UNIFORM APPROXIMATION OF METRICS BY GRAPHS

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ABSTRACT. We say that a metric graph is uniformly bounded if the degrees
of all vertices are uniformly bounded and the lengths of edges are pinched
between two positive constants; a metric space is approximable by a uniform
graph if there is one within a finite Gromov-Hausdorff distance. We show that
the Euclidean plane and Gromov hyperbolic geodesic spaces with bounded
geometry are approximable by uniform graphs, and pose a number of open
problems.

1. INTRODUCTION

In this paper we are concerned with graph approximations of Riemannian manifold as metric spaces. We will address problems of spectral approximation elsewhere.

By a *metric graph* we mean an undirected graph whose edges are labeled by positive numbers called *edge lengths*. This naturally turns the set of vertices of the graph into a metric space (where some distances may be infinite). Namely one defines the length of a path in a metric graph as the sum of edge lengths along the path, and then the distance $d_\Gamma(p, q)$ between vertices p and q of a metric graph Γ is defined as the infimum of lengths of paths connecting p and q.

We say that a metric graph is *uniform* if there are positive constants M, D and δ such that the degree of every vertex is no greater than M and the length of every edge is between δ and D.

We say that a metric space is approximable by a uniform graph if there exists a uniform graph which is within finite Gromov–Hausdorff distance from the space.

For this paper, the reader does not need to even know what the Gromov–Hausdorff distance is. We say that two metrics on the same set are *additively close* if there exists a constant C such that the difference of the two distances between every two points is at most C. Also recall that a *net* in metric space is a subset which is an ε-net for some $\varepsilon > 0$ (for example $\mathbb{Z}^2$ is a net in $\mathbb{R}^2$). Now the definition can be reformulated in a more concrete way by means of the following trivial lemma:

Lemma 1.1. *A manifold is approximable by a uniform graph if and only if there exists a uniform graph whose vertices form a net in the manifold and whose distance*

function on the set of vertices is additively close to the restriction of the distance function on the manifold. □

The general question which remains widely open is the following problem.

Problem. What complete (Riemannian or even Finsler) manifolds are approximable by uniform graphs?

Let us emphasize that we are interested in an approximation with a bounded *additive* error. Say, for $\mathbb{R}^2$, the question is trivial for quasi-isometries. Indeed, the standard grid approximates the Euclidean metric up to a factor of $\sqrt{2}$. The question of making the same cone at infinity as that of the Euclidean plane is more subtle though it is still much easier (see [3] and the references there). However, let us reiterate that we want the difference between distances to be bounded.

We do not have a single example of a manifold with bounded geometry for which we can prove that it is not approximable by a uniform graph (one can easily construct such examples with sectional curvature rapidly going to $-\infty$). However, so far we cannot even prove that $\mathbb{R}^3$ with its standard metric is approximable by a uniform graph.

If the answer is affirmative for a certain class of manifolds, one also wonders how the constants in the definition of the uniform graph and the additive error C depend on the class (dimension, injectivity radius and such). This makes this problem meaningful for compact manifolds. Formally, one can consider a disjoint union of manifolds from a certain class and ask if it is approximable by a uniform graph. For instance, asking if the spheres are approximable by uniform graphs with the same constants is the same as asking if the disjoint union of spheres with integer radii is approximable by a uniform graph.

One can easily see that if $\mathbb{R}^n$ is approximable by a uniform graph, then the graph can be chosen so that its vertices form the standard integer lattice and the degree of every vertex does not exceed three. We do not use this fact in the sequel, so we give only a punchline of a (rather easy) argument. Modifying the graph to move vertices to a lattice and reducing their degrees to at most three can be done in three steps. First, one can get rid of vertices of higher degree by replacing their neighborhoods by graphs with controlled parameters. For each vertex, one just cuts all outgoing edges in the middle, removes the vertex and replaces it by an appropriate degree 3 metric graph. It is crucial to make sure that all distances are exactly preserved. Next, one chooses a very fine square lattice and moves each vertex to the nearest lattice point. Lengths of edges stay the same. Still, many points of the lattice are not occupied by vertices. To fix this, it is enough to add a whole bunch of vertices of degree two on existing edges and move them to lattice points, having made the edges to pass through all lattice points. Finally, one rescales the lattice to get the integer one and multiplies the edge lengths by the same factor. Of course, the constant would deteriorate drastically, but one still gets a uniform approximation.

The main result of the paper is the following:

Theorem 1. *There exists a uniform metric graph Γ whose set of vertices is the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, such that its distance function d_Γ is additively close to the standard Euclidean metric:*

$$\left| d_\Gamma(p,q) - |p-q| \right| \leq C$$

for some constant $C > 0$ and all $p, q \in \mathbb{Z}^2$.

Even though we do not know the answer to the problem even for $\mathbb{R}^3$ (the next paragraph partially explains a difficulty), the problem becomes much easier for Gromov hyperbolic spaces:

Theorem 2. *Every simply connected complete Riemannian manifold M whose sectional curvature is negative and bounded away from 0 and $-\infty$ (more generally, every Gromov hyperbolic geodesic space of bounded coarse geometry) is approximable by a uniform graph.*

To illustrate some difficulties arising in approximating even $\mathbb{R}^2$ let us consider approximations by a periodic graph, that is, by a graph Γ invariant under two integer translations $(x, y) \mapsto (x + m, y)$ and $(x, y) \mapsto (x, y + n)$. One can think of constructing such a graph as first choosing a graph inside a large rectangle and then repeating it periodically to tile the entire plane. In this case, one can show that there exists a norm $\| \cdot \|$ on $\mathbb{R}^2$ such that the distance d_Γ is additively close to the norm: there is a constant C such that for every two vertices p, q of the graph, $|d_\Gamma(p, q) - \|p - q\|| < C$. Furthermore, one can see that the unit ball of the norm is a polygon (with polynomially many sides). Hence a periodic graph not only cannot be additively close to the Euclidean plane, but actually the difference between its metric and the Euclidean one grows linearly with the distance between points.

The rest of the paper is organized as follows. In Section 3 we derive Theorem 1 from analytic lemmas proven in Section 4. The proof of Theorem 2 is contained in Section 5. Section 2 is not needed for understanding the proofs. It informally discusses several problems in Dynamics and Analysis motivated by the proof of Theorem 1. In particular, resolving some of these problems could possibly help to handle dimensions higher than two.

