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AN OCTONIONIC CONSTRUCTION OF THE KAC SUPERALGEBRA $\mathbf{K}_{10}$

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Dedicated to Kevin McCrimmon on his seventieth birthday

ABSTRACT. The split Kac Jordan superalgebra K_{10}, as well as its forms, are embedded in $\mathcal{M}_2(\mathcal{O})$, the 2 by 2 matrices with entries in the split octonion algebra $\mathcal{O}$.

1. INTRODUCTION

In his classification of simple Jordan superalgebras, Kac [K, HK] introduced a simple exceptional Jordan superalgebra of dimension 10, now denoted K_{10}. Benkart and Elduque [BE] gave a construction of K_{10} based on the Kaplansky superalgebra and used it to give a direct proof that the superalgebra K_{10} is Jordan. As a byproduct of their classification of pseudo-composition superalgebras, Elduque and Okubo [EO] constructed the forms of K_{10} and showed that they are in one-to-one correspondence with the square classes of the base field. A more Jordan algebraic variant of their proof is given in [RZ, Proposition 30]. Since all forms of the Kaplansky superalgebra are split, the forms of K_{10} cannot be obtained by the construction of Benkart and Elduque. Octonion algebras are usually associated with exceptional algebras, sometimes in a nonobvious way. Our aim is to give an "orthosymplectic" construction of the split K_{10} using the split octonion algebra $\mathcal{O}$ and also to embed the forms of K_{10} in $\mathcal{M}_2(\mathcal{O})$.

We first recall the orthosymplectic Jordan superalgebras. Let F be a field of characteristic not 2 and $V = V_{\bar{0}} + V_{\bar{1}}$ a $\mathbf{Z}_2$-graded F-vector space with $\dim V_{\bar{0}} = n$ and $\dim V_{\bar{1}} = 2m$. The endomorphism ring $\mathrm{End}(V)$ is an associative superalgebra: $\mathrm{End}(V) = \mathrm{End}(V)_{\bar{0}} \oplus \mathrm{End}(V)_{\bar{1}}$, where

$$\mathrm{End}(V)_{\bar{i}} = \{a \in \mathrm{End}(V) \mid V_{\bar{j}} a \subseteq V_{\bar{j}+\bar{i}}\} \qquad \bar{i}, \bar{j} \in \mathbf{Z}_2,$$

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writing the operators on the right. With respect to a homogeneous basis of V, these endomorphisms correspond to the following matrices:

$$\mathrm{End}(V)_{\bar 0} = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \mid A \text{ an } n\times n \text{ block, } D \text{ a } 2m\times 2m \text{ block} \right\},$$
$$\mathrm{End}(V)_{\bar 1} = \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \mid B \text{ an } n\times 2m \text{ block, } C \text{ a } 2m\times n \text{ block} \right\}.$$

This associative superalgebra is sometimes denoted $\mathcal{M}_{n+2m}(F)$.

For homogeneous elements $a_{\bar i}$, $b_{\bar j}$, the Jordan superproduct

$$a_{\bar i} \cdot b_{\bar j} := \frac{1}{2}(a_{\bar i} b_{\bar j} + (-1)^{\bar i \bar j} b_{\bar j} a_{\bar i})$$

yields a simple Jordan superalgebra denoted $\mathcal{M}_{n,2m}(F)$.

Let $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} I_n & 0 \\ 0 & S_m \end{bmatrix}$, where I_n is the $n\times n$ identity matrix and S_m is the $2m\times 2m$-matrix with m copies of S along the diagonal. The matrix H induces a nondegenerate form $h(\ ,\ )$ on V, which in turn induces a superinvolution $*$ on $\mathcal{M}_{n+2m}(F)$

$$h(u_{\bar i} a_{\bar j}, w_{\bar k}) = (-1)^{\bar j \bar k} h(u_{\bar i}, w_{\bar k} a^*_{\bar j}).$$

We have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^t & C^t S_m \\ S_m B^t & D^s \end{bmatrix},$$

where t and s are the transpose and symplectic involutions respectively. $*$ is the *orthosymplectic superinvolution*, and, for $n > 0$, $m > 0$, the symmetric elements of $\mathcal{M}_{n+2m}$ with respect to $*$ endowed with the Jordan superproduct form a simple Jordan subsuperalgebra of $\mathcal{M}_{n,2m}$, $osp(n, 2m)$: the *orthosymplectic Jordan superalgebra.*

$$osp(n, 2m) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid A^t = A, D^s = D, C = S_m B^t \right\}.$$

For quaternion and octonion algebras, the standard involution is the unique (not just up to isomorphism) symplectic involution. Note that, if $n = 2$ and $m = 1$ above, we are dealing with $\mathcal{M}_4(F) = \mathcal{M}_2(\mathcal{M}_2(F))$, that is 2×2-matrices with entries in the split quaternions. For $A, B, D \in \mathcal{M}_2(F)$, $B^s = SB^tS^{-1}$ and

$$\begin{aligned} osp(2, 2) &= \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid A^t = A, D^s = D, C = SB^t \right\} \\ &= \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid A^t = A, D^s = D, C = B^s S \right\}. \end{aligned}$$

Both the even and odd parts of $osp(2, 2)$ have dimension 4, and one checks that the commutators of odd elements span a subspace of dimension 3 of the even part.

