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PROJECTIVE AND INJECTIVE SYMMETRIC CATEGORICAL GROUPS AND DUALITY

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Abstract. We prove that the 2-category of symmetric categorical groups have enough projective and injective objects.

1. Introduction

Symmetric categorical groups, also known as Picard categories or abelian 2-groups, play the same role in the 2-dimensional algebra as abelian groups play in the classical algebra. In [3] D. Bourn and E. Vitale defined the notion of projective and injective symmetric categorical groups, and they asked whether there are enough projective symmetric categorical groups (see p. 104 in [3]).

In this paper we show that there are enough projective and injective symmetric categorical groups, thereby giving an affirmative answer to the question of D. Bourn and E. Vitale. Moreover, we construct a projective symmetric categorical group Φ and we prove that any projective symmetric categorical group is a coproduct of copies of Φ, thus any projective symmetric categorical group is free in some sense.

As an application we construct a self-duality of the 2-category of finite symmetric categorical groups. For some other applications of these facts see [5].

2. Basic notions

Recall that a *groupoid* is a small category such that all morphisms are isomorphisms. For a groupoid $\mathbb{G}$ and an object $x \in \mathbb{G}$ we let $\pi_0(\mathbb{G})$ and $\pi_1(\mathbb{G}, x)$ be the set of connected components of $\mathbb{G}$ and the group of automorphisms of x in $\mathbb{G}$ respectively.

The notion of a symmetric categorical group is a categorification of the notion of an abelian group. More precisely, let $(\mathbb{A}, +, 0, a, l, r, c)$ be a symmetric monoidal category, where $+ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is the composition law, 0 is the neutral element, a is the associative constraint, c is the commutativity constraint and $l : \mathsf{Id} \to 0 + \mathsf{Id}$ and $r : \mathsf{Id} \to \mathsf{Id} + 0$ are natural transformations satisfying well-known properties [6]. We will say that $\mathbb{A}$ is a *symmetric categorical group* or *Picard category* provided $\mathbb{A}$ is a groupoid and for any object x the endofunctor $x+ : \mathbb{A} \to \mathbb{A}$ is an equivalence of categories. It follows that $\pi_0(\mathbb{A})$ is an abelian group and the endofunctor $x+ : \mathbb{A} \to \mathbb{A}$ yields an isomorphism $\pi_1(\mathbb{A}, 0) \cong \pi_1(\mathbb{A}, x)$ of abelian groups. In what follows we will write $\pi_1(\mathbb{A})$ instead of $\pi_1(\mathbb{A}, 0)$.

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For symmetric categorical groups $\mathbb{S}_1$ and $\mathbb{S}_2$ we have a groupoid (in fact a symmetric categorical group [3]) $\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)$. Hence symmetric categorical groups form a groupoid enriched category $\mathfrak{SCG}$. Objects of the groupoid $\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)$ are symmetric monoidal functors, called *morphisms* of symmetric categorical groups, while morphisms of the groupoid $\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)$ are monoidal natural transformations, called *tracks*. We let $\mathbf{Ho}(\mathfrak{SCG})$, or simply $\mathbf{Ho}$, be the additive category, with the same objects as $\mathfrak{SCG}$, and

$$\mathsf{Hom}_{\mathbf{Ho}}(\mathbb{S}_1, \mathbb{S}_2) = \pi_0(\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)).$$

Recall that a morphism $f : \mathbb{A} \to \mathbb{B}$ of symmetric categorical groups is *essentially surjective* provided $\pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$ is a surjective homomorphism of abelian groups. A symmetric categorical group $\mathbb{P}$ is *projective* provided for any essentially surjective morphism $f : \mathbb{A} \to \mathbb{B}$ of symmetric categorical groups the induced morphism $\mathbf{Hom}(\mathbb{P}, \mathbb{A}) \to \mathbf{Hom}(\mathbb{P}, \mathbb{B})$ is essentially surjective. Equivalently, the map $\mathsf{Hom}_{\mathbf{Ho}}(\mathbb{P}, \mathbb{A}) \to \mathsf{Hom}_{\mathbf{Ho}}(\mathbb{P}, \mathbb{B})$ is surjective.

Dually, a morphism $g : \mathbb{C} \to \mathbb{D}$ of symmetric categorical groups is *faithful* provided the induced map $\pi_1\mathbb{C} \to \pi_1\mathbb{D}$ is injective. A symmetric categorical group $\mathbb{I}$ is called *injective* provided for any faithful morphism $F : \mathbb{C} \to \mathbb{D}$ the induced map

$$\mathsf{Hom}_{\mathbf{Ho}}(\mathbb{D}, \mathbb{I}) \to \mathsf{Hom}_{\mathbf{Ho}}(\mathbb{C}, \mathbb{I})$$

is surjective.

