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THE CLUSTER VALUE PROBLEM IN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We study the cluster value problem for certain Banach algebras of holomorphic functions defined on the unit ball of a complex Banach space X. The main results are for spaces of the form $X = C(K)$.

1. Preliminaries

A cluster value problem for a complex Banach space X is a weak version of the corona problem for the open unit ball B of X, which is a long-standing open problem in complex analysis when X has dimension at least 2. Instead of studying when B is dense in the spectrum of a uniform algebra H of bounded analytic functions on B in the weak topology induced by H(corona problem), the cluster value problem investigates the following situation:

Let $\bar{B}^{**}$ be the closed unit ball of the bidual X^{**}, and let M_H be the spectrum (i.e., maximal ideal space) of a uniform algebra H of norm continuous functions on B with $H \supset X^*$. Then $\pi : M_H \to \bar{B}^{**}$, given by $\pi(\tau) = \tau|_{X^*}$ for $\tau \in M_H$, is surjective (as a consequence of the results in Chapter 2 of [10]). For each $x^{**} \in \bar{B}^{**}$, $M_{x^{**}}(B) = \pi^{-1}(x^{**})$ is called the fiber of M_H over x^{**}. Aron, Carando, Gamelin, Lassalle and Maestre observed in [4] that for every $x^{**} \in \bar{B}^{**}$ we have the inclusion

$$Cl_B(f, x^{**}) \subset \widehat{f}(M_{x^{**}}(B)),\ \forall f \in H, \tag{1}$$

where $Cl_B(f, x^{**})$, *the cluster set of* f *at* x^{**}, stands for the set of all limits of values of f along nets in B converging weak-star to x^{**}, while $\widehat{f}$ represents the Gelfand transform of f. There they formulated the cluster value problem for H : for which Banach spaces X is there equality in (1) for all $x^{**} \in \bar{B}^{**}$? When there is equality in (1) for a certain $x^{**} \in \bar{B}^{**}$, we say X satisfies *the cluster value theorem for* H *at* x^{**}.

As was pointed out in [4], it is easy to check that the cluster value theorem for H at all points in $\bar{B}^{**}$ is indeed weaker than the corona problem for B and H : Given $x^{**} \in \bar{B}^{**}$, if $\tau \in M_{x^{**}}(B)$ were the weak-star limit of the net $(x_\alpha) \subset B$, then $\lim_\alpha f(x_\alpha) = \widehat{f}(\tau)$ for all $f \in H$, and in particular $\lim_\alpha x^*(x_\alpha) = \widehat{x^*}(\tau) = x^*(x^{**})$ for all $x^* \in X^*$, i.e., x^{**} would be the weak-star limit of (x_α), and so $\widehat{f}(\tau) \in Cl_B(f, x^{**})$.

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In an effort to research the corona problem, we investigate conditions that guarantee the simpler cluster value theorem for a Banach algebra of analytic functions defined on the unit ball of a complex Banach space X. In particular, we generalize some of the results in [4]. Our main results are for the spaces of the form $X = C(K)$, including a translation result of a cluster value problem for certain algebras H at any point in $B_{C(K)}$ to the origin.

2. Cluster value problem in finite-codimensional subspaces

In [4], the authors obtain a cluster value theorem at the origin for Banach spaces with shrinking 1-unconditional bases for the algebra $H = A_u(B)$ of bounded analytic functions on B that are also uniformly continuous. Slight modifications of their arguments in Section 3 of [4] yield the following:

Proposition 2.1. *Let S be a finite rank operator on X, so that $P = I - S$ has norm one. If $\phi \in M_0(B)$, then $\hat{f}(\phi) = \widehat{f \circ P}(\phi)$, for all $f \in A_u(B)$.*

Proposition 2.2. *Suppose that for each finite-dimensional subspace E of X^* and $\epsilon > 0$ there exists a finite rank operator S on X so that $||(I - S^*)|_E|| < \epsilon$ and $||I - S|| = 1$. Then the cluster value theorem holds for $A_u(B)$ at 0.*