2. Involutions of octonion algebras

We need some results on involutions of octonion algebras. While some of them are known, there seems to be no convenient reference. Let $\mathcal{O}$ be an octonion algebra over a field F of characteristic not 2 and $^-$ its *standard involution*. The *norm form*

$$n(x)1 = x\bar{x} = \bar{x}x$$

and its bilinearization the *trace form*

$$t(x,y)1 = x\bar{y} + y\bar{x} = \bar{x}y + \bar{y}x.$$

The *trace* $t(x) := t(x,1)$ and, for any $x \in \mathcal{O}$,

$$x^2 - t(x)x + n(x)1 = 0,$$

$$\bar{x} = t(x)1 - x.$$

For any subset $E \subseteq \mathcal{O}$ denote its elements of trace 0 by $E_0 = \{x \in E \mid t(x) = 0\} = (F1)^\perp \cap E$. Recall that the symmetric elements $\mathcal{H}(\mathcal{O}, ^-) = \{a \in \mathcal{O} \mid \bar{a} = a\} = F1$ and the skew-symmetric elements $\mathcal{S}(\mathcal{O}, ^-) = \{a \in \mathcal{O} \mid \bar{a} = -a\} = \mathcal{O}_0$, the octonions of trace 0. An element $x \in \mathcal{O}$ is *anisotropic* if $n(x) \neq 0$.

Lemma 1. *Any involution ι of $\mathcal{O}$ commutes with the standard involution $^-$. For all x, $y \in \mathcal{O}$,*

$$t(x^\iota, y) = t(x, y^\iota) \tag{1}$$

and hence

$$\mathcal{H}(\mathcal{O}, \iota) \perp \mathcal{S}(\mathcal{O}, \iota), \tag{2}$$

where $\perp$ denotes orthogonality with respect to the trace form.

Proof. Let ι be an arbitrary involution of $\mathcal{O}$. Then $1^\iota = 1$. Let $x_0 \in \mathcal{O}_0$ and ${x_0}^\iota = \alpha 1 + y_0$, $y_0 \in \mathcal{O}_0$. But ${x_0}^2 = -n(x_0)1$, so $({x_0}^\iota)^2 = ({x_0}^2)^\iota = -n(x_0)1$ while $(\alpha 1 + y_0)^2 = \alpha^2 1 + 2\alpha x_0 + x_0^2 = \alpha^2 1 + 2\alpha x_0 - n(x_0)1$. Thus $\alpha = 0$, ${\mathcal{O}_0}^\iota \subseteq \mathcal{O}_0$ and ${\mathcal{O}_0}^\iota = \mathcal{O}_0$. The restriction of $^-$ to $\mathcal{O}_0$ is minus the identity, and the standard involution $^-$ commutes with ι.

Since $t(x^\iota, y)1 \in F1$ and ι commutes with $^-$,

$$x^\iota \bar{y} + y\overline{x^\iota} = t(x^\iota, y)1 = (t(x^\iota, y)1)^\iota = \bar{y}^\iota x + \bar{x} y^\iota = t(x, y^\iota)1.$$

If $x \in \mathcal{H}(\mathcal{O}, \iota)$ and $y \in \mathcal{S}(\mathcal{O}, \iota)$, then $t(x,y) = t(x^\iota, y) = t(x, y^\iota) = -t(x,y)$ and $\mathcal{H}(\mathcal{O}, \iota) \perp \mathcal{S}(\mathcal{O}, \iota)$. □

Recall that if $\mathcal{Q}$ is a quaternion subalgebra of $\mathcal{O}$ and $v \in \mathcal{Q}^\perp$ with $v^2 = \lambda 1 \neq 0$, then $\mathcal{O} = \mathcal{Q} + \mathcal{Q}v$ can be obtained via the Cayley-Dickson process and the product is given by

$$(a + bv)(c + dv) = ac + \lambda \bar{d}b + (b\bar{c} + da)v, \qquad a, b, c, d \in \mathcal{Q}. \tag{3}$$