We will prove that the 2-category $\mathfrak{SCG}$ has *enough injective and projective* objects, meaning that for any symmetric categorical group $\mathbb{A}$ there exists a faithful morphism $\mathbb{A} \to \mathbb{I}$ with injective object $\mathbb{I}$ and an essentially surjective morphism $\mathbb{P} \to \mathbb{A}$, with projective $\mathbb{P}$.

Recall also the notion of direct sum of symmetric categorical groups. Assume $\mathbb{S}_\alpha$, $\alpha \in A$, are symmetric categorical groups. A *coproduct* $\bigoplus_{\alpha\in A} \mathbb{S}_\alpha$ is a symmetric categorical group $\mathbb{S}$, together with morphisms of symmetric categorical groups $i_\alpha : S_\alpha \to \mathbb{S}$, such that for any symmetric categorical group $\mathbb{A}$, the obvious functor

$$\mathbf{Hom}(\mathbb{S}, \mathbb{A}) \to \prod_\alpha \mathbf{Hom}(S_\alpha, \mathbb{A})$$

is an equivalence of categories. In a similar way one can define a *product* $\prod_{\alpha\in A} \mathbb{S}_\alpha$ of symmetric categorical groups S_α. As in the case of abelian groups, finite product and coproduct are equivalent. One easily sees that coproduct and product exist and

$$\pi_i\left(\bigoplus_{\alpha\in A} \mathbb{S}_\alpha\right) = \bigoplus_{\alpha\in A} \pi_i(\mathbb{S}_\alpha), \quad \pi_i\left(\prod_{\alpha\in A} \mathbb{S}_\alpha\right) = \prod_{\alpha\in A} \pi_i(\mathbb{S}_\alpha), \quad i = 0, 1.$$

Thus product and coproduct of symmetric categorical groups yield the usual product and coproduct in the homotopy category $\mathbf{Ho}$.

3. The homotopy category of the 2-category of symmetric categorical groups

It follows from the results of [7] that the 2-category $\mathfrak{SCG}$ of symmetric categorical groups is 2-equivalent to the 2-category of 2-stage spectra (see also Proposition B.12 in [4]). Hence we can use the classical facts of algebraic topology to study $\mathfrak{SCG}$.

Let $\mathbf{\Gamma AB}$ be the category of triples (A, B, a) where A and B are abelian groups and

$$a \in \mathsf{Hom}(A/2A, B) = \mathsf{Hom}(A, {}_2B)$$

where ${}_2B = \{b \in B \mid 2b = 0\}$. A morphism $(A, B, a) \to (A_1, B_1, a_1)$ is a pair (f, g) where $f : A \to A_1$ and $g : B \to B_1$ are homomorphisms, such that $a_1 f = ga$. The functor

$$\mathsf{k} : \mathbf{Ho}(\mathfrak{SCG}) \to \mathbf{\Gamma AB}$$

is defined by

$$\mathsf{k}(\mathbb{S}) := (\pi_0(\mathbb{S}), \pi_1(\mathbb{S}), k_{\mathbb{S}})$$

where $\mathbb{S}$ is a symmetric categorical group and $k_{\mathbb{S}} : \pi_0(\mathbb{S}) \to {}_2\pi_1(\mathbb{S})$ is the homomorphism induced by the commutativity constraints in $\mathbb{S}$:

$$x \mapsto c_{x,x} \in \pi_1(\mathbb{S}, x + x) \cong \pi_1(\mathbb{S}).$$

Proposition 1. *For any symmetric categorical groups $\mathbb{S}_1$ and $\mathbb{S}_2$, one has a short exact sequence of abelian groups*

$$(1) \quad 0 \to \mathsf{Ext}(\pi_0(\mathbb{S}_1), \pi_1(\mathbb{S}_2)) \to \pi_0(\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)) \xrightarrow{\gamma} \mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathsf{k}(\mathbb{S}_1), \mathsf{k}(\mathbb{S}_2)) \to 0$$

Furthermore, one has also an isomorphism of abelian groups

$$(2) \qquad \pi_1(\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)) \cong \mathsf{Hom}(\pi_0(\mathbb{S}_1), \pi_1(\mathbb{S}_2)).$$

Moreover, for the k-invariant one has the equality:

$$(3) \qquad k_{\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)} = \alpha\gamma,$$

where α sends a pair $f_0 : \pi_0(\mathbb{S}_1) \to \pi_0(\mathbb{S}_2), f_1 : \pi_0(\mathbb{S}_1) \to \pi_0(\mathbb{S}_2)$ to the composite

$$\pi_0(\mathbb{S}_1) \xrightarrow{f_0} \pi_0(\mathbb{S}_2) \xrightarrow{k_{\mathbb{S}_2}} {}_2\pi_1(\mathbb{S}_1).$$