Proof. Suppose that $0 \notin Cl_B(f, 0)$. We must show that $0 \notin \hat{f}(M_0)$. Since $0 \notin Cl_B(f, 0)$, there exist $\delta > 0$ and a weak neighborhood U of 0 in X such that $|f| \geq \delta$ on $U \cap B$. Without loss of generality we may assume $U = \bigcap_{i=1}^{n} \{x \in X : \ |x_i^*| < \epsilon_0\}$ for some $x_1^*, \cdots, x_n^* \in B_{X^*}$ and $\epsilon_0 > 0$. Let $E = \operatorname{span}\{x_1^*, \cdots, x_n^*\}$ and let S be as in the statement for $\epsilon = \epsilon_0$. Then $|f \circ (I - S)| \geq \delta$ on B, because for every $x \in B$ we have that $(I - S)x \in U$, indeed:

$$|\langle x_i^*, (I - S)x\rangle| = |\langle (I - S^*)x_i^*, x\rangle| < \epsilon_0, \text{ for } i = 1, \cdots, n.$$

Consequently $f \circ (I - S)$ is invertible in $A_u(B)$. Hence $\widehat{f \circ (I - S)} \neq 0$ on the fiber of the spectrum of $A_u(B)$ over 0. From the preceding lemma we then obtain $\hat{f} \neq 0$ on M_0, that is, $0 \notin \hat{f}(M_0)$. ☐

Since Proposition 2.2 builds on Proposition 2.1, one naturally wonders if Proposition 2.1 can be extended to the larger algebra $H^\infty(B)$ of all bounded analytic functions on B. The answer is no in general, as shown by the following example of Aron.

Example 2.3. There exists a finite rank operator S on ℓ_2 so that $P = I - S$ has norm one, and there exists $\phi \in M_0(B_{\ell_2})$ as well as $f \in H^\infty(B_{\ell_2})$ so that $\hat{f}(\phi) \neq \widehat{f \circ P}(\phi)$.

Proof. Let $S : \ell_2 \to \ell_2$ be given by $S(x) = (x_1, 0, 0, \cdots)$.

Clearly S is a finite rank operator and $P = I - S$ has norm one.

Let (r_j) and (s_j) be sequences of positive real numbers, such that $(r_j) \downarrow 0$ and $(s_j) \uparrow 1$ in such a way that each $r_j^2 + s_j^2 < 1$ and $r_j^2 + s_j^2 \to 1^-$. For each $j = 1, 2, 3, \cdots$, let $\delta_{r_j e_1 + s_j e_j}$ be the usual point evaluation homomorphism from $H^\infty(B_{\ell_2}) \to \mathbb{C}$. Let $\phi : H^\infty(B_{\ell_2}) \to \mathbb{C}$ be an accumulation point of $\{\delta_{r_j e_1 + s_j e_j}\}$ in the spectrum of $H^\infty(B_{\ell_2})$. Let $f : B_{\ell_2} \to \mathbb{C}$ be the H^∞ function given by

$$f(x) = \frac{x_1}{\sqrt{1 - \sum_{j=2}^{\infty} x_j^2}},$$

where the square root is taken with respect to the usual logarithm branch. Then $\phi(f) = 1$. However $\phi(f \circ P) = 0$ since $f \circ P \equiv 0$. □

When a Banach space has a shrinking reverse monotone finite-dimensional decomposition (FDD), that is, a shrinking FDD so that the natural projections are at distance one from the identity operator, we have that the condition in Proposition 2.2 holds, and therefore we obtain a cluster value theorem:

Corollary 2.4. *If X is a Banach space with a shrinking reverse monotone FDD, then the cluster value theorem holds for $A_u(B)$ at 0.*

The operators P considered in Propositions 2.1 and 2.2 have finite-codimensional rank, which suggests that the cluster value problem at the origin of a Banach space can be studied by considering the same problem in its finite-codimensional subspaces. We established the following relationship with the help of Aron and Maestre:

Proposition 2.5. *If Y is a closed finite-codimensional subspace of X and $f \in A_u(B)$, then $Cl_B(f, 0) = Cl_{B_Y}(f|_Y, 0)$, where B_Y is the unit ball of Y.*

Proof. $A_u(B)$ coincides with the uniform limits on $\bar{B}$ of continuous polynomials on X (see Theorem 7.13 in [11] and p. 56 in [3]), where polynomials are finite linear combinations of symmetric multi-linear mappings (of possibly distinct degrees) restricted to the diagonal. Thus, by passing to the uniform limit on $\bar{B}$, we may assume f is an m-homogeneous polynomial, with associated symmetric m-linear functional F. Let (x_α) be a weakly null net in B such that $f(x_\alpha) \to \lambda$.