Theorem 2. *If ι is an involution of an octonion algebra $\mathcal{O}$ over a field F of characteristic not 2, then either ι is the standard involution $^-$ or there exist a quaternion subalgebra $\mathcal{Q}$ of $\mathcal{O}$ such that the restriction $^\iota|_\mathcal{Q} = {}^-|_\mathcal{Q}$ and an anisotropic element $v \in \mathcal{Q}^\perp \cap \mathcal{H}(\mathcal{O}, \iota)$. In the second case, $\mathcal{O}$ is obtained from $\mathcal{Q}$ and v by the Cayley-Dickson process and $\mathcal{H}(\mathcal{O}, \iota) = F1 + \mathcal{Q}v$, $\mathcal{S}(\mathcal{O}, \iota) = \mathcal{S}(\mathcal{Q}, ^-)$.*

Proof. Since the characteristic is not 2, we may choose an orthogonal basis of $\mathcal{O}$ compatible with the orthogonal decomposition $\mathcal{O} = F1 \perp \mathcal{H}(\mathcal{O}, \iota)_0 \perp \mathcal{S}(\mathcal{O}, \iota)$. Since n is nondegenerate, we may assume this basis consists of anisotropic vectors. Note that if x, y are orthogonal vectors of $\mathcal{H}(\mathcal{O}, \iota)_0$ or $\mathcal{S}(\mathcal{O}, \iota)$, then $0 = t(x,y)1 = -(xy + yx)$ and x and y anticommute. The dimension of $\mathcal{S}(\mathcal{O}, \iota)$ is at least 2 since if x_1, x_2, x_3 are mutually orthogonal anisotropic vectors of $\mathcal{H}(\mathcal{O}, \iota)_0$, then

$x_ix_j = -x_jx_i$, $i \neq j$, and x_1x_2, x_1x_3 are orthogonal anisotropic vectors of $\mathcal{S}(\mathcal{O}, \iota)$. Therefore we may choose y_1, y_2 orthogonal anisotropic vectors of $\mathcal{S}(\mathcal{O}, \iota)$. Again $y_2y_1 = -y_1y_2$ and $y_1y_2 \in \mathcal{S}(\mathcal{O}, \iota)$. Let $\mathcal{Q}$ be the quaternion subalgebra of $\mathcal{O}$ generated by y_1, y_2. On $\mathcal{Q}$, the involutions ι and $^-$ coincide.

If ι is distinct from $^-$, then $\mathcal{H}(\mathcal{O}, \iota)$ has dimension at least 2 and, by (2), must contain an anisotropic v orthogonal to $\mathcal{Q}$. If $a \in \mathcal{S}(\mathcal{Q}, \iota)$ then, by (3), $(av)^\iota = -va = -\bar{a}v = av$ and $\mathcal{Q}v \subset \mathcal{H}(\mathcal{O}, \iota)$. □

The above result could also be obtained by considering the automorphism $x \mapsto \bar{x}^\iota$ and applying the fact that for an octonion algebra, automorphisms of order 2 coincide with reflections in a quaternion subalgebra [J].

3. Embedding K_{10} in $\mathcal{M}_2(\mathcal{O})$

Following [RZ], we write $\mathrm{K}_{10} = A + M$, where the even part A is the direct sum of a one-dimensional simple Jordan algebra Ff, f, an idempotent, and a five-dimensional Jordan algebra $J(V, Q)$ of a nondegenerate quadratic form Q of maximal Witt index on a four-dimensional space V, while the odd part M is a four-dimensional bimodule of A.

Denote by $\mathcal{M}_2(\mathcal{O})$ the superalgebra of 2×2 matrices with entries in an octonion algebra $\mathcal{O}$ with even part $\mathcal{O}_{11} + \mathcal{O}_{22}$ and odd part $\mathcal{O}_{12} + \mathcal{O}_{21}$, where $\mathcal{O}_{ij}$ denotes the matrices with arbitrary entries from $\mathcal{O}$ in the ij^{th} position and 0 elsewhere. We can embed $\mathcal{O} \oplus \mathcal{O}$ diagonally in $\mathcal{M}_2(\mathcal{O})$. Consider the involution $*$ on $\mathcal{O} \oplus \mathcal{O}$ given by $\iota \oplus {}^-$, where $\iota \neq {}^-$, and $^-$ is the standard involution. Therefore there exists a unique quaternion subalgebra $\mathcal{Q} \subset \mathcal{O}$ and a $v \in \mathcal{Q}^\perp$ with $v^2 = \lambda \in F^\times$ such that $\mathcal{O} = \mathcal{Q} + \mathcal{Q}v$, $\iota|_\mathcal{Q} = {}^-|_\mathcal{Q}$, $\iota|_{\mathcal{Q}v} = id$ and $\mathcal{H}(\mathcal{O}, \iota) = F1 + \mathcal{Q}v$.