Proof. The second isomorphism is obvious, while the first one is Proposition 7.1.6 in [2]. The statement on k-invariant follows from the expression of symmetric constant on $\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)$. □

We see that both categories $\mathbf{Ho}$ and $\mathbf{\Gamma AB}$ are additive and the functor

$$\mathsf{k} : \mathbf{Ho} \to \mathbf{\Gamma AB}$$

preserves coproducts and products. Thus it is additive. Moreover k is a part of a linear extension of categories (see Lemma 7.2.4 and Theorem 7.2.7 in [2]). It follows from the properties of linear extensions of categories [1] that the functor k is full, reflects isomorphisms, is essentially surjective on objects and it induces a bijection on the isomorphism classes of objects. Moreover, the kernel of k (the class of morphisms which go to zero) is a square zero ideal of $\mathbf{Ho}$. Hence, for a given object $\mathbb{A}$ of the category $\mathbf{\Gamma AB}$ we can choose a symmetric categorical group $K(\mathbb{A})$ such that $\mathsf{k}(K(\mathbb{A})) = \mathbb{A}$. Such an object exists and is unique up to equivalence. Moreover, for any morphism $f : \mathbb{A} \to \mathbb{B}$ we can choose a morphism of symmetric categorical groups $K(f) : K(\mathbb{A}) \to K(\mathbb{B})$, such that $\mathsf{k}(K(f)) = f$. The reader must be aware that the assignments $\mathbb{A} \mapsto K(\mathbb{A})$, $f \mapsto K(f)$ do NOT define a functor $\mathbf{\Gamma AB} \to \mathbf{Ho}$. Having in mind the relation with spectra, the construction K for the objects of the form $(A, 0, 0)$ coincides with the Eilenberg-Mac Lane spectrum and in the general case is consistent with Definition 7.1.5 in [2].

We let Φ be the symmetric categorical group corresponding to the object $(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, k = \mathsf{Id}_{\mathbb{Z}/2\mathbb{Z}})$. We can assume that the objects of the groupoid Φ are integers; if n and m are integers, then there are no morphisms from n to m, if $m \neq n$, while the automorphism group of the object n is the cyclic group of order two $\{\pm 1\}$. The monoidal structure on objects is induced by the group structure of integers and the monoidal structure on morphisms is induced by the multiplication on the cyclic group of order two. The associativity and unitality constraints are the identity morphisms, while the commutativity constraint $n + m \to m + n$ is $(-1)^{nm}$. It follows from Proposition 1 that for any symmetric categorical group $\mathbb{S}$ the symmetric categorical groups $\mathbf{Hom}(\Phi, \mathbb{S})$ and $\mathbb{S}$ are equivalent.

4. Projective objects in $\mathfrak{SCG}$

In this section we prove the following theorem.

Theorem 2. *There are enough projective symmetric categorical groups. Moreover, any projective symmetric categorical group is equivalent to a coproduct of copies of* Φ.

This result is a consequence of Lemma 3 proved below.

A morphism $f = (f_0, f_1)$ in $\mathbf{\Gamma AB}$ is *essentially surjective* if f_0 is an epimorphism of abelian groups. Moreover an object $\mathbb{P}$ in $\mathbf{\Gamma AB}$ is *projective* if for any essentially surjective morphism $f : \mathbb{A} \to \mathbb{B}$ in $\mathbf{\Gamma AB}$ the induced map

$$\mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathbb{P}, \mathbb{A}) \to \mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathbb{P}, \mathbb{B})$$

is surjective.

It is clear that a morphism $F : \mathbb{S}_1 \to \mathbb{S}_2$ of symmetric categorical groups is *essentially surjective* iff $\mathsf{k}(F) : \mathsf{k}(\mathbb{S}_1) \to \mathsf{k}(\mathbb{S}_2)$ is so in $\mathbf{\Gamma AB}$. For an abelian group M we introduce two objects in $\mathbf{\Gamma AB}$:

$$l(M) := (M, M/2M, id_{M/2M}),$$
$$M[0] = (M, 0, 0).$$

Lemma 3. *i) If M is an abelian group and $\mathbb{A} = (A_0, A_1, \alpha)$ is an object in $\mathbf{\Gamma AB}$, then one has the following functorial isomorphism of abelian groups:*

$$\mathsf{Hom}_{\mathbf{\Gamma AB}}(l(M), \mathbb{A}) = \mathsf{Hom}(M, A_0).$$

ii) An object $\mathbb{P}$ is projective in $\mathbf{\Gamma AB}$ iff it is isomorphic to the object of the form $l(P)$ with free abelian group P.

iii) Φ is a projective object in $\mathfrak{SCG}$ and any projective object in $\mathfrak{SCG}$ is equivalent to a coproduct of copies of Φ.

iv) The 2-category $\mathfrak{SCG}$ of symmetric categorical groups has enough projective objects.