Remark 1.2. There is a problem that sounds rather similar. It asks if one can approximate the Euclidean distance function between points of the integer lattice in the plane by connecting them by edges of unit length. It is easy to see that this is equivalent to approximating $\mathbb{R}^2$ by a uniform graph whose edge length are integers. We heard about this problem from Bruce Kleiner. Apparently it goes back to Erdős; see [3]. This is however a rather different question due to its Number Theory aspects. Nonetheless in the hyperbolic case (Theorem 2) our construction is very robust and it is easy to see that all edges can be assigned integral; see Remark 5.1 at the end of Section 5.

2. Analytic problems motivated by the proof

In this section we discuss several problems that emerged from the proof of Theorem 1 (and more specifically of the analytic lemmas in Section 4) and our attempts to generalize it to higher dimensions. The problems have to do with uniformly distributed sequences and approximating integrals of functions from certain classes by finite sums of their values along an infinite sequence.

First of all, up to minor nuances, the key analytic Lemma 4.3 tells us the following. Given a smooth convex function $f \colon [0, 1] \to \mathbb{R}$ with appropriate boundary conditions, there exists a sequence $\{x_i\}$, $x_i \in [0, 1]$ and a constant C such that $|\sum_{i=1}^{n} |x - x_i| - nf(x)| < C$ for all $n \in N$ and $x \in [0, 1]$. This means that not only averages of distance functions to x_i's converge to f uniformly, but also that the convergence is extremely fast. This suggests considering a similar question in higher dimensions; however, it is even unlikely that all smooth convex functions

on a square (or a disc) can be approximated by averages of distance functions to a sequence of points. Needless to say, obtaining approximations so fast is hardly possible. There might be however a reasonable class of functions which admit such approximations. Actually, for higher-dimensional generalizations of Theorem 1 one would probably need to average not distance functions to points but rather translations of somewhat different functions such as piecewise linear ones. This circle of problems seems to be widely open.

Furthermore, if one looks into the "guts" of the proof of Lemma 4.3, it becomes clear that it is closely related to a whole line of research which perhaps starts with Corput's Conjecture proven by Aardenne–Ehrenfest and further advanced by K. Roth, W. Schmidt and many others. There is an excellent account of this topic, including historical remarks, in [1], so we refer the reader to this book for all the detail. Theorems of Aardenne–Ehrenfest, Roth, Schmidt and others show that there are no uniformly distributed sequences on $[0, 1]$ (there are infinitely many values of n such that there are two intervals of the same length but the number of visits to them by the sequence until the nth member differs by at least $c \log n$). This implies that, unlike the distance functions, characteristic functions cannot be used for very fast approximations by averages: if one wants to approximate $f(x) = x$ on $[0, 1]$ by a sum $\frac{1}{n} \sum_1^n \chi_{[x_i, 1]}$, where $\{x_i\}$ is an infinite sequence, then for every $C > 0$ there are infinitely many n such that $\max_{x \in [0,1]} \left| \sum_1^n \chi_{[x_i, 1]}(x) - nf(x) \right| > C$. Furthermore, the proof of Lemma 4.3 is based on approximating the integral of a function $f \colon [0, 1] \to \mathbb{R}$ by averages $\frac{1}{n} \sum_1^n f(x_i)$ along some sequence of points $x_i \in [0, 1]$. Let us say that this approximation is super-fast (for a class of functions f) if for all f, n and x, $\left| \sum_1^n f(x_i) - n \int_0^1 f \right| \leq C$. Non-existence of very uniformly distributed sequences by Schmidt et al. imply that the class of characteristic functions of intervals does not admit super-fast approximations of integrals. However, Lemmas 4.1 and 4.2 imply that such an approximation exists for a class of functions satisfying appropriate regularity conditions (this class includes the functions $x \mapsto |x - a|$ but not characteristic functions of intervals). Hence we wonder: How fast can we approximate functions in several variables (from a certain regularity class) by averages along a sequence?

3. The construction

The goal of this section in to prove Theorem 1 modulo a technical lemma (Lemma 3.7) which is proven in the next section.

We divide $\mathbb{Z}^2$ into two lattices L and L' where

$$L = \{(i, j) \in \mathbb{Z}^2 : i + j \text{ is even}\},$$
$$L' = \{(i, j) \in \mathbb{Z}^2 : i + j \text{ is odd}\}.$$

Consider a graph whose set of vertices is L and whose edges connect each node (i, j) to its four diagonal neighbors $(i \pm 1, j \pm 1)$. Our plan is to assign lengths to the edges of this graph so that the resulting metric d_L on L majorizes the Euclidean norm and is additively close to it on the set of vectors $(x, y) \in \mathbb{R}^2$ such that $|y| \geq |x|$.

Then similarly one can construct an analogous metric graph on L' whose metric $d_{L'}$ is additively close to the Euclidean one on vectors (x, y) with $|x| \geq |y|$. By joining each point $(i, j) \in L$ with $(i, j+1) \in L'$ by an edge of a sufficiently large fixed length one gets a desired metric graph Γ whose distance function d_Γ is additively close to the Euclidean one.

We construct the graph metric d_L on L as follows. We choose sequences $\{u_j\}_{j \in \mathbb{Z}}$ and $\{v_j\}_{j \in \mathbb{Z}}$ of positive numbers (bounded away from 0 and ∞) and for every $i, j \in \mathbb{Z}$ assign length u_j to the edge from (i, j) to $(i+1, j+1)$ and length v_j to the edge from (i, j) to $(i-1, j+1)$. Note that the resulting metric is invariant under horizontal translations (by even integer vectors). The sequences $\{u_j\}$ and $\{v_j\}$ are explicit but the expression is too cumbersome to be presented here. An important feature of the construction is that $u_j + v_j$ is a constant independent of j.

In this section we express (almost explicitly) the graph distance d_L via the sequences $\{u_j\}$ and $\{v_j\}$; see Lemma 3.6. In the next section we deal with the choice of the sequences and prove estimates (encapsulated in Lemma 3.7) that control the difference between d_L and the Euclidean metric.

We introduce the following notation and terminology. By e_1 and e_2 we denote the standard basis vectors: $e_1 = (1, 0)$, $e_2 = (0, 1)$. The coordinates of a point $p \in \mathbb{R}^2$ are denoted by $x(p)$ and $y(p)$. For $j \in \mathbb{Z}$, we denote by S_j the horizontal strip

$$S_j = \{p \in \mathbb{R}^2 : j \le y(p) \le j + 1\}.$$

Definition 3.1. Let $\| \cdot \|$ be a norm on $\mathbb{R}^2$ and d a metric on L. We say that d *realizes* $\| \cdot \|$ *between levels* m *and* n if $d(p, q) = \|p - q\|$ for any $p, q \in L$ such that $y(p) = m$ and $y(q) = n$.