While octonion algebras do not have nontrivial one-sided ideals, split ones contain nontrivial $\mathcal{H}(\mathcal{O}, \iota)$-modules. Since ι yields a bijection between right and left $\mathcal{H}(\mathcal{O}, \iota)$-modules, we need only consider left modules. Writing $\mathcal{O} = \mathcal{Q} + \mathcal{Q}v$, for any left $\mathcal{H}(\mathcal{O}, \iota)$-submodule M of $\mathcal{O}$ we have two projections

$$\pi_1 : M \to \mathcal{Q} \qquad \text{and} \qquad \pi_2 : M \to \mathcal{Q}v.$$

Since

$$(qv)(c + dv) = \lambda \bar{d} q + (q\bar{c})v, \tag{3'}$$

$\pi_2(M) = Lv$ and $\pi_1(M) = \bar{L}$, where L is a left ideal of $\mathcal{Q}$ and of course $\bar{L}$ is the corresponding right ideal. Also, $M \cap \mathcal{Q}$ contains an invertible element if and only if $M \cap \mathcal{Q}v$ contains an invertible element. In fact neither can occur if M is proper.

Lemma 3. *Let $\iota \neq {}^-$ be an involution of an octonion algebra $\mathcal{O}$ and $\mathcal{Q}$ the quaternion subalgebra determined by ι. If M is a proper left $\mathcal{H}(\mathcal{O}, \iota)$-submodule of $\mathcal{O}$, then $M \cap \mathcal{Q}$ and $M \cap (\mathcal{Q}v)$ do not contain invertible elements.*

Proof. If $c \in M \cap \mathcal{Q}$ is invertible, then $(\mathcal{Q}\bar{c})v = \mathcal{Q}v \subset M$ and, applying (3′) again, $\mathcal{Q} \subset M$. So $M = \mathcal{O}$ is not proper. □

Having a proper $\mathcal{H}(\mathcal{O}, \iota)$-module is very restrictive.

Proposition 4. *Let $\iota \neq {}^-$ be an involution of an octonion algebra $\mathcal{O}$ and $\mathcal{Q}$ the quaternion subalgebra determined by ι. $\mathcal{O}$ contains a proper $\mathcal{H}(\mathcal{O}, \iota)$-submodule if and only if $\mathcal{Q}$ is split, in which case any proper left $\mathcal{H}(\mathcal{O}, \iota)$-submodule M of $\mathcal{O}$*

is of the form

$$M = \bar{L} + Lv,$$

where L is a minimal left ideal of $\mathcal{Q}$.

Proof. Let M be a proper left $\mathcal{H}(\mathcal{O},\iota)$-submodule of $\mathcal{O}$. We wish to show that $\pi_1(M) \neq \mathcal{Q}$.

Assume that $\pi_1(M) = \mathcal{Q}$. Then $m = 1 + dv \in M$ for some $d \in \mathcal{Q}$ which, by Lemma 3, cannot be 0. Now

$$(dv)(1+dv) = \lambda n(d) + dv \in M.$$

If d is not invertible, $n(d) = 0$, $dv \in M$ and hence $1 \in M$, contradicting Lemma 3. So d is invertible and $\lambda n(d) - 1 \in M \cap \mathcal{Q}$ must be 0 by Lemma 3. So

$$1 = \lambda n(d). \tag{4}$$

Now, for p, $q \in \mathcal{Q}$,

$$(pv)((qv)(1+dv)) = (pv)(\lambda\bar{d}q + qv) = \lambda\bar{q}p + \lambda(p\bar{q}d)v \in M$$

while

$$(\lambda p\bar{q}dv)(1+dv) = \lambda\bar{d}\lambda p\bar{q}d + \lambda(p\bar{q}d)v \in M$$

and

$$\lambda(\lambda\bar{d}p\bar{q}d - \bar{q}p) \in M \cap \mathcal{Q} \qquad \forall p, q \in \mathcal{Q}.$$

By (4), the commutator

$$[\lambda\bar{d}p, \bar{q}d] = \lambda\bar{d}p\bar{q}d - \bar{q}p$$

and, since d is invertible, $\mathcal{Q}d = \mathcal{Q}$. It follows that $[\mathcal{Q},\mathcal{Q}] \subseteq M \cap \mathcal{Q}$. But since $[\mathcal{Q},\mathcal{Q}]$ has dimension 3, it contains an invertible element, contradicting Lemma 3. Therefore $\pi_1(M) \neq \mathcal{Q}$.