Proof. i) Assume $f = (f_0, f_1) : l(M) \to \mathbb{A}$ is a morphism in $\mathbf{\Gamma AB}$. So $f_0 : M \to A_0$ and $f_1 : M/2M \to A_1$ are homomorphisms of abelian groups and the following diagram commutes:

$$\begin{array}{ccc} M/2M & \xrightarrow{\mathsf{Id}} & M/2M \\ \downarrow{\scriptstyle \hat{f}_0} & & \downarrow{\scriptstyle f_1} \\ A_0/2A_0 & \xrightarrow[\alpha]{} & A_1 \end{array}$$

Here $\hat{f}_0$ is induced by f_0. It follows that f_1 is completely determined by f_0. This proves the result.

ii) Let P be a free abelian group and let $\mathbb{A} \to \mathbb{B}$ be an essentially surjective morphism in $\mathbf{\Gamma AB}$. Thus $A_0 \to B_0$ is an epimorphism. It follows that $\mathsf{Hom}(P, A_0) \to \mathsf{Hom}(P, B_0)$ is an epimorphism as well. Hence, by virtue of i), the map $\mathsf{Hom}_{\mathbf{\Gamma AB}}(l(P), \mathbb{A}) \to \mathsf{Hom}_{\mathbf{\Gamma AB}}(l(P), \mathbb{B})$ is surjective. Thus $l(P)$ is projective in $\mathbf{\Gamma AB}$.

Conversely, assume $\mathbb{P} = (P_0, P_1, \pi)$ is a projective object in $\mathbf{\Gamma AB}$. We claim that P_0 is a free abelian group. In fact it suffices to show that it is a projective object in the category $\mathfrak{Ab}$ of abelian groups. Take any epimorphism of abelian groups $f_0 : A \to B$ and any homomorphism of abelian groups $g_0 : P_0 \to B$. We have to show that g_0 has a lift to A. Observe that $f = (f_0, 0) : A[0] \to B[0]$ is essentially surjective in $\mathbf{\Gamma AB}$ and $g = (g_0, 0) : \mathbb{P} \to B[0]$ is a well-defined morphism in $\mathbf{\Gamma AB}$. By our assumption we can lift g to a morphism $\tilde{g} : \mathbb{P} \to A[0]$. It is clear that $\tilde{g} = (\tilde{g_0}, 0)$ for some $\tilde{g_0} : P_0 \to A$. Clearly, $g_0 = f_0 \circ \tilde{g_0}$. It follows that P_0 is a free abelian group. Hence $l(P_0)$ is a projective object in $\mathbf{\Gamma AB}$. By i) the identity map defines a canonical morphism $i = (\mathsf{Id}_{P_0}, i_1) : l(P_0) \to \mathbb{P}$, which is obviously essentially surjective in $\mathbf{\Gamma AB}$. Since $\mathbb{P}$ is projective, it follows that there exists a morphism $p = (\mathsf{Id}_{P_0}, p_1) : \mathbb{P} \to l(P)$ such that $i \circ p = \mathsf{Id}_{\mathbb{P}}$. Thus, we have a commutative diagram

$$\begin{array}{ccc}
P_0/2P_0 & \xrightarrow{\pi} & P_1 \\
\downarrow{\scriptstyle \mathsf{Id}} & & \downarrow{\scriptstyle p_1} \\
P_0/2P_0 & \xrightarrow{\mathsf{Id}} & P_0/2P_0 \\
\downarrow{\scriptstyle \mathsf{Id}} & & \downarrow{\scriptstyle i_1} \\
P_0/2P_0 & \xrightarrow{\pi} & P_1
\end{array}$$

with $i_1 p_1 = \mathsf{Id}_{P_1}$. It follows that p_1 and i_1 are mutually inverse isomorphisms of abelian groups. Hence $p : \mathbb{P} \to l(P)$ and $l : l(P) \to \mathbb{P}$ are mutually inverse isomorphisms in $\mathbf{\Gamma AB}$.

iii) First of all observe that k preserves coproduct and $\mathsf{k}(\Phi) = l(\mathbb{Z})$. Hence our assertion is equivalent to the following one: For any free abelian group P the symmetric categorical group $K(l(P))$ is a projective symmetric categorical group and conversely, if $\mathbb{S}$ is a projective symmetric categorical group, then $\pi_0(\mathbb{S})$ is a free abelian group and $\mathbb{S}$ is equivalent to $K(l(\pi_0(\mathbb{S})))$.