Each x_α can be written uniquely as $y_\alpha + u_\alpha$, where $y_\alpha \in Y$ and u_α is from a fixed finite-dimensional complement of Y in X. Then,

$$\begin{aligned} & f(x_\alpha) \\ =& F(x_\alpha, \cdots, x_\alpha) \\ =& f(y_\alpha) + mF(y_\alpha, \cdots, y_\alpha, u_\alpha) + [m(m-1)/2]F(x_\alpha, \cdots, x_\alpha, u_\alpha, u_\alpha) + \cdots + f(u_\alpha). \end{aligned}$$

Now, since (x_α) is weakly null, the same holds for (y_α) and (u_α). However, since (u_α) belongs to a finite-dimensional space, it follows that $||u_\alpha|| \to 0$. Thus $F(y_\alpha, \cdots, y_\alpha, u_\alpha)$, $F(y_\alpha, \cdots, y_\alpha, u_\alpha, u_\alpha), \cdots, f(u_\alpha)$ all go to 0. Thus $f(y_\alpha) \to \lambda$. Finally, since each $||y_\alpha|| \leq ||x_\alpha|| + ||-u_\alpha|| < 1 + ||u_\alpha||$, then by defining $t_\alpha = \frac{1}{1+||u_\alpha||}$ we get that $||t_\alpha y_\alpha|| < 1$ for all α and $t_\alpha \to 1$, and consequently, $\lim f(t_\alpha y_\alpha) = \lim t_\alpha^m f(y_\alpha) = \lambda$. Hence $\lambda \in Cl_{B_Y}(f|_Y, 0)$. □

As a consequence we obtain that the cluster sets of an element f of $A_u(B)$ at 0 can be described in terms of the Gelfand transforms of $f|_{B_Y}$ as Y ranges over finite-codimensional subspaces of X:

Proposition 2.6. *For every Banach space X,*

$$Cl_B(f, 0) = \bigcap_{Y \subset X, \dim(X/Y) < \infty} \widehat{f|_{B_Y}}(M_0(B_Y)), \ \forall f \in A_u(B).$$

Proof. From Proposition 2.5 and the inclusion in (1), for every finite-codimensional subspace Y of X,

$$Cl_B(f, 0) = Cl_{B_Y}(f|_{B_Y}, 0) \subset \widehat{f|_{B_Y}}(M_0(B_Y)).$$

For the reverse inclusion, suppose $0 \notin Cl_B(f,0)$. Then there are $\epsilon > 0$ and a weak neighborhood U of 0 such that $|f| > \epsilon$ on $U \cap B$. U contains a closed finite-codimensional subspace Y_0 of X, so $|f|_{B_{Y_0}}| > \epsilon$. Hence $\widehat{f|_{B_{Y_0}}}$ is invertible, which implies that $0 \notin \widehat{f|_{B_{Y_0}}}(M_0(B_{Y_0}))$. □

Going back to Proposition 2.2, we see that having the cluster value property at 0 only requires the existence of a certain type of finite rank operators at distance one from the identity operator. However simple this condition may seem, it is impossible in the case of the Banach space c of continuous functions on ω, also seen as the subspace of l^∞ of convergent sequences:

Example 2.7. Let $L \in B_{c^*}$ be given by

$$L((c_n)_n) = \lim_{n\to\infty} c_n.$$

If $S : c \to c$ is a finite rank operator with $||(S^* - I_{c^*})L|| < \epsilon$, then $||S - I_c|| \geq 2 - \epsilon$.

Proof. For each $k \in \mathbb{N}$, consider $L_k \in B_{c^*}$ given by

$$L_k((c_n)_n) = (\lim_{n\to\infty} c_n - c_k)/2.$$

Let us show that $||S^*(L_k)|| \to 0$ as $k \to \infty$. For every $x \in B_c$, $S^*(L_k)x = L_k(Sx) \to 0$ as $k \to \infty$. Moreover, since S has finite rank, $\{Sx : x \in B_c\}$ is pre-compact. Thus $S^*L_k = L_k \circ S$ converges to zero uniformly on B_c, i.e., $||S^*L_k|| \to 0$ as $k \to \infty$.

