q-p+1} \right)^p \int_{\Omega} H(w \nabla \eta)^p \, dx + \frac{q^p p-1}{q-p+1} \int_{\Omega} g(x) w^p \eta^p \, dx$$

holds for all $0 \leq \eta \in C^{\infty}_{c}(\Omega)$. 

Proof. Let $v$ be a positive subsolution of (3.1) in $\Omega$. Fix a nonnegative function $\eta \in C_c^\infty(\Omega)$. Then $u := v^{q/p}\eta$ belongs to $W^{1,p}_c \cap C(\Omega)$ and we can use it as a test function in (3.4) to get

$$J_H(v^{q/p}\eta; \Omega) \leq \int_{\Omega} H(\nabla(v^{q/p}\eta))^p dx + (p-1) \int_{\Omega} v^{q-p} H(\eta \nabla v)^p dx$$

$$-p \int_{\Omega} (v^{q(p-1)}/p\eta)^{p-1} \langle H(\nabla v)^{p-1} \nabla \xi H(\nabla v), \nabla (v^{q/p}\eta) \rangle dx$$

$$= \int_{\Omega} H(\nabla(v^{q/p}\eta))^p dx - (q-p+1) \int_{\Omega} v^{q-p} H(\eta \nabla v)^p dx$$

$$-p \int_{\Omega} (H(v^{q(p-1)/p}\eta \nabla v)^{p-1} \nabla \xi H(v^{q(p-1)/p}\eta \nabla v), v^{q/p} \nabla \eta) dx,$$

where we have used $\nabla (v^{q/p}\eta) = (q/p)v^{q(p-1)/p+1} \nabla v + v^{q/p} \nabla \eta$ and the property (2.5) of the norm $H$. To estimate the latter integral, we use the generalized Hölder inequality (2.9), (2.5) and Young’s inequality in the form

$$ab^{p-1} \leq \frac{1}{pr^{p-1}} a^p + \frac{p-1}{p} r b^p, \quad a, b \geq 0, \quad \tau > 0,$$

applied to $a = H(v^{q/p} \nabla \eta)$ and $b = H(v^{q(p-1)/p+1} \nabla v)$. As a result we obtain

$$J_H(v^{q/p}\eta; \Omega) \leq \int_{\Omega} H(\nabla(v^{q/p}\eta))^p dx - [q-p+1-\tau(p-1)] \int_{\Omega} H(\nabla(v^{q(p-1)/p+1}\eta \nabla v))^p dx$$

$$+ \frac{1}{\tau p-1} \int_{\Omega} H(v^{q/p} \nabla \eta)^p dx.$$  

Now, making use of the definition of $J_H$, we get

$$\int_{\Omega} v^{q-p} H(\eta \nabla v)^p dx \leq \frac{\tau^{1-p}}{q-p+1-\tau(p-1)} \int_{\Omega} H(v^{q(p-1)/p} \nabla \eta)^p dx$$

$$+ \frac{1}{q-p+1-\tau(p-1)} \int_{\Omega} g(x) v^{q(p-1)/p} \eta^p dx,$$

which after choosing the constant $\tau$ appropriately leads to

$$\int_{\Omega} v^{q-p} H(\eta \nabla v)^p dx \leq \left( \frac{p}{q-p+1} \right)^p \int_{\Omega} H(v^{q(p-1)/p} \nabla \eta)^p dx + \frac{p}{q-p+1} \int_{\Omega} g(x) v^{q(p-1)/p} \eta^p dx.$$  

Finally, the substitution $w = v^{q/p}$ yields (3.6) as claimed. \hfill \Box

**Corollary 3.1.** Let $q = p$ and $g(x) \equiv 0$ in $\Omega$. If $v > 0$ is a weak subsolution of (3.1) in $\Omega$, then

$$\int_{\Omega} H(\eta \nabla v)^p dx \leq p \int_{\Omega} H(v \nabla \eta)^p dx$$

for any nonnegative $\eta \in C_c^\infty(\Omega)$.  

For the next corollary, let $B(x_0; r)$ be the Wulff ball with the radius $r > 0$ centered at $x_0$, i.e., the set $\{ x \in \mathbb{R}^n : H_0(x - x_0) \leq r \}$, and denote by $B_r$ and $B_{2r}$ two concentric $H_0$-balls with $x_0 \in \Omega$.

**Corollary 3.2.** If the conditions of Corollary 3.1 are satisfied and $B_{2r} \subset \Omega$, then

$$\int_{B_r} H(\nabla v)^p dx \leq (p/r)^p \int_{B_{2r}\setminus B_r} v^p dx.$$
Proof. To get the estimate (3.10), choose in (3.9) an $H_0$-radial test function $\eta$ which satisfies $\eta \equiv 1$ in $B_r$, $H_0(\nabla \eta) \leq 1/r$ in $B_{2r} \setminus B_r$ and $\eta \equiv 0$ in $\Omega \setminus B_{2r}$. □

An analogous result as Theorem 3.1 above holds for positive supersolutions of (3.1) and a range of $q$ smaller than $p - 1$. The proof is left to the reader. (If necessary, use $v(x) + \epsilon, \epsilon > 0$ instead of $v(x)$.)

**Theorem 3.2.** Let $v > 0$ be a supersolution of (3.1) in $\Omega$. Then, for any fixed $q < p - 1$, the inequality

\[
\int_{\Omega} v^{q-p} H(\eta \nabla v)^p dx \leq \left( \frac{p}{p-q-1} \right)^p \int_{\Omega} H(\eta^{q/p} \nabla \eta)^p dx - \frac{p}{p-1} \int_{\Omega} g(x)v \eta^{p} dx
\]

holds for all $\eta \in C^\infty_c(\Omega)$ with $\eta \geq 0$.

The particular case $q = 0$ of the above theorem is interesting in the sense that the right-hand side of (3.11) does not contain $v$. The result specializes as follows.

**Corollary 3.3** (Logarithmic Caccioppoli inequality). Let $v > 0$ be a weak supersolution of (3.1) in $\Omega$. Then

\[
\int_{\Omega} H(\eta \nabla \log v)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_{\Omega} H(\nabla \eta)^p dx - \frac{p}{p-1} \int_{\Omega} g(x)|\eta|^p dx,
\]

whenever $0 \leq \eta \in C^\infty_c(\Omega)$.

If $H(\xi) = ||\xi||_2, \xi \in \mathbb{R}^n$, and $g(x) \equiv 0$ in $\Omega$, then (3.12) reduces to the well-known logarithmic Caccioppoli inequality for positive $p$-superharmonic functions (cf. [12]).

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**References**