To prove the last assertion, let $F : \mathbb{S}_1 \to \mathbb{S}_2$ be an essentially surjective morphism of symmetric categorical groups and let $G : K(l(P)) \to \mathbb{S}_2$ be a morphism of symmetric categorical groups. Apply the functor k to get morphisms $\mathsf{k}(F) : \mathsf{k}(\mathbb{S}_1) \to \mathsf{k}(\mathbb{S}_2)$ and $\mathsf{k}(G) : l(P) \to \mathsf{k}(\mathbb{S}_2)$ in $\mathbf{\Gamma AB}$. Since $\pi_0(F) : \pi_0(\mathbb{S}_1) \to \pi_0(\mathbb{S}_2)$ is an epimorphism of abelian groups, it follows that $\mathsf{k}(F) : \mathsf{k}(\mathbb{S}_1) \to \mathsf{k}(\mathbb{S}_2)$ is an essentially surjective morphism in $\mathbf{\Gamma AB}$. Since P is a free abelian group, $l(P)$ is projective in $\mathbf{\Gamma AB}$ by ii). Thus we can lift $\mathsf{k}(F)$ to get a morphism $\hat{g} : l(P) \to \mathsf{k}(\mathbb{S}_1)$ such that $\mathsf{k}(F) \circ \hat{g} = \mathsf{k}(G)$ holds in $\mathbf{\Gamma AB}$. Since $P = \pi_0(K(l(P))$ is a free abelian group, the Ext-term in the exact sequence (1) disappears and we get the isomorphism

$$\pi_0(\mathbf{Hom}(K(l(P)), \mathbb{S}_i)) \cong \mathsf{Hom}_{\mathbf{\Gamma AB}}(l(P), \mathsf{k}(\mathbb{S}_i)), \quad i = 0, 1. \tag{4}$$

Take a morphism $L : K(l(P)) \to \mathbb{S}_1$ of symmetric categorical groups which corresponds to the morphism $\hat{g} : l(P) \to \mathsf{k}(\mathbb{S}_1)$. By our construction one has an equality $\mathsf{k}(FL) = \mathsf{k}(G)$ in $\mathsf{Hom}\mathbf{\Gamma AB}(l(P), \mathsf{k}(\mathbb{S}_2)) = \pi_0(\mathbf{Hom}(H(l(P)), \mathbb{S}_2))$. Thus the classes of FL and of G in $\pi_0(\mathbf{Hom}(K(l(P)),\mathbb{S}_1))$ are the same. Hence there exists a track from FL to G. This shows that $K(l(P))$ is a projective symmetric categorical group.

Conversely assume $\mathbb{S}$ is a projective symmetric categorical group. Since $\mathbb{S}$ and $K(\mathsf{k}(\mathbb{S}))$ are equivalent, it follows that $K(\mathsf{k}(\mathbb{S}))$ is also projective. We claim that $\mathsf{k}(\mathbb{S})$ is projective in $\mathbf{\Gamma AB}$. In fact, take any essentially surjective morphism $f = (f_0, f_1) : \mathbb{A} \to \mathbb{B}$ and any morphism $g : \mathsf{k}(\mathbb{S}) \to \mathbb{B}$ in $\mathbf{\Gamma AB}$. Then $K(f) : K(\mathbb{A}) \to K(\mathbb{B})$ is essentially surjective in $\mathfrak{SCG}$. Hence for $K(g) : K(\mathsf{k}(\mathbb{S})) \to K(\mathbb{B})$ we have a morphism $\tilde{G} : K(\mathsf{k}(\mathbb{S})) \to K(\mathbb{A})$ and a track $K(f) \circ \tilde{G} \to K(g)$. Thus $K(f) \circ \tilde{G} = K(g)$ in $\pi_0(\mathbf{Hom}(K(\mathsf{k}(\mathbb{S})), K(\mathbb{B})))$. Now apply the functor k to get the equality $f \circ \mathsf{k}(\tilde{G}) = g$, showing that $\mathsf{k}(\mathbb{S})$ is projective in $\mathbf{\Gamma AB}$. Hence $\mathsf{k}(\mathbb{S})$ is isomorphic to $l(P)$ for a free abelian group P. Thus $\mathbb{S}$ and $K(l(P))$ are equivalent.

iv) Let $\mathbb{S}$ be a symmetric categorical group. Choose a free abelian group P and an epimorphism of abelian groups $f_0 : P \to \pi_0(\mathbb{S})$. By part i) of Lemma 3, f_0 has a unique extension to a morphism $f = (f_0, f_1) : l(P) \to \mathsf{k}(\mathbb{S})$ which is essentially surjective. Since P is a free abelian group, we have the isomorphism (4), which shows that there exists a morphism of symmetric categorical groups $K(l(P)) \to \mathbb{S}$ which realizes f_0 on the level of π_0. Clearly this morphism does the job. □

5. Injective objects

In this section we prove the following result.