Now note that $||L - 2L_k|| = 1$ for each k, so

$$||S^* - I_{c^*}|| \geq ||(S^* - I_{c^*})(L - 2L_k)|| \geq ||2L_k - 2\cdot S^*(L_k)|| - \epsilon \geq 2 - \epsilon - 2||S^*(L_k)||.$$

Since $S^*(L_k) \to 0$, then $||S - I_c|| = ||S^* - I_{c^*}|| \geq 2 - \epsilon$. □

The reader may check that the condition is also impossible for L_p, $1 \leq p \neq 2 < \infty$.

However, note that since c_0 is one-codimensional in c, Proposition 2.5 implies that for all $f \in A_u(B_c)$,

$$Cl_{B_c}(f,0) = Cl_{B_{c_0}}(f|_{B_{c_0}}, 0).$$

Also, Propositions 1.59 and 2.8 of [8] imply that all functions in $A_u(B_{c_0})$ can be uniformly approximated on B by polynomials in the functions in X^*, which in turn implies that each fiber at $x \in \bar{B}^{**}$ consists only of x, so the cluster value theorem for $A_u(B_{c_0})$ holds, and in particular

$$Cl_{B_{c_0}}(f|_{B_{c_0}}, 0) = \widehat{f|_{B_{c_0}}}(M_0(B_{c_0})), \ \ \forall f \in A_u(B_c).$$

Hence we are left to compare $\widehat{f|_{B_{c_0}}}(M_0(B_{c_0}))$ with $\widehat{f}(M_0(B_c))$ for $f \in A_u(B_c)$. Note that an inclusion is evident:

Proposition 2.8. *For a Banach space X and for Y a subspace of X,*

$$\widehat{f|_{B_Y}}(M_0(B_Y)) \subset \widehat{f}(M_0(B)), \ \ \forall f \in A_u(B).$$

Proof. Let $f \in A_u(B)$ and $\tau \in M_0(B_Y)$. Since $\phi_1 : A_u(B) \to A_u(B_Y)$ given by $\phi(g) = g|_Y$ for all $g \in A_u(B)$ is a continuous homomorphism that maps $A(B)$ into

$A(B_Y)$, the mapping $\tilde{\tau} : A_u(B) \to \mathbb{C}$ given by $\tilde{\tau}(g) = \tau(g|_Y)$ for all $g \in A_u(B)$ is in the fiber $M_0(B)$. Moreover,

$$\widehat{f|_Y}(\tau) = \hat{f}(\tilde{\tau}).$$

□

The reverse inclusion is unclear. However, the space c also has the property of being isomorphic to c_0, which implies, as we will see, that c has the cluster value property too.

Let $P(X)$ denote the continuous polynomials on X, $P_f(X)$ the polynomials in the functions of X^* (known as finite type polynomials), and $A(B_X)$ the uniform algebra of uniform limits of elements in $P_f(X)$.

Lemma 2.9. *Let X be a Banach space so that $A_u(B_X) = A(B_X)$. If the Banach space Y is isomorphic to X, then also $A_u(B_Y) = A(B_Y)$.*

Proof. Let $T : Y \to X$ be the Banach space isomorphism between Y and X.

Let $f \in A_u(B_Y)$. Then there exists a sequence of polynomials $P_n \in \mathcal{P}(Y)$ such that $||P_n - f||_{B_Y} \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $P_n \circ T^{-1} \in \mathcal{P}(X)$, so there exists a polynomial $Q_n \in \mathcal{P}_f(X)$ such that $||P_n \circ T^{-1} - Q_n||_{B_X} < \frac{1}{n \cdot ||T||}$, and consequently $||P_n - Q_n \circ T||_{B_Y} < \frac{1}{n}$, where $Q_n \circ T \in \mathcal{P}_f(Y)$.

Consequently, the sequence of polynomials $Q_n \circ T \in \mathcal{P}_f(Y)$ converges to f uniformly on B_Y, so $f \in A(B_Y)$. □

Corollary 2.10. *The Banach space c satisfies the cluster value theorem for $A_u(B_c)$ at all points in $\overline{B_c}^{**}$.*

3. Cluster value problem in $C(K) \ncong c$

Bessaga and Pełczyński proved in [6] that, when $\alpha \geq \omega^\omega$ is a countable ordinal, $C(\alpha)$ is not isomorphic to $c = C(\omega)$. Therefore we no longer can use Lemma 2.9 to obtain a cluster value theorem on such spaces of continuous functions.