Since $\pi_1(M) \neq \{0\}$, $\pi_1(M)$ must be a proper right ideal of $\mathcal{Q}$, which must therefore be split. So $\pi_1(M) = \bar{L}$ and $\pi_2(M) = Lv$, where $L = \mathcal{Q}e$ for some primitive idempotent $e \in \mathcal{Q}$. Therefore $M \subseteq \bar{L} + Lv$ and it remains to show that $\bar{L} + Lv$ has no proper $\mathcal{H}(\mathcal{O},\iota)$-submodule N. By the irreducibility of L (respectively $\bar{L}$) as left (respectively right) $\mathcal{Q}$-modules, as soon as $N \cap \mathcal{Q} \neq \{0\}$ or $N \cap (\mathcal{Q}v) \neq \{0\}$ we obtain $N = M$. If $x + yv \in N$, $x \neq 0$, then $\bar{e} + dv \in N$ for some $d \in \mathcal{Q}e$. Arguing as above $(dv)(\bar{e} + dv) = dv \in N$ and we are done. □

Lemma 5. *For a fixed invertible $z = z_0 + z_1 v \in \mathcal{O}$, $z_i \in \mathcal{Q}$, the map $m \mapsto \bar{m}z$ is an F linear isomorphism of the left $\mathcal{H}(\mathcal{O},\iota)$-submodule M of $\mathcal{O}$ onto the right $\mathcal{H}(\mathcal{O},\iota)$-submodule $\overline{M}z$ if and only if $z_1 = 0$ and $t(z_0) = 0$.*

Proof. The map $m \mapsto \bar{m}z$ is an isomorphism of the left $\mathcal{H}(\mathcal{O},\iota)$-module $M = \bar{L} + Lv$ onto the right $\mathcal{H}(\mathcal{O},\iota)$-module $\overline{M}z$ if and only if $(qv)m \mapsto (\bar{m}z)(qv)$ for all $m \in M$, $q \in \mathcal{Q}$. For $\ell \in L$,

$$(qv)\bar{\ell} = (q\ell)v \mapsto (-(q\ell)v)z = (-(q\ell)v)(z_0 + z_1 v) = -\lambda\bar{z}_1 q\ell - (q\ell\bar{z}_0)v,$$

while

$$\bar{\ell} \mapsto \ell z \quad \text{and} \quad (\ell z)(qv) = (\ell z_0 + (z_1\ell)v)(qv) = \lambda\bar{q}z_1\ell + (q\ell z_0)v.$$

If the map $m \mapsto \bar{m}z$ is an isomorphism, then

$$\lambda\bar{q}z_1\ell = -\lambda\bar{z}_1 q\ell \quad \text{and} \quad q\ell z_0 = -q\ell\bar{z}_0, \quad \forall q \in \mathcal{Q},\ \ell \in L,$$

or equivalently

$$\lambda t(q, z_1)\ell = 0 \quad \text{and} \quad t(z_0)q\ell = 0, \quad \forall q \in \mathcal{Q},\ \ell \in L.$$

Thus $z_1 = 0$ and $t(z_0) = 0$, which is equivalent to $\bar{z}_0 = -z_0$. These conditions are also sufficient since

$$(qv)(\ell v) = \lambda\bar{\ell}q \mapsto \lambda\bar{q}\ell z_0$$

while

$$((\overline{\ell v})z)(qv) = ((-\ell v)z_0)(qv) = ((-\ell\bar{z}_0)v)(qv) = \lambda\bar{q}\ell z_0. \qquad \square$$

We may therefore embed $\mathrm{K}_{10} = A + M$ in $\mathcal{M}_2(\mathcal{O})$ by identifying A with $\mathcal{H}(\mathcal{O} \oplus \mathcal{O}, *) = \mathcal{H}(\mathcal{O}, \iota) \oplus \mathcal{H}(\mathcal{O}, {}^-)$ along the diagonal and $M = \bar{L} + Lv$ by mapping $m \in M$ to $m_{12} + (\bar{m}z)_{21}$ for a fixed $z \in \mathcal{Q}$ of trace 0. Unfortunately the involution $*$ of $\mathcal{O} \oplus \mathcal{O}$ does not extend to a superinvolution $*$ of $\mathcal{M}_2(\mathcal{O})$. If it did

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^\iota & c^\psi \\ b^\phi & \bar{d} \end{pmatrix},$$

for some vector space isomorphisms $\phi, \psi : \mathcal{O} \to \mathcal{O}$. Since $1_{12}1_{21} = 1_{11}$,

$$1^\psi 1^\phi = -1 \quad \text{and} \quad 1^\psi = -(1^\phi)^{-1}.$$

Applying $*$ to $b_{12} = 1_{12}b_{22}$ and $c_{21} = c_{22}1_{21}$,

$$b^\phi = \bar{b}1^\phi, \qquad c^\psi = 1^\psi\bar{c}.$$

Similarly

$$b^\phi = 1^\phi b^\iota, \qquad c^\psi = c^\iota 1^\psi.$$

Let $1^\phi = x + yv$. We have

$$\begin{aligned} 1^\phi(c + dv)^\iota &= (\overline{c + dv})1^\phi \\ (x + yv)(\bar{c} + dv) &= (\bar{c} - dv)(x + yv) \\ (x\bar{c} + \lambda\bar{d}y) + (yc + dx)v &= (\bar{c}x - \lambda\bar{y}d) + (-d\bar{x} + y\bar{c})v, \quad \forall c, d \in \mathcal{Q}. \end{aligned}$$