Definition 3.2. Let $u, v > 0$. The *rhombus norm* with parameters u, v is the norm $\| \cdot \|_{u,v}$ defined as follows: for a vector $p \in \mathbb{R}^2$,

$$\|p\|_{u,v} = u|p_1| + v|p_2|$$

where p_1 and p_2 are the components of p in the basis made of vectors $(1, 1)$ and $(1, -1)$, that is, $p = p_1 \cdot (1, 1) + p_2 \cdot (1, -1)$.

In other words, $\| \cdot \|_{u,v}$ is the norm whose unit ball is a rhombus with vertices $\pm(1/u, 1/u)$ and $\pm(1/v, -1/v)$.

Consider a graph metric d_L on L obtained from sequences $\{u_j\}$ and $\{v_j\}$ as explained above. Assume that $u_j + v_j = 2D$ for all j, where D is a constant independent of j. This assumption implies that for every $p, q \in L$ with $y(p) \ne y(q)$, the distance $d_L(p, q)$ is realized by a path confined between the horizontal lines through p and q. It follows that for every $j \in \mathbb{Z}$, the metric d_L realizes the rhombus norm $\| \cdot \|_j := \| \cdot \|_{u_j, v_j}$ between levels j and $j + 1$. For points $p, q \in L$ lying on the same horizontal line, we have $d_L(p, q) = D|x(p) - x(q)|$.

As the first step, we show that the distances in the graph metric are the same as the distances in a certain metric on $\mathbb{R}^2$. Namely, consider the following length metric d on $\mathbb{R}^2$: in each strip S_j, $j \in \mathbb{Z}$, the metric is the restriction of the rhombus norm $\| \cdot \|_j$, and the metric d on $\mathbb{R}^2$ is the metric gluing of the metrics in the strips.

The *metric gluing* of strips is defined as follows. For points $p, q \in \mathbb{R}^2$, the distance $d(p, q)$ is the infimum of lengths of broken lines connecting p and q. The length of a broken line γ is the sum of lengths of its parts $\gamma \cap S_j$, and the length of each part is measured in the metric of the respective strip. Since $u_j + v_j = 2D$ for all j, we have $\|e_1\|_j = D$ for all j and hence any two neighboring strips determine the same length on their common boundary line.

Lemma 3.3. *Let* d_L *and* d *be as above. Then* $d(p, q) = d_L(p, q)$ *for every* $p, q \in L$.

Proof. Since the length of a horizontal vector is the same in all norms $\| \cdot \|_j$, the distance $d(p, q)$ between any two points $p, q \in \mathbb{R}^2$ is realized by a broken line whose

y-coordinate is monotone and whose internal vertices have integral y-coordinates. In particular, horizontal lines are the shortest paths of d, and this implies that the assertion of the lemma holds if p and q lie in the same horizontal line.

Now consider $p, q \in L$ with $y(q) = m$ and $y(p) = m + k$ where $k > 0$. The distance $d(p, q)$ is realized by a broken line with vertices $p = p_0, p_1, \ldots, p_k = q$ such that $y(p_j) = m + j$ for all j. That is,

$$d(p, q) = \sum_{j=0}^{k-1} \|p_j - p_{j+1}\|_{m+j}.$$

If p_j and p_{j+1} belong to L, then $\|p_j - p_{j+1}\|_{m+j} = d_L(p_j, p_{j+1})$. Therefore it suffices to show that the points $p_1, \ldots, p_{k-1}$ can be chosen from our lattice L.

We prove this by induction in k. The base $k = 1$ is obvious. For $k \geq 2$, fix points $p_1, \ldots, p_{k-1}$ as above. For every $t \in \mathbb{R}$ and $j = 1, \ldots, k-1$, define $p_j(t) = p_j + te_1$. Let $p_0(t) = p$ and $p_k(t) = q$ for all t. Consider a function $f \colon \mathbb{R} \to \mathbb{R}_+$ given by

$$f(t) = \sum_{j=0}^{k-1} \|p_j(t) - p_{j+1}(t)\|_{m+j}.$$

Clearly $f(0) = d(p, q)$ is the minimum of f. On the other hand, f is piecewise linear and its break points occur only if one of the segments $[p, p_1(t)]$ and $[p_{k-1}(t), q]$ is an edge of L. Indeed, in the definition of f all summands but those for $j = 0$ and $j = k-1$ are independent of t, and the summands $\|p_0 - p_1(t)\|_m$ and $\|p_{k-1}(t) - p_k\|_{m+k-1}$ are piecewise linear in t with break points corresponding to the diagonal directions.

Since the minimum of a piecewise linear function f is attained at a break point, we can replace $p_1, \ldots, p_{k-1}$ by $p_1(t_0), \ldots, p_{k-1}(t_0)$ where t_0 is a break point of f such that $f(t_0) = f(0)$. Now at least one of the new points $p_1(t_0)$ and $p_{k-1}(t_0)$ belongs to L, and we apply the induction hypothesis to the distance from $p_1(t_0)$ to q or from p to $p_{k-1}(t_0)$. □

The next step is to figure out the distances in the metric d glued from strips. To do this, we associate to every norm $\| \cdot \|$ on $\mathbb{R}^2$ a concave function $h = h_{\|\cdot\|}$ referred to as the *dual profile* of the norm. It turns out that the metric obtained by gluing normed strips is given by arithmetic averages of dual profiles.

Definition 3.4. Let $\| \cdot \|$ be a norm on $\mathbb{R}^2$ and $D = \|e_1\|$. The *dual profile* of $\| \cdot \|$ is a function $h = h_{\|\cdot\|} \colon [-D, D] \to \mathbb{R}$ defined as follows. Let B be the unit ball of $\| \cdot \|$ and B^* the dual body to B, i.e.,

$$B^* = \{v \in \mathbb{R}^2 : \forall p \in B \ \langle v, p \rangle \leq 1\}.$$

By duality, the horizontal width of B^* equals $2D$, that is, $[-D, D] \times \mathbb{R}$ is the minimal vertical strip containing B^*. We define

$$h(\xi) = \sup\{\eta \in \mathbb{R} : (\xi, \eta) \in B^*\}, \qquad \xi \in [-D, D].$$

The definition implies that B^* is enclosed between the vertical lines $\{\xi = D\}$, $\{\xi = -D\}$ and the graphs $\{\eta = h(\xi)\}$ and $\{\eta = -h(-\xi)\}$ in the $\xi\eta$-plane. Therefore, h uniquely determines the norm $\| \cdot \|$.