Nevertheless, for α a countable ordinal, the intervals $[1, \alpha]$ are always compact, Hausdorff and dispersed (they contain no perfect nonvoid subset). The compact, Hausdorff and dispersed sets K satisfy, from the Main Theorem in [12], that $X = C(K)$ contains no isomorphic copy of l_1. Moreover, from Theorem 5.4.5 in [1], $X = C(K)$ has the Dunford-Pettis property. Therefore, for dispersed K, the continuous polynomials on $X = C(K)$ are weakly (uniformly) continuous on bounded sets by Corollary 2.37 in [8].

Moreover, since $X^* = l_1(K)$ has the approximation property, Proposition 2.8 in [8] and the conclusion in the former paragraph now yield that all continuous polynomials on X can be uniformly approximated, on bounded sets, by polynomials of finite type. Thus the elements of $A_u(B)$ can be approximated, uniformly on B, by polynomials of finite type. Hence $A_u(B) = A(B)$, so each fiber at $x \in \bar{B}^{**}$ is the singleton $\{x\}$, and then X satisfies the cluster value theorem for the algebra $A_u(B)$.

We now consider the cluster value problem on X for the algebra of all bounded analytic functions $H^\infty(B)$. Following the line of proof of Theorem 5.1 in [4], we still get a cluster value theorem.

Theorem 3.1. *If X is the Banach space $C(K)$, for K compact, Hausdorff and dispersed, then the cluster value theorem holds for $H^\infty(B)$ at every $x \in \bar{B}^{**}$.*

Proof. Fix $f \in H^\infty(B)$ and $w = (w_t)_{t\in K} \in \bar{B}^{**}$ (where $C(K)^{**} = l_\infty(K)$). Suppose $0 \notin Cl_B(f, w)$. It suffices to show that $0 \notin \hat{f}(M_w)$.

Since 0 is not a cluster value of f at w, there exists a weak-star neighborhood U of w such that $0 \notin \overline{f(U \cap B)}$, where

$$U \cap B \supset \bigcap_{i=1}^{n} \{z \in B : |\langle (z - w), x_i^* \rangle| < \epsilon\},$$

for some $\epsilon > 0$ and $x_1^*, \cdots, x_n^* \in X^* = l_1(K)$.

We have that $x_i^* = (x_i^*(t))_{t\in K}$ has countably many nonzero coordinates $\{x_i^*(t)\}_{t\in F_i}$ for $i = 1, \cdots, n$. Thus,

$$U \cap B \supset \bigcap_{i=1}^{n} \{z \in B : |\sum_{t\in K} (z_t - w_t) x_i^*(t)| < \epsilon\},$$

and there is a finite set $F \subset \bigcup_{i=1}^n F_i$ so that $\sum_{t\notin F} |x_i^*(t)| < \epsilon/4$, for $i = 1, \cdots, n$. Then,

$$U \cap B \supset \bigcap_{t\in F} \{z \in B : |z_t - w_t| < \delta\},$$

where

$$\delta = \min_{1\le i\le n, t\in F} \frac{\epsilon}{(2|F|)|x_i^*(t)|}.$$

In summary, there exist $c > 0$, $\delta > 0$ and a finite set $F \subset K$ such that if $z \in B$ satisfies $|z_t - w_t| < \delta$ for $t \in F$, then $|f(z)| \ge c$. Relabel the indices in F as $t_1, \cdots, t_m$, where $m = |F|$. Then proceed as in the proof of Theorem 5.1 in [4]:

For $0 \le k \le m-1$, define $U_k = \{z \in B : |z_{t_j} - w_{t_j}| < \delta, k+1 \le j \le m\}$, and set $U_m = B$. Note that $1/f$ is bounded and analytic on U_0.

We claim that for each k, $1 \le k \le m$, there are functions g_k and $h_{k,j}$, $1 \le j \le k$, in $H^\infty(U_k)$ that satisfy

$$f(z)g_k(z) = 1 + (z_{t_1} - w_{t_1})h_{k1}(z) + \cdots + (z_{t_k} - w_{t_k})h_{kk}(z), \quad z \in U_k. \tag{2}$$

Once this claim is established, the proof is easily completed as follows. The functions g_m and h_{mj} belong to $H^\infty(B)$ and satisfy

$$\widehat{f}\widehat{g_m} = \widehat{1} + \sum_{j=1}^{m} (\widehat{z_{t_j} - w_{t_j}})\widehat{h_{mj}}.$$

Since each $\widehat{z_{t_j}} - w_{t_j}$ vanishes on M_w (by the definition of M_w), we obtain $\widehat{f}\widehat{g_m} = 1$ on M_w, and consequently $\widehat{f}$ does not vanish on M_w, as required.