Letting $d = 0$ or $c = 0$,

$$\begin{aligned} &x\bar{c} = \bar{c}x, \quad \text{so} \quad x = \alpha 1, \\ &dx = -d\bar{x}, \quad \text{so} \quad \bar{x} = -x \quad \text{and} \quad x = 0. \\ &yc = y\bar{c}. \end{aligned}$$

If $\bar{c} = -c \neq 0$, $2yc = 0$ and $y = 0$. Thus $1^\phi = 0$, a contradiction.

Nonetheless we would like to have a product $x_{\bar{i}} \cdot y_{\bar{j}} := \frac{1}{2}(x_{\bar{i}}y_{\bar{j}} + (-1)^{\bar{i}\bar{j}}y_{\bar{j}}x_{\bar{i}})$ for homogeneous elements $x_{\bar{i}}$, $y_{\bar{j}} \in A \cup M$.

Recall that K_{10}, the split *Kac superalgebra*: $A = (Fe + \sum_{1\leq i\leq 4} Fv_i) \oplus Ff$ and $M = \sum_{i=1,2}(Fx_i + Fy_i)$; the following basis for the Kac algebra $A + M$ was obtained in [RZ] by scaling the basis given in [HK]: e, v_1, v_2, v_3, v_4, $f \in A$, x_1, y_1, x_2, $y_2 \in M$, where

$$e^2 = e, \quad e.v_i = v_i, \quad v_1.v_2 = 2e = v_3.v_4, \tag{5}$$

$$f^2 = f, \quad f.x_j = \frac{1}{2}x_j, \quad f.y_j = \frac{1}{2}y_j, \quad j = 1, 2, \tag{5'}$$

$$\begin{matrix} e.x_j = 1/2x_j, & y_1.v_1 = x_2, & y_2.v_1 = -x_1, & x_1.v_2 = -y_2, & x_2.v_2 = y_1, \\ e.y_j = 1/2y_j, & x_2.v_3 = x_1, & y_1.v_3 = y_2, & x_1.v_4 = x_2, & y_2.v_4 = y_1 \end{matrix} \tag{6}$$

(7)

$$[x_i,\ y_i] = e - 3f, \quad [x_1,\ x_2] = v_1, \quad [x_1,\ y_2] = v_3, \quad [x_2,\ y_1] = v_4, \quad [y_1, y_2] = v_2$$

and every other product is zero or is obtained by the symmetry or skew-symmetry of one of the above products. As in [RZ], $\cdot$ indicates a commutative product and $[\ ,\ ]$ a skew-commutative product. While we were inspired by the embedding of $osp(2,2)$ in 2×2 matrices with entries in the split quaternions, one should note that $osp(2,2)$ cannot be embedded in K_{10}. Indeed M is of dimension 4 and equation (7) shows that $[M,M]$ spans a subspace of A of dimension 5 and, as was observed before, the odd part of $osp(2,2)$ is also of dimension 4, but its products span a subspace of dimension 3.

Let $\mathcal{Q}=\mathcal{M}_2(F)$, $v\in\mathcal{Q}^\perp$ with $v^2=1$, $\mathcal{O}=\mathcal{Q}+\mathcal{Q}v$ and $L=\mathcal{Q}e_{11}=Fe_{11}+Fe_{21}$. So $\bar{L}=e_{22}\mathcal{Q}=Fe_{22}+Fe_{21}$. Let

$$e=1\in\mathcal{H}(\mathcal{O},\iota),\quad v_1=2e_{12}v,\quad v_2=-2e_{21}v,\quad v_3=2e_{22}v,\quad v_4=2e_{11}v\in\mathcal{H}(\mathcal{O},\iota).$$

One checks that equations (5) are satisfied. Let $L=\mathcal{Q}e_{11}$,

$$f=1\in\mathcal{H}(\mathcal{O},^-),\quad x_1=e_{22},\quad y_1=-e_{21},\quad x_2=e_{11}v,\quad y_2=e_{21}v\in M=\bar{L}+Lv.$$