Lemma 3.5. *Let a metric d on $\mathbb{R}^2$ be the metric gluing of strips S_j, $j \in \mathbb{Z}$, where each strip S_j is equipped with a norm $\| \cdot \|_j$ such that $\|e_1\|_j = D$ (where $D > 0$ is independent of j). Then for every $m, n \in \mathbb{Z}$ such that $m < n$, d realizes some norm*

$\|\cdot\|^{m,n}$ *between levels* m *and* n *(see Definition* 3.1*) with* $\|e_1\|^{m,n} = D$. *The dual profile* $h^{m,n}$ *of* $\|\cdot\|^{m,n}$ *is given by*

$$h^{m,n}(\xi) = \frac{1}{n-m} \sum_{j=m}^{n-1} h_j(\xi), \qquad \xi \in [-D, D]$$

where h_j *is the dual profile of* $\|\cdot\|_j$.

Proof. First we associate yet another function to an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^2$. Namely, define $f = f_{\|\cdot\|} \colon \mathbb{R} \to \mathbb{R}_+$ by $f(x) = \|(x, 1)\|$. Clearly, $f_{\|\cdot\|}$ is a convex function with linear asymptotics at $+\infty$ and $-\infty$. More precisely, if $\|e_1\| = D$, then

$$(3.1) \qquad f(x) \sim f(-x) \sim Dx, \qquad x \to +\infty.$$

Conversely, every positive convex function f satisfying (3.1) equals $f_{\|\cdot\|}$ for some norm $\|\cdot\|$ such that $\|e_1\| = D$.

Let us express the dual profile $h = h_{\|\cdot\|}$ of $\|\cdot\|$ in terms of $f = f_{\|\cdot\|}$. Fix $\xi \in [-D, D]$. First we show that

$$(3.2) \qquad h(\xi) = \sup\{\eta \in \mathbb{R} : (\xi, \eta) \in (B \cap H_+)^*\}$$

where $H_+ = \{(x, y) : y \geq 0\} \subset \mathbb{R}^2$ is the upper half-plane. By duality we have

$$(B \cap H_+)^* = \text{closure}(\text{conv}(B^* \cup H_+^*)) = \text{closure}(\text{conv}(B^* \cup Y_-)) = B^* + Y_-$$

where $Y_- = \{(0, y) : y \leq 0\}$, conv denotes the convex hull, and $B^* + Y_-$ is the Minkowski sum of B^* and Y_-. Therefore

$$\sup\{\eta \in \mathbb{R} : (\xi, \eta) \in (B \cap H_+)^*\} = \sup\{\eta \in \mathbb{R} : (\xi, \eta) \in B^*\}.$$

Since the right-hand side equals $h(\xi)$ by definition, (3.2) follows.

We rewrite (3.2) as follows.

$$\begin{aligned}
h(\xi) &= \sup\{\eta : \xi x + \eta y \leq 1 \text{ for all } (x, y) \in B \cap H_+\} \\
&= \sup\{\eta : \xi x + \eta y \leq \|(x, y)\| \text{ for all } x \in \mathbb{R}, y \geq 0\} \\
&= \sup\{\eta : \xi x + \eta y \leq \|(x, y)\| \text{ for all } x \in \mathbb{R}, y > 0\} \\
&= \sup\{\eta : \xi x + \eta \leq \|(x, 1)\| \text{ for all } x \in \mathbb{R}\} \\
&= \sup\{\eta : \xi x + \eta \leq f(x) \text{ for all } x \in \mathbb{R}\} \\
&= \inf_{x \in \mathbb{R}}\{f(x) - \xi x\}.
\end{aligned}$$

Here we subsequently use the definition of duality, positive homogeneity of $\|\cdot\|$, the fact that if $y = 0$, then $\xi x + \eta y = \xi x \leq Dx = \|(x, 0)\|$, again the positive homogeneity of $\|\cdot\|$, the definition of $f = f_{\|\cdot\|}$, and the definition of infimum. Thus

$$(3.3) \qquad h(\xi) = \inf_{x \in \mathbb{R}}\{f(x) - \xi x\}.$$

Now we proceed with the proof of the lemma. Without loss of generality we assume that $m = 0$. Define $f_j = f_{\|\cdot\|_j}$. The distance between points $p = (a, 0)$ and $q = (b, n)$, where $a, b \in \mathbb{R}$, is given by $d(p, q) = g(b - a)$ where $g \colon \mathbb{R} \to \mathbb{R}_+$ is a function defined by

$$g(x) = \inf_{\{x_j\}} \left\{ \sum_{j=0}^{n-1} f_j(x_j) : \{x_j\} \text{ such that } \sum_{j=0}^{n-1} x_j = x \right\}.$$

Indeed, to get from $(a, 0)$ to $(a + x, n)$ one has to traverse the strips S_j, $j = 0, \ldots, n-1$, so that the total displacement in the horizontal direction equals x.

Define $f(x) = \frac{1}{n} g(nx)$. It is easy to see that the function f is convex and $f(x) \sim f(-x) \sim Dx$ as $x \to +\infty$. Therefore $f = f_{\|\cdot\|}$ for some norm $\|\cdot\|$ such that $\|e_1\| = D$. By definition, d realizes $\|\cdot\|$ between levels 0 and n. It remains to prove that the dual profile $h = h_{\|\cdot\|}$ satisfies $h = \frac{1}{n} \sum h_j$.