Just as in [4], the claim is established by induction on k. The first step, the construction of g_1 and h_{11}, is as follows. We regard $1/f((z_t)_{t\in K})$ as a bounded analytic function of z_{t_1} for $|z_{t_1}| < 1$ and $|z_{t_1} - w_{t_1}| < \delta$, with z_t, $t \in K - \{t_1\}$, as analytic parameters in the range $|z_t| < 1$ for $t \in K - \{t_1\}$, and $|z_{t_j} - w_{t_j}| < \delta$ for $2 \le j \le m$. According to Lemma 5.3 in [4], we can express

$$\frac{1}{f(z)} = g_1(z) + (z_{t_1} - w_{t_1})h(z), \quad z \in U_0,$$

where $g_1 \in H^\infty(U_1)$. We set

$$h_{11}(z) = [f(z)g_1(z) - 1]/(z_{t_1} - w_{t_1}), \quad z \in U_1,$$

so that (2) is valid for $k = 1$. Note that $h_{11} = -hf$ on U_0. Consequently h_{11} is bounded and analytic on U_0. The defining formula then shows that h_{11} is analytic on all of U_1, and since $|z_{t_1} - w_{t_1}| \geq \delta$ on $U_1 - U_0$, h_{11} is bounded on U_1.

Now suppose that $2 \leq k \leq m$, and that there are functions g_{k-1} and $h_{k-1,j}$ $(1 \leq j \leq k-1)$ that satisfy (2) and are appropriately analytic. We apply Lemma 5.3 in [4] to these as functions of z_{t_k}, with the other variables regarded as analytic parameters, to obtain decompositions

$$g_{k-1}(z) = g_k(z) + (z_{t_k} - w_{t_k})G_k(z)$$

and

$$h_{k-1,j}(z) = h_{k,j}(z) + (z_{t_k} - w_{t_k})H_{k,j}(z), \quad 1 \leq j \leq m-1,$$

where g_k and the h_{kj}'s are in $H^\infty(U_{k-1})$, and G_k and the H_{kj}'s are in $H^\infty(U_{k-1})$. From the identity (2), with k replaced with $k-1$, we obtain

$$fg_k = 1 + \sum_{j=1}^{k-1}(z_{t_j} - w_{t_j})h_{kj} + (z_{t_k} - w_{t_k})[-fG_k + \sum_{j=1}^{k-1}(z_{t_j} - w_{t_j})H_{kj}]$$

on U_{k-1}. We define

$$h_{kk} = [fg_k - 1 - \sum_{j=1}^{k-1}(z_{t_j} - w_{t_j})h_{kj}]/(z_{t_k} - w_{t_k}), \quad z \in U_k.$$

Then (2) is valid. On U_{k-1} we have

$$h_{kk} = -fG_k + \sum_{j=1}^{k-1}(z_{t_j} - w_{t_j})H_{kj},$$

so that h_{kk} is bounded and analytic on U_{k-1}. Since $|z_{t_k} - w_{t_k}| \geq \delta$ on $U_k - U_{k-1}$, we see from the defining formula that $h_{kk} \in H^\infty(U_k)$. This establishes the induction step, and the proof is complete. $\square$

We do not know the answer to the cluster value problem for other spaces $C(K)$.

Consider the following cluster value problem: Given $f_0^{**} \in \overline{B}^{**}$, the cluster value problem for $H^\infty(B)$ over $A_u(B)$ at f_0^{**} asks whether for all $\psi \in H^\infty(B)$ and $\tau \in \mathcal{M}_{f_0^{**}}(B)$ ($\mathcal{M}_{f_0^{**}}(B) = \pi^{-1}(\delta_{f_0^{**}})$ for the restriction map $\pi : M_{H^\infty(B)} \to M_{A_u(B)}$), can we find a net $(f_\alpha) \subset B$ such that $\psi(f_\alpha) \to \tau(\psi)$ and f_α converges to f_0^{**} in the polynomial-star topology, i.e. the smallest topology that makes every extension of a polynomial on X to X^{**} continuous (that we denote by $\tau(\psi) \in \mathsf{Cl}_B(\psi, f_0^{**})$)? As before, clearly $\mathsf{Cl}_B(\psi, f_0^{**}) \subset \widehat{\psi}(\mathcal{M}_{f_0^{**}}(B))$, $\forall \psi \in H^\infty(B)$.