Theorem 4. *The groupoid enriched category $\mathfrak{SCG}$ has enough injective objects.*

This is just part iv) of Lemma 5 proved below.

A morphism $f = (f_0, f_1)$ in $\mathbf{\Gamma AB}$ is *faithful* provided f_1 is injective and an object $\mathbb{I} = (I_0, I_1, \iota)$ of $\mathbf{\Gamma AB}$ is *injective* if for any faithful morphism $f : \mathbb{A} \to \mathbb{B}$ in $\mathbf{\Gamma AB}$ the induced map

$$\mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathbb{B}, \mathbb{I}) \to \mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathbb{A}, \mathbb{I})$$

is surjective. It is clear that a morphism $F : \mathbb{S}_1 \to \mathbb{S}_2$ of symmetric categorical groups is faithful iff $\mathsf{k}(F) : \mathsf{k}(\mathbb{S}_1) \to \mathsf{k}(\mathbb{S}_2)$ is faithful in $\mathbf{\Gamma AB}$. For an abelian group M we introduce two objects in $\mathbf{\Gamma AB}$:

$$r(M) = ({}_2M, M, id_{{}_2M}),$$

$$M[1] = (0, M, 0).$$

Lemma 5. *i) If M is an abelian group and $\mathbb{A} = (A_0, A_1, \alpha)$ is an object in $\mathbf{\Gamma AB}$, then one has the following functorial isomorphism of abelian groups*

$$\mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathbb{A}, r(M)) = \mathsf{Hom}(A_1, M).$$

ii) An object $\mathbb{D}$ is an injective object in $\mathbf{\Gamma AB}$ iff it is isomorphic to the object of the form $r(D)$, with divisible abelian group D.

iii) For any divisible abelian group D the symmetric categorical group $K(r(D))$ is injective. Conversely, if $\mathbb{S}$ is an injective categorical group, then $\pi_1(\mathbb{S})$ is a divisible abelian group and $\mathbb{S}$ is equivalent to $K(r(\pi_1(\mathbb{S})))$.

iv) The 2-category $\mathfrak{SCG}$ of symmetric categorical groups has enough injective objects.

Proof. i) Assume $g = (g_0, g_1) : \mathbb{A} \to r(M)$ is a morphism in $\mathbf{\Gamma AB}$. So $g_0 : A_0 \to {}_2M$ and $g_1 : A_1 \to M$ are homomorphisms of abelian groups and we have a commutative diagram

$$\begin{array}{ccccc} A_0 & \xrightarrow{\alpha} & {}_2A_1 & \xrightarrow{i} & A_1 \\ {\scriptstyle g_0}\downarrow & & {\scriptstyle \bar{g_1}}\downarrow & & \downarrow{\scriptstyle g_1} \\ {}_2M & \xrightarrow{\mathsf{Id}} & {}_2M & \xrightarrow{j} & M \end{array}$$

where $\bar{g_1}$ is induced by g_1 and i, j are inclusions. It follows that g_0 is completely determined by g_1 and the result follows.

ii) Let D be a divisible abelian group and let $\mathbb{A} \to \mathbb{B}$ be a faithful morphism in $\mathbf{\Gamma AB}$. Thus $A_1 \to B_1$ is a monomorphism. Since D is an injective object in $\mathfrak{Ab}$ it follows that $\mathsf{Hom}(B_1, D) \to \mathsf{Hom}(A_1, D)$ is an epimorphism of abelian groups. So by i) the map $\mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathbb{B}, r(D)) \to \mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathbb{A}, r(D))$ is surjective. Thus $r(D)$ is injective in $\mathbf{\Gamma AB}$.

Conversely, assume $\mathbb{D} = (D_0, D_1, \chi)$ is injective in $\mathbf{\Gamma AB}$. We claim that D_1 is a divisible abelian group. In fact, it suffices to show that it is an injective object in the category $\mathfrak{Ab}$. Take any monomorphism of abelian groups $f_1 : A \to B$ and any homomorphism of abelian groups $g_1 : A_1 \to D_0$. We have to show that g_1 has a lift to B_1. Observe that $f = (0, f_1) : A[1] \to B[1]$ is faithful in $\mathbf{\Gamma AB}$ and $g = (0, g_1) : A[1] \to \mathbb{D}$ is a well-defined morphism in $\mathbf{\Gamma AB}$. By assumption there exists a morphism $\tilde{g} : B[1] \to \mathbb{D}$. It is clear that $\tilde{g} = (0, \tilde{g_1})$ for some $\tilde{g_1} : B \to D_1$. Thus D_1 is a divisible abelian group and $r(D_1)$ is an injective object in $\mathbf{\Gamma AB}$. By i) the identity map defines a canonical morphism $i = (i_0, \mathsf{Id}_{D_1}) : \mathbb{D} \to r(D_1)$, which is obviously faithful in $\mathbf{\Gamma AB}$. Since $\mathbb{D}$ is injective, it follows that there exists a morphism $q = (q_0, \mathsf{Id}_{D_1}) : r(D_1) \to \mathbb{D}$ such that $q \circ i = \mathsf{Id}_{\mathbb{Q}}$. Thus we have a commutative diagram