Let $\xi \in [-D, D]$. By (3.3), we have $h(\xi) = \inf_{x \in \mathbb{R}} \{f(x) - \xi x\}$. Plugging in the definition of f yields

$$h(\xi) = \inf_{x \in \mathbb{R}} \left\{ \inf_{\{x_j\}} \left\{ \frac{1}{n} \sum_{j=0}^{n-1} f_j(x_j) : \{x_j\} \text{ such that } \sum_{j=0}^{n-1} x_j = nx \right\} - \xi x \right\}$$

$$= \inf_{x \in \mathbb{R}} \inf_{\{x_j\}} \left\{ \left(\frac{1}{n} \sum_{j=0}^{n-1} f_j(x_j) \right) - \xi x : \{x_j\} \text{ such that } \sum_{j=0}^{n-1} x_j = nx \right\}$$

$$= \frac{1}{n} \inf_{x \in \mathbb{R}} \inf_{\{x_j\}} \left\{ \sum_{j=0}^{n-1} f_j(x_j) - \xi \sum_{j=0}^{n-1} x_j : \{x_j\} \text{ such that } \sum_{j=0}^{n-1} x_j = nx \right\}$$

$$= \frac{1}{n} \inf_{x \in \mathbb{R}} \inf_{\{x_j\}} \left\{ \sum_{j=0}^{n-1} \left(f_j(x_j) - \xi x_j \right) : \{x_j\} \text{ such that } \sum_{j=0}^{n-1} x_j = nx \right\}$$

$$= \frac{1}{n} \inf_{\{x_j\}} \left\{ \sum_{j=0}^{n-1} \left(f_j(x_j) - \xi x_j \right) : x_0, \ldots, x_{n-1} \in \mathbb{R} \right\}$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} \inf_{x_j \in \mathbb{R}} \{f_j(x_j) - \xi x_j\} = \frac{1}{n} \sum_{j=0}^{n-1} h_j(\xi).$$

The lemma follows. $\qquad\square$

Now we return to our special case when each $\|\cdot\|_j$ is a rhombus norm $\|\cdot\|_{u_j, v_j}$ with $u_j + v_j = 2D$. A direct computation shows that the dual profile $h_j := h_{\|\cdot\|_j}$ has the form $h_j(\xi) = D - |\xi - \beta_j|$ where $\beta_j = \frac{u_j - v_j}{2}$. By combining Lemma 3.3 and Lemma 3.5 we get the following:

Lemma 3.6. *Let a graph metric d_L on our lattice L be defined as above, using sequences $\{u_j\}$ and $\{v_j\}$ such that $u_j + v_j = 2D$ for all j. Then for any two points $p, q \in L$ with $y(p) = m$ and $y(q) = n$ where $m < n$, one has*

$$d_L(p, q) = \|p - q\|^{m,n}$$

where $\|\cdot\|^{m,n}$ is a norm on $\mathbb{R}^2$ whose dual profile $h^{m,n}$ is given by

$$h^{m,n}(\xi) = \frac{1}{n - m} \sum_{j=m}^{n-1} \left(D - |\xi - \beta_j| \right)$$

where $\beta_j = \frac{u_j - v_j}{2}$.

In addition, for $p, q \in L$ with $y(p) = y(q)$, one has $d_L(p, q) = D|x(p) - x(q)|$. $\quad\square$

Note that for every $D > 0$ and $\beta_j \in (-D, D)$ there exist positive u_j and v_j with $u_j + v_j = 2D$ and $\frac{u_j - v_j}{2} = \beta_j$. Namely, $u_j = D + \beta_j$ and $v_j = D - \beta_j$. Therefore

rather than operating with the sequences $\{u_j\}$ and $\{v_j\}$, one can work with a sequence $\{\beta_j\}$ in the interval $(-D, D)$ after having fixed a constant $D > 0$. The resulting metric graph is uniform if and only if $\{\beta_j\}$ is separated from $\{D, -D\}$.

From now on we fix $D = \sqrt{2}$ and introduce a function $h^0 \colon [-D, D] \to \mathbb{R}$ by

$$(3.4) \qquad h^0(\xi) = \begin{cases} 1 - \sqrt{1 - \xi^2}, & |\xi| \leq \frac{\sqrt{2}}{2}, \\ \sqrt{2} - |\xi|, & \frac{\sqrt{2}}{2} \leq |\xi| \leq \sqrt{2}. \end{cases}$$

Clearly h^0 is C^1 smooth and is the dual profile of a norm $\| \cdot \|^0$ given by

$$\|(x, y)\|^0 = \max\{|(x, y)|, \sqrt{2}|x|\} = \begin{cases} |(x, y)|, & |x| \leq |y|, \\ \sqrt{2}|x|, & |x| \geq |y|, \end{cases}$$

where $|(x, y)| = \sqrt{x^2 + y^2}$.

Lemma 3.7. *There exist a constant $C > 0$ and a sequence $\{\beta_j\}_{j \in \mathbb{Z}}$ such that $\beta_j \in [-D/2, D/2])$ for all j and for every $m, n \in \mathbb{Z}$ with $m < n$ one has*

$$\left| (n - m)h^0(\xi) - \sum_{j=m}^{n-1} \left(D - |\xi - \beta_j| \right) \right| \leq C.$$

The proof of this lemma is given in Section 4. The lemma is proved by analytic methods including Fourier series and the theory of rational approximations.

Applying Lemma 3.6 to the sequence constructed in Lemma 3.7 yields that the dual profiles $h^{m,n}$ (determining the distances between levels m and n in our metric graph) satisfy the following inequality:

$$(3.5) \qquad \left| h^{m,n}(\xi) - h^0(\xi) \right| \leq \frac{C}{n - m}.$$

Let us show that this inequality implies that the distance d_L is additively close to the norm $\| \cdot \|^0$. We need the following lemma.

Lemma 3.8. *Let $\| \cdot \|^1$, $\| \cdot \|^2$ be norms on $\mathbb{R}^2$ such that $\|e_1\|^1 = \|e_1\|^2 = D$, and let h^1, h^2 be their dual profiles. Then for any $(x, y) \in \mathbb{R}^2$ one has*

$$\left| \|(x, y)\|^1 - \|(x, y)\|^2 \right| \leq |y| \cdot \sup_{[-D, D]} |h^1 - h^2|.$$

Proof. If $y = 0$, then $\|(x, y)\|^1 = \|(x, y)\|^2 = D|x|$. Therefore we may assume that $y > 0$. Every norm $\| \cdot \|$ is expressed via its dual profile $h = h_{\|\cdot\|}$ as follows. First observe that

$$\|(x, y)\| = \sup_{(\xi, \eta) \in B^*} \{\xi x + \eta y\}$$

where B^* is dual to the unit ball of $\| \cdot \|$. If $y > 0$, the right-hand side equals

$$\sup_{\xi \in [-D, D]} \{\xi x + h(\xi)y\}.$$

Therefore

$$\left| \|(x, y)\|^1 - \|(x, y)\|^2 \right| = \left| \sup_{\xi \in [-D, D]} \{\xi x + h^1(\xi)y\} - \sup_{\xi \in [-D, D]} \{\xi x + h^2(\xi)y\} \right|$$

$$\leq \sup_{\xi \in [-D, D]} \{|h^1(\xi)y - h^2(\xi)y|\}.$$

The lemma follows. $\qquad \square$

This lemma and (3.5) imply that for any points $p = (x_1, m)$ and $q = (x_2, n)$ with $m, n \in \mathbb{Z}$ and $m < n$, one has

$$\left| \|p - q\|^{m,n} - \|p - q\|^0 \right| \leq C.$$

This and Lemma 3.6 imply that our graph metric d_L is additively close to $\| \cdot \|^0$. Recall that $\|(x, y)\|^0 \geq |(x, y)|$ for all $(x, y) \in \mathbb{R}^2$ and $\|(x, y)\|^0 = |(x, y)|$ if $|x| \leq |y|$.