Embedding M in $\mathcal{M}_2(\mathcal{O})$ via $m\mapsto m_{12}+\bar{m}z_{21}$, where $z=e_{12}-e_{21}=S\in\mathcal{Q}$, one checks that equations (5′) and (6) are satisfied. In general, for $m,\ m'\in M$, $[m,m']\notin\mathcal{H}(\mathcal{O},\iota)\oplus\mathcal{H}(\mathcal{O},^-)$ but the 11-component of $[m,m']+[m,m']^*$ agrees with equation (7) while the 22-component must be multiplied by 3 to agree with (7). In the above notation, we have therefore shown

Theorem 6. *Let F be a field of characteristic not 2 or 3, $\mathcal{O}$ the split octonions over F, $^-$ the standard, i.e. symplectic, involution of $\mathcal{O}$ and ι an orthogonal involution of $\mathcal{O}$. Then the subspace*

$$\mathcal{K}=\left\{\begin{pmatrix} a_{11} & m_{12}\\ (\bar{m}S)_{21} & b_{22}\end{pmatrix}\,\middle|\, a\in\mathcal{H}(\mathcal{O},\iota), b\in\mathcal{H}(\mathcal{O},^-), m\in M\right\}\subset\mathcal{M}_2(\mathcal{O}),$$

where M is a proper left $\mathcal{H}(\mathcal{O},\iota)$-module in $\mathcal{O}$ and $S=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, can be endowed with a product

$$c_{\bar{i}}\cdot d_{\bar{j}}:=\frac{1}{2}(c_{\bar{i}}d_{\bar{j}}+(-1)^{\bar{i}\bar{j}}c_{\bar{j}}d_{\bar{i}})^\pi, \tag{8}$$

where π is the identity of at least one of $\bar{i}$, $\bar{j}$ is $\bar{0}$ and

$$(c_{\bar{1}}\cdot d_{\bar{1}})^\pi:=((c_{\bar{1}}\cdot d_{\bar{1}})+(c_{\bar{1}}\cdot d_{\bar{1}})^*)_{11}+3((c_{\bar{1}}\cdot d_{\bar{1}})+(c_{\bar{1}}\cdot d_{\bar{1}})^*)_{22},$$

which gives $\mathcal{K}$ the structure of a split Kac superalgebra K_{10}.

4. Embedding forms of K_{10} in $\mathcal{M}_2(\mathcal{O})$

We recall a few facts concerning forms of K_{10} [RZ]. Let $d\in F^\times$. The Jordan superalgebra with basis $\{f,e,u_1,u_2,u_3,u_4,z_1,z_2,z_3,z_4\}$ and products

$$e^2=e,\quad e.u_i=u_i,\quad u_1^2=4e,\quad u_2^2=-4de,\quad u_3.u_4=2e, \tag{9}$$

$$f^2=f,\quad f.z_j=\frac{1}{2}z_j=e.z_j,\quad j=1,\dots,4, \tag{10}$$

$$\begin{aligned} &z_1.u_1=z_1, && z_2.u_1=-z_2, && z_3.u_1=-z_3, && z_4.u_1=z_4,\\ &z_1.u_2=-z_2, && z_2.u_2=dz_1, && z_3.u_2=z_4, && z_4.u_2=-dz_3,\\ &z_3.u_3=z_1, && z_4.u_3=z_2, && z_1.u_4=z_3, && z_2.u_4=z_4,\end{aligned} \tag{11}$$

$$[z_1,\ z_2]=-2u_3,\quad [z_3,\ z_4]=-2u_4,\quad [z_1,\ z_3]=d^{-1}u_2,\quad [z_2,\ z_4]=u_2, \tag{12}$$

(13) $$[z_1,\ z_4] = -u_1 - 2(e - 3f), \qquad [z_2,\ z_3] = -u_1 + 2(e - 3f),$$

where all other products are 0 or obtained by symmetry or skew-symmetry, yields a form of K_{10}. These are determined up to isomorphism by the class of $d \in F^\times/F^{\times 2}$, K_{10} corresponding to the class of 1. The even part is the sum of Ff and $J(U,Q)$, where U is the span of $\{u_1, u_2, u_3, u_4\}$ and Q is a nondegenerate quadratic form of discriminant d and Witt index 2 or 1 according as d is a square or not. Since a quaternion algebra is either a division algebra or split and hence its norm is of Witt index 0 or 2, if d is not a square, then we cannot hope to embed $J(U,Q)$ in an octonion algebra as the symmetric elements of an orthogonal involution. Denote by $\mathcal{O}_d$ the octonion algebra obtained by repeated application of the Cayley-Dickson process using the scalars d, 1 and 1. So it has generators a_1, a_2, a_3 satisfying

$$a_1^2 = d, \qquad a_2^2 = 1, \qquad a_3^2 = 1,$$

and a basis $\{1, a_1, a_2, a_1a_2, a_3, a_1a_3, a_2a_3, (a_1a_2)a_3\}$. If we let

$$e = 1, \qquad u_1 = 2a_2, \qquad u_2 = 2a_1a_2, \qquad u_3 = a_3 + a_2a_3, \qquad u_4 = a_3 - a_2a_3,$$

we can check that (9) is satisfied, including the fact that other Jordan products are 0. In particular the fact that some nonzero elements square to 0 shows that for any $d \in F^\times$, $\mathcal{O}_d$ is a split octonion algebra. Let