$$\begin{array}{ccc} D_0 & \xrightarrow{\chi} & {}_2D_1 \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle \mathsf{Id}} \\ {}_2D_1 & \xrightarrow{\mathsf{Id}} & {}_2D_1 \\ \downarrow{\scriptstyle q_0} & & \downarrow{\scriptstyle \mathsf{Id}} \\ D_0 & \xrightarrow{\chi} & {}_2D_1 \end{array}$$

with $q_0 i_0 = \mathsf{Id}_{D_0}$. It follows that i_0 is an isomorphism. Hence $i : \mathbb{D} \to r(D_1)$ is an isomorphism and we are done.

iii) Let $F : \mathbb{S}_1 \to \mathbb{S}_2$ be a faithful morphism of symmetric categorical groups and let $G : \mathbb{S}_1 \to K(r(D))$ be a morphism of symmetric categorical groups. Apply the functor k to get morphisms $\mathsf{k}(F) : \mathsf{k}(\mathbb{S}_1) \to \mathsf{k}(\mathbb{S}_2)$ and $\mathsf{k}(G) : \mathsf{k}(\mathbb{S}_1) \to r(D)$ in $\mathbf{\Gamma AB}$. Since $\pi_1(F) : \pi_1(\mathbb{S}_1) \to \pi_1(\mathbb{S}_2)$ is a monomorphism of abelian groups, it follows that $\mathsf{k}(F) : \mathsf{k}(\mathbb{S}_1) \to \mathsf{k}(\mathbb{S}_2)$ is a faithful morphism in $\mathbf{\Gamma AB}$. Since D is a divisible abelian group, $r(D)$ is injective in $\mathbf{\Gamma AB}$ by ii) and we can extend $\mathsf{k}(F)$ to get a morphism $\hat{g} : \mathsf{k}(\mathbb{S}_2) \to r(D)$. Thus we have the equality $\hat{g} \circ \mathsf{k}(F) = \mathsf{k}(G)$ in $\mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathsf{k}(\mathbb{S}_1), r(Q))$. Since $D = \pi_1(K(r(D)))$ is a divisible abelian group, the Ext-term in the exact sequence (1) disappears and we get the isomorphism

$$\pi_0(\mathbf{Hom}(\mathbb{S}_i, K(r(D)))) \cong \mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathsf{k}(\mathbb{S}_i), r(D)), \quad i = 0, 1. \tag{5}$$

Take a morphism $L : \mathbb{S}_2 \to K(r(D))$ of symmetric categorical groups which corresponds to the morphism $\hat{g} : \mathsf{k}(\mathbb{S}_2) \to r(D)$. By our construction, one has the equality $\mathsf{k}(LF) = \mathsf{k}(G)$. This is an equality in $\pi_0(\mathbf{Hom}(\mathbb{S}_1, K(r(D))))$, which implies that the classes of LF and of G in $\pi_0(\mathbf{Hom}(\mathbb{S}_1, K(r(D))))$ are the same. Thus there exists a track from LF to G. This shows that $K(r(D))$ is an injective symmetric categorical group. Conversely, assume $\mathbb{S}$ is an injective symmetric categorical group. Since $\mathbb{S}$ and $K(\mathsf{k}(\mathbb{S}))$ are equivalent, it follows that $K(\mathsf{k}(\mathbb{S}))$ is also projective. We claim that $\mathsf{k}(\mathbb{S})$ is injective in $\mathbf{\Gamma AB}$. In fact, take any faithful morphism $f = (f_0, f_1) : \mathbb{A} \to \mathbb{B}$ in $\mathbf{\Gamma AB}$ and any morphism $g : \mathbb{A} \to \mathsf{k}(\mathbb{S})$ in $\mathbf{\Gamma AB}$. Then $K(f) : K(\mathbb{A}) \to K(\mathbb{B})$ is faithful in $\mathfrak{SCG}$. Hence for $K(g) : K(\mathbb{A}) \to K(\mathsf{k}(\mathbb{S}))$ we have a morphism $\tilde{G} : K(\mathbb{B}) \to K(\mathsf{k}(\mathbb{S}))$ and a track $\tilde{G} \circ K(f) \to K(g)$. Thus $\tilde{G} \circ K(f) = K(g)$ in $\pi_0(\mathbf{Hom}(K(\mathsf{k}(\mathbb{S})), K(\mathbb{B})))$. Now apply the functor k to get the equality $\mathsf{k}(\tilde{G}) \circ f = g$, showing that $\mathsf{k}(\mathbb{S})$ is injective in $\mathbf{\Gamma AB}$. Hence $\mathsf{k}(\mathbb{S})$ is isomorphic to $r(D)$ for a divisible abelian group D. Thus $\mathbb{S}$ and $K(r(D))$ are equivalent.