Thus we have constructed a uniform graph on L such that its distance function d_L is additively close to the Euclidean metric on vectors with $|y| \geq |x|$ and no less than the Euclidean distance minus some constant C for all vectors. Similarly, one can construct a graph with vertices in L' whose distance is additively close to the Euclidean one on vectors with $|x| \geq |y|$ and also no less than Euclidean distance minus C for all vectors. Now we "glue" the two graphs by choosing a sufficiently large number M (e.g., $M = 2C + 1$ works) and connecting (i, j) with $(i, j + 1)$ by an edge of length M for all $(i, j) \in L$. One easily sees that the distance function of the resulting graph is additively close to the Euclidean one. This completes the proof of Theorem 1.

Remark 3.9. The above construction is a special case of the following general observation. If one has a finite collection of uniform graphs Γ_i such the distance function of each Γ_i is additively close to metrics of some norms $\| \cdot \|_i$, then there exists a uniform graph Γ whose distance function is additively close to the metric of the norm whose unit ball is the convex hull of the unit balls of $\| \cdot \|_i$. This graph is easily constructed by connecting sufficiently close vertices of Γ_i's by edges of a sufficiently large fixed length.

4. Approximation of functions in one variable

Lemma 4.1. *Let $f \colon S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be a piecewise C^1 function with $\int f = 0$ and $V(f') \leq M < \infty$ where $V(f')$ denotes the variation of the derivative f' on S^1. Let α be a quadratic irrational. Then for every positive integer n and every $x \in S^1$, one has*

$$\left| \sum_{j=0}^{n-1} f(x + j\alpha) \right| \leq C(\alpha) \cdot M$$

for some constant $C(\alpha)$.

Proof. Consider the Fourier series $f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}$ for f. Note that $a_0 = 0$ since $\int f = 0$. Since f' is of bounded variation, the Fourier series converges uniformly and moreover $|a_k| \leq M/k^2$; see e.g. [4, Ch.II,§4]. The Fourier series for the sum on the left-hand side of the desired inequality has the form

$$\sum_{j=0}^{n-1} f(x + j\alpha) = \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k \sum_{j=0}^{n-1} e^{2\pi i k (x + j\alpha)} = \sum_{k \in \mathbb{Z} \setminus \{0\}} b_k(n) e^{2\pi i k x}$$

where

$$b_k(n) = a_k \sum_{j=0}^{n-1} \left(e^{2\pi i k \alpha} \right)^j = a_k \cdot \frac{1 - e^{2\pi i k \alpha n}}{1 - e^{2\pi i k \alpha}}.$$

Therefore

$$\left| \sum_{j=0}^{n-1} f(x + j\alpha) \right| \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} |b_k(n) e^{2\pi i k x}| = \sum_{k \in \mathbb{Z} \setminus \{0\}} |b_k(n)|.$$

Thus it suffices to prove that

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}|b_k(n)|\leq C(\alpha)\cdot M$$

for all n.

For $t\in\mathbb{R}$, we denote by $d(t,\mathbb{Z})$ the distance from t to the nearest integer. Note that $d(-t,\mathbb{Z})=d(t,\mathbb{Z})$. One easily sees that $|1-e^{2\pi it}|\geq 4d(t,\mathbb{Z})$ for all $t\in\mathbb{R}$. Substituting $t=k\alpha$ yields $|1-e^{2\pi ik\alpha}|\geq 4d(k\alpha,\mathbb{Z})$. Since $|a_k|\leq M/k^2$, it follows that

$$|b_k(n)|\leq\frac{2a_k}{|1-e^{2\pi ik\alpha}|}\leq\frac{M}{2k^2\cdot d(k\alpha,\mathbb{Z})}.$$

It remains to prove that

(4.1) $$\sum_{k=1}^{\infty}\frac{1}{k^2\cdot d(k\alpha,\mathbb{Z})}<\infty.$$

Indeed, the left-hand side depends on α only, so if it is finite, then we can just denote it by $C(\alpha)$ and the lemma follows.

Since α is a quadratic irrational, Liouville's Approximation Theorem asserts that

$$\left|\alpha-\frac{p}{k}\right|>\frac{c}{k^2}$$

for some constant $c=c(\alpha)>0$ and all $p,k\in\mathbb{Z}$. Multiplying this by k we get

(4.2) $$d(k\alpha,\mathbb{Z})=\min_{p\in\mathbb{Z}}|k\alpha-p|>\frac{c}{k}$$

for every positive integer k.

Consider the partition of $\mathbb{N}$ into sets N_l, $l=1,2,\ldots$, defined by

$$N_l=\{k\in\mathbb{N}:2^{-l-1}<d(k\alpha,\mathbb{Z})\leq 2^{-l}\}.$$

For every $k\in N_l$, (4.2) implies that

$$k>\frac{c}{d(k\alpha,\mathbb{Z})}\geq c\cdot 2^l.$$

For any distinct $k_1,k_2\in N_l$, we have

$$d(k_1\alpha-k_2\alpha,\mathbb{Z})\leq d(k_1\alpha,\mathbb{Z})+d(k_2\alpha,\mathbb{Z})\leq 2^{1-l}.$$