(14)
$$z_1 = 1 + a_2, \qquad z_2 = -(a_1 + a_1a_2), \qquad z_3 = a_3 - a_2a_3, \qquad z_4 = a_1a_3 + (a_1a_2)a_3,$$

and M be the span of $\{z_1, z_2, z_3, z_4\}$. Embedding $J(U,Q)$ in $\mathcal{O}_{11} \subset \mathcal{M}_2(\mathcal{O})$ and M in $\mathcal{O}_{12} \subset \mathcal{M}_2(\mathcal{O})$, one checks that, for the product (8), equations (11) are satisfied.

Let μ be the linear map on $\mathcal{O}_d$ given by

(15)
$$\begin{array}{cccc} 1^\mu = 1, & (a_1)^\mu = -a_1, & (a_2)^\mu = a_2, & (a_1a_2)^\mu = a_1a_2, \\ (a_3)^\mu = a_3, & (a_1a_3)^\mu = a_1a_3, & (a_2a_3)^\mu = a_2a_3, & ((a_1a_2)a_3)^\mu = -(a_1a_2)a_3. \end{array}$$

While μ is of period 2 and is an involution of the quaternion subalgebra generated by a_1 and a_2, it is not an involution of $\mathcal{O}_d$. For example, ${a_3}^\mu {a_2}^\mu = a_3a_2 = -a_2a_3$ but $(a_2a_3)^\mu = a_2a_3$.

Arguing along the lines of Theorem 6, one obtains

Theorem 7. *Let F be a field of characteristic not 2 or 3. The split octonions over F may be obtained by repeated applications of the Cayley-Dickson process starting with F using generators a_1, a_2 and a_3 satisfying*

$${a_1}^2 = d, \qquad {a_2}^2 = 1, \qquad {a_3}^2 = 1.$$

Denote the split octonions obtained in this fashion by $\mathcal{O}_d$. Let M be the subspace of $\mathcal{O}_d$ spanned by z_1, z_2, z_3, and z_4 as in equation (14) and $\mu : \mathcal{O}_d \to \mathcal{O}_d$, the linear map defined in (15). Let $w = \frac{1}{2}d^{-1}(a_1a_2)a_3$. Then the subspace of $\mathcal{M}_2(\mathcal{O}_d)$,

$$\mathcal{K}_d = \left\{ \begin{pmatrix} a_{11} & m_{12} \\ (\bar{m}w)_{21} & b_{22} \end{pmatrix} \mid a \in \mathcal{O}_d, a^\mu = a, b \in \mathcal{H}(\mathcal{O}_d, ^-), m \in M \right\} \subset \mathcal{M}_2(\mathcal{O}_d),$$

can be endowed with a product

$$c_{\bar{i}} \cdot d_{\bar{j}} := \frac{1}{2}(c_{\bar{i}}d_{\bar{j}} + (-1)^{\bar{i}\bar{j}} c_{\bar{j}} d_{\bar{i}})^\pi,$$

where π is the identity map of at least one of $\bar{i}$, $\bar{j}$ is $\bar{0}$ and

$$(c_{\bar{1}}d_{\bar{1}} - d_{\bar{1}}c_{\bar{1}})^\pi := ([c_{\bar{1}},\ d_{\bar{1}}] + [c_{\bar{1}},\ d_{\bar{1}}]^*)_{11} + 3([c_{\bar{1}},\ d_{\bar{1}}] + [c_{\bar{1}},\ d_{\bar{1}}]^*)_{22},$$

for * *the map on* $diag\mathcal{M}_2(\mathcal{O}_d)$ *given by* $(g_{11}, h_{11})^* := ((g^\mu)_{11}, \bar{h}_{22})$. *This product gives* $\mathcal{K}_d$ *the structure of a form of the split Kac superalgebra* K_{10}. *All forms can be obtained this way and any* $\mathcal{K}_d$ *is isomorphic to* $\mathcal{K}_{d'}$ *if and only if* $d = d'$ *in* $F^\times/F^{\times 2}$.

If d is a square, this provides for an embedding of the split Kac superalgebra in $\mathcal{M}_2(\mathcal{O})$. If the characteristic is 3, then K_{10} is not simple but the subalgebra of codimension 1, $\mathrm{K}_9 = J(V, Q) + M$ is simple [RZ]. The embedings above allow us to embed K_9 and its forms in $\mathcal{M}_2(\mathcal{O})$.

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