iv) Let $\mathbb{S}$ be a symmetric categorical group. Choose a divisible abelian group D and a monomorphism of abelian groups $f_1 : \pi_1(\mathbb{S}) \to D$. By Lemma 5 f_1 has a unique extension to a morphism $f = (f_0, f_1) : \mathsf{k}(\mathbb{S}) \to r(D)$ which is essentially surjective. Since D is a divisible abelian group, we have the isomorphism (5), which shows that there exists a morphism of symmetric categorical groups $\mathbb{S} \to K(r(D))$ which realizes f_1 on the level of π_1 and we get the result. □

6. Duality

For an abelian group A we set $d(A) = \mathsf{Hom}(A, \mathbb{Q}/\mathbb{Z})$, where Hom is taken in the category of abelian groups. It is well known that d yields a duality of the category of finite abelain groups. We will prove that there is a similar duality of the 2-category of finite symmetric categorical groups. Here, a symmetric categorical group $\mathbb{S}$ is called *finite* provided $\pi_i(\mathbb{S})$ is finite, $i = 0, 1$. To describe the new duality we put $\mathbb{J} = K(r(\mathbb{Q}/\mathbb{Z}))$. By Lemma 5 we know that $\mathbb{J}$ is an injective object of the 2-category $\mathfrak{SCG}$. For a symmetric categorical group $\mathbb{S}$ we set

$$D(\mathbb{S}) = \mathbf{Hom}(\mathbb{S}, \mathbb{J}).$$

Lemma 6. *i) For any symmetric categorical group* $\mathbb{S}$ *one has*

$$\pi_0(D(\mathbb{S})) = d(\pi_1(\mathbb{S})), \quad \pi_1(D(\mathbb{S})) = d(\pi_0(\mathbb{S})) \quad \text{and} \quad k_{D(\mathbb{S})} = d(k_S).$$

ii) If $\mathbb{S}$ *is finite, then* $D(\mathbb{S})$ *is also finite.*

Proof. ii) follows immediately from i). Since $\pi_1(\mathbb{J}) = \mathbb{Q}/\mathbb{Z}$ it follows from the exact sequence (1) that $\pi_0(D(\mathbb{S})) = \mathsf{Hom}_{\mathbf{\Gamma AB}}(\mathsf{k}(\mathbb{S}), \mathsf{k}(\mathbb{J})) = d(\pi_1(\mathbb{S}))$. The last equality follows from part i) of Lemma 5. The isomorphism $\pi_1(D(\mathbb{S})) = d(\pi_0(\mathbb{S}))$ follows directly from the isomorphism (2). To prove the equation $\pi_1(D(\mathbb{S})) = d(\pi_0(\mathbb{S}))$, let us recall that $k_{\mathbb{S}}$ is a homomorphism $\pi_0(\mathbb{S})/2\pi_0(\mathbb{S}) \to \pi_1(\mathbb{S})$, hence $d(k_S)$ is the homomorphism

$$d(\pi_1(\mathbb{S})) \to d(\pi_0(\mathbb{S})/2\pi_0(\mathbb{S})) = {}_2d(\pi_0(\mathbb{S})).$$

The fact on k-invariants follows from the equality (3). □

Theorem 7. *The 2-functor*

$$D : \mathfrak{SCG}^{op} \to \mathfrak{SCG}$$

yields duality on finite symmetric categorical groups.

Proof. For any symmetric categorical group $\mathbb{S}$ the obvious evaluation map yields a morphism of symmetric categorical groups

$$\mathbb{S} \to D(D(\mathbb{S})).$$

We claim that this map is an equivalence, provided $\mathbb{S}$ is finite. For this it suffices to show that it induces an isomorphism on π_0 and π_1. But this directly follows from Lemma 6. In fact the map $\pi_0(\mathbb{S}) \to \pi_0(DD(\mathbb{S}))$ equals to $\pi_0(\mathbb{S}) \to d(\pi_1(D(\mathbb{S})))$, which is the same as $\pi_0(\mathbb{S}) \to d(d(\pi_0(S)))$, which is an isomorphism, by the classical duality of finite abelian groups. Same for π_1. □

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