On the other hand, applying (4.2) to $|k_1-k_2|$ in place of k yields

$$d(k_1\alpha-k_2\alpha,\mathbb{Z})=d(|k_1-k_2|\alpha,\mathbb{Z})\geq\frac{c}{|k_1-k_2|},$$

therefore $|k_1-k_2|\geq c\cdot 2^{l-1}$ for any distinct $k_1,k_2\in N_l$. Thus the nth smallest element of the set N_l is bounded below by $cn\cdot 2^{l-1}$, hence

$$\sum_{k\in N_l}\frac{1}{k^2\cdot d(k\alpha,\mathbb{Z})}\leq 2^{l+1}\sum_{k\in N_l}\frac{1}{k^2}\leq 2^{l+1}\sum_{n=1}^{\infty}\frac{1}{(cn\cdot 2^{l-1})^2}=\frac{8}{2^lc^2}\cdot\frac{\pi^2}{6}$$

where $\frac{\pi^2}{6}$ denotes the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Summing these inequalities for $l=1,2,\ldots$, we get

$$\sum_{k\in\mathbb{N}}\frac{1}{k^2\cdot d(k\alpha,\mathbb{Z})}\leq\frac{8}{c^2}\cdot\frac{\pi^2}{6}<\infty.$$

This completes the proof of (4.1) and hence of the lemma. $\qquad\square$

Lemma 4.2. *There exists a constant $C > 0$ and a sequence $\{\alpha_j\}_{j=-\infty}^{\infty}$ of points in $[0, 1]$ such that the following holds. For every piecewise smooth function $f\colon [0, 1] \to \mathbb{R}$ and every $m, n \in \mathbb{Z}$ such that $m < n$, one has*

$$\left| \sum_{j=m}^{n-1} f(\alpha_j) - (n - m) \int_0^1 f \right| \leq C(V_0^1(f') + |f'(0)| + |f'(1)|).$$

Here $V_0^1(f')$ denotes the variation of f' on $[0, 1]$.

Proof. Fix a quadratic irrational α and define

$$\alpha_j = 2d(j\alpha, \mathbb{Z}) = \begin{cases} 2\{j\alpha\} & \text{if } \{j\alpha\} \leq 1/2, \\ 2 - 2\{j\alpha\} & \text{if } \{j\alpha\} \geq 1/2, \end{cases}$$

where $\{j\alpha\}$ denotes the fractional part of $j\alpha$. We claim that this sequence works.

Let $A = \int_0^1 f$ and $f_0 = f - A$; then $\int_0^1 f_0 = 0$. Define $g\colon [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f_0(2x) & \text{if } 0 \leq x \leq 1/2, \\ f_0(2 - 2x) & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Observe that $g(0) = g(1)$. Therefore g (unlike f) descends to a continuous and hence piecewise smooth function $\bar{g}$ on the circle $\mathbb{R}/\mathbb{Z}$. From definitions by term-by-term comparison we get

$$\sum_{j=m}^{n-1} f(\alpha_j) - (n - m) \int_0^1 f = \sum_{j=m}^{n-1} f_0(\alpha_j) = \sum_{j=m}^{n-1} g(\{j\alpha\}) = \sum_{j=m}^{n-1} \bar{g}(j\alpha).$$

Since $\int \bar{g} = 0$, the previous lemma implies that

$$\left| \sum_{j=m}^{n-1} \bar{g}(j\alpha) \right| = \left| \sum_{j=0}^{n-m-1} \bar{g}(m\alpha + j\alpha) \right| \leq C(\alpha)V(\bar{g}') = 4C(\alpha)(V_0^1(f') + |f'(0)| + |f'(1)|).$$

The lemma follows. $\qquad\square$

Lemma 4.3. *Let $h\colon [a, b] \to \mathbb{R}$ be a smooth function such that $h'' > 0$ everywhere on $[a, b]$, $h'(a) = -1$, $h'(b) = 1$ and $h(a) + h(b) = b - a$. Then there exists a constant C and a sequence $\{\beta_j\}_{j=-\infty}^{\infty}$ of points in $[a, b]$ such that for every $x \in [a, b]$ and every $m, n \in \mathbb{Z}$ such that $m < n$, one has*

$$\left| \sum_{j=m}^{n-1} |x - \beta_j| - (n - m)h(x) \right| \leq C.$$

Proof. Define a map $\varphi\colon [a, b] \to \mathbb{R}$ by

$$\varphi(t) = \frac{1}{2}\big(h'(t) + 1\big).$$

The assumptions that $h'' > 0$, $h'(a) = -1$ and $h'(b) = 1$ imply that φ is a diffeomorphism from $[a, b]$ onto $[0, 1]$. Let $\beta_j = \varphi^{-1}(\alpha_j)$ where $\{\alpha_j\}$ is the sequence from

the previous lemma. Then, for every $x \in [a, b]$ we can write

$$\sum_{j=m}^{n-1} |x - \beta_j| = \sum_{j=m}^{n-1} f_x(\alpha_j)$$

where f_x is a function on $[0, 1]$ defined by

$$f_x(y) = |x - \varphi^{-1}(y)|.$$

Since φ^{-1} is smooth, the variations $V_0^1(f_x')$ are bounded above by some constant C_0 which is independent of x. Therefore, by the previous lemma,

$$\left| \sum_{j=m}^{n-1} f_x(\alpha_j) - (n - m) \int_0^1 f_x \right| \leq C_1 \cdot C_0 =: C$$

where C_1 is the constant from the previous lemma. To complete the proof, it now suffices to show that

$$\int_0^1 f_x = h(x)$$

for every $x \in [a, b]$. This is shown by the following computation:

$$\int_0^1 f_x(y) \, dy = \int_0^1 |x - \varphi^{-1}(y)| \, dy$$

$$= \int_a^b |x - t| \, \varphi'(t) \, dt \qquad \text{by substitution } y = \varphi(t)$$

$$= \frac{1}{2} \int_a^b |x - t| \, h''(t) \, dt \qquad \text{by the definition of } \varphi$$

$$= \frac{1}{2} \left(\int_a^x (x - t) h''(t) \, dt + \int_x^b (t - x) h''(t) \, dt \right)$$

$$= \frac{1}{2} \left(\int_0^x h'(t) \, dt - (x - a) h'(a) - \int_x^b h'(t) \, dt + (b - x) h'(b) \right) \qquad \text{by parts}$$

$$= \frac{1}{2} \left(h(x) - h(a) + (x - a) - h(b) + h(x) + (b - x) \right) = h(x).$$

The last two equalities employ the assumptions that $h'(a) = -1$, $h'(b) = 1$ and $h(a) + h(b) = b - a$. □

Now we are in a position to prove Lemma 3.7.

Proof of Lemma 3.7. We apply Lemma 4.3 to the interval $[a, b] = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and the function h given by

$$h(x) = \sqrt{2} - \sqrt{1 - x^2}, \qquad t \in [-\sqrt{2}/2, \sqrt{2}/2].$$

It is easy to see that these data satisfy the assumptions of Lemma 4.3. Hence there exist a constant $C > 0$ and a sequence $\{\beta_j\} \subset [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ such that

$$\left| (n-m)h(x) - \sum_{j=m}^{n-1} |x - \beta_j| \right| \leq C$$

for all $x \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and all $m, n \in \mathbb{Z}$ such that $m < n$. Hence

$$\left| (n-m)(\sqrt{2} - h(x)) - \sum_{j=m}^{n-1} (\sqrt{2} - |x - \beta_j|) \right| \leq C$$

or, equivalently,

$$\left| (n-m)\sqrt{1 - x^2} - \sum_{j=m}^{n-1} (\sqrt{2} - |x - \beta_j|) \right| \leq C$$

for all $x \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and all $m, n \in \mathbb{Z}$ such that $m < n$.

On the intervals $[\frac{\sqrt{2}}{2}, \sqrt{2}]$ and $[-\sqrt{2}, -\frac{\sqrt{2}}{2}]$, the functions $x \mapsto \sqrt{2} - |x - \beta_j|$ are linear with slopes -1 and 1, respectively. Therefore

$$\left| (n-m)h^0(x) - \sum_{j=m}^{n-1} (\sqrt{2} - |x - \beta_j|) \right| \leq C$$

for all $x \in [-\sqrt{2}, \sqrt{2}]$, where the function h^0 is defined by (3.4). $\qquad\square$

5. The hyperbolic case

In this section we prove Theorem 2. Let us indicate that the proof uses only δ-hyperbolicity, Gromov's Morse Lemma for Gromov hyperbolic spaces (see [2] for basic definitions and the Morse Lemma), and a very weak corollary of bounded geometry: given any ε, there is an ε-net such that for every $R > 0$ there is a constant $C = C(R, \varepsilon)$ such that every ball of radius R contains at most C points from the net.

We set $\varepsilon = \delta$ and fix such an ε-net X. This is the set of vertices of our graph.

We construct the graph in two steps. Fix a point $p \in M$. First, we build a tree such that all distances in this tree from p to points in X are (exactly) equal to distances in M, and distances in M between other pairs of points in X do not exceed distances in the tree up to an additive constant.

To achieve this, for every point $q \in X \setminus \{p\}$ we choose a $q' \in X$ such that:

(1) q' lies within distance ε from the geodesic segment $[pq]$;
(2) $d(q, p) - 15\varepsilon < d(q', p) < d(q, p) - 5\varepsilon$ if $d(p, q) > 5\varepsilon$;
(3) if $d(q, p) \leq 5\varepsilon$, then $q' = p$.

The existence of such q' follows simply from the triangle inequality and the definition of ε-nets. Note that we choose just one point q' (a "parent") for every q in X.

We connect every $q \in X$ to its "parent" q' by an edge and set the length of this edge to be $d(q, p) - d(q', p)$. The resulting graph T is a tree. We denote the distance in T by d_T. By construction, $d_T(p, q) = d(p, q)$ for every $q \in X$. Moreover

$d_T(q_1, q_2) = d(q_1, p) - d(q_2, p)$ for any $q_1, q_2 \in X$ such that q_2 lies on the T-path from q_1 to p.

Note that since each point q' is connected to points which are no further than 100ε away, the degree of vertices in this tree is uniformly bounded due to the bounded geometry assumption.

Let us make an important though obvious observation here. For a shortest path from p to q in the tree, consider a broken geodesic line obtained by joining adjacent vertices along this path by shortest segments in M. These broken lines are quasi-geodesics with the same quasi-geodesic constant (say, 10) and hence by the Morse Lemma there is a constant D such that every shortest path from q to p in the tree stays within the D-neighborhood of a shortest path in M. This implies that for any $q_1, q_2 \in X$ such that q_2 lies on the T-path between q_1 and p, we have $|d_T(q_1, q_2) - d(q_1, q_2)| \leq 2D$.

Now for any two $q_1, q_2 \in X$ the T-path from q_1 to q_2 contains a point q which lies on both T-paths connecting q_1 and q_2 to p. Hence

$$d_T(q_1, q_2) = d_T(q_1, q) + d_T(q_2, q) \geq d(q_1, q) + d(q_2, q) - 4D \geq d(q_1, q_2) - 4D$$

by the triangle inequality. Thus the distance between q_1 and q_2 in M cannot exceed that in the tree by more than an additive constant.

For any two $q_1, q_2 \in X$, by (one of the definitions of) δ-hyperbolicity there is a point p' on the M-geodesic $[q_1, q_2]$ such that the distance from p' to geodesic segments connecting q_1 and q_2 to p is less than δ. Thus there are points $q_1', q_2' \in X$ such that q_i' lies between q_i and p in T and $d(q_i', p) < D_1 := D + 100\varepsilon + \delta$ for $i = 1, 2$.

Now we are prepared for Step 2. We connect by a new edge every pair of points $x, y \in X$ such that $d(x, y) < 2D_1$. By our bounded geometry assumption, the degrees of vertices in the graph remain uniformly bounded. We set the length of each of the new edges to be $2D_1 + 4D$. This guarantees that the distances in the new graph still cannot be shorter than those in M by more than $4D$. Now for $q_1, q_2 \in X$ consider the points q_1' and q_2' constructed in the previous paragraph. They are connected by a new edge, hence the distance between q_1 and q_2 in the new graph is bounded above by

$$d_T(q_1, q_1') + d_T(q_2, q_2') + 2D_1 + 4D \leq d(q_1, q_1') + d(q_2, q_2') + 2D_1 + 12D.$$

Since q_1' and q_2' lie within distance δ from a point p' on the geodesic $[q_1 q_2]$, we have $d(q_1, q_1') + d(q_2, q_2') \leq d(q_1, q_2) + 2\delta$. Thus that distance in the graph and in M differ by no more than by an additive constant, namely by $2D_1 + 12D + 2\delta$. This finishes the proof of Theorem 2.

Remark 5.1. The construction can be modified in a trivial way to assign integral lengths to all edges. First, one sets ε and δ larger than say 10. Next, at Step 1, we set the length of the edge from q to q' to be $[d(q, p)] - [d(q', p)]$ where $[\cdot]$ denotes the integral part. At Step 2, we take any integer greater than $2D_1 + 4D$ and assign this length to all new edges. It is easy to check that the argument goes through exactly the same way.

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