The cluster value problem for $H^\infty(B)$ over $A_u(B)$ coincides with the cluster value problem for $H^\infty(B)$ when $A_u(B) = A(B)$. Thus when K is compact, Hausdorff and dispersed, we have a positive answer to the previous cluster value problem for such $C(K)$ spaces.

The previous problem seems to be highly nontrivial. Since for every infinite compact Hausdorff space K, $C(K)$ contains a subspace Y isometric to c_0 (Proposition 4.3.11 in [1]), the fiber $\mathcal{M}_0(B_{C(K)})$ is huge (and from Lemma 3.3, also each fiber $\mathcal{M}_{f_0}(B_{C(K)})$ for $f_0 \in B_{C(K)}$). Indeed, according to Theorem 6.6 in [7], there is a family of distinct characters $\{\tau_\alpha\}_{\alpha \in B_{\ell_\infty}}$, such that each $\tau_\alpha : H^\infty(B_Y) \to \mathbb{C}$ satisfies

$\delta_0 = \tau_\alpha|_{A(B_Y)} = \tau_\alpha|_{A_u(B_Y)}$ (because Y is isometric to c_0, so $A(B_Y) = A_u(B_Y)$). Hence $\{\tau_\alpha\}_{\alpha\in B_{\ell_\infty}} \subset \mathcal{M}_0(B_Y)$ and therefore $\{\tau_\alpha \circ R\}_{\alpha\in B_{\ell_\infty}} \subset \mathcal{M}_0(B_{C(K)})$, where R is the restriction mapping $R : H^\infty(B_{C(K)}) \to H^\infty(B_Y)$, which is clearly a homomorphism. Note that the characters $\{\tau_\alpha \circ R\}_{\alpha\in B_{\ell_\infty}}$ are all distinct due to Theorem 1.1 in [2], because ℓ_∞ is an isometrically injective space (Proposition 2.5.2 in [1]), so there exists a norm-one linear map $S : C(K) \to \ell_\infty$ such that $S|_{c_0} = I_{c_0}$.

We prove in Corollary 3.4 that if the latter cluster value problem has an affirmative answer at some point of $B_{C(K)}$, then it has an affirmative answer at all points of $B_{C(K)}$. For that let us first establish the following lemmas, the first of which is a folklore result mentioned, e.g., in [14] and [5], but since there seems to be no proof in the literature we will sketch the proof.

Lemma 3.2. *Let $f_0 \in B = B_{C(K)}$. Then $T : B \to B$ given by*

$$T(f) = \frac{f - f_0}{1 - \overline{f_0}\cdot f} \quad \forall f \in B$$

is biholomorphic.

Proof. Set $\delta_0 = ||f_0||$.

Let us start by showing that T is well defined, i.e., $||Tf|| < 1$ when $||f|| < 1$.

Let $f \in B$. We can find $\delta \in (\delta_0, 1)$ such that $||f|| \leq \delta$.

For every $t_0 \in K$, let $z = f(t_0)$ and $c = f_0(t_0)$, so that $T(f)(t_0) = \frac{z-c}{1-\overline{c}z}$.

Let Δ denote the open unit disk in the complex plane $\mathbb{C}$.

Since $\sigma : (\delta\cdot\overline{\Delta}) \times (\delta_0\cdot\overline{\Delta}) \to \Delta$ given by $\sigma(z, c) = \frac{z-c}{1-\overline{c}z}$ is continuous, then $\sigma((\delta\cdot\overline{\Delta}) \times (\delta_0\cdot\overline{\Delta}))$ is a compact subset of Δ, so there exists $\delta_1 < 1$ so that $\sigma((\delta\cdot\overline{\Delta}) \times (\delta_0\cdot\overline{\Delta})) \subset \delta_1\overline{\Delta}$.

Thus $||Tf|| \leq \delta_1 < 1$.

Let us now show that T is also holomorphic, or equivalently, $\mathbb{C}$-differentiable. For $f \in B$ fixed, the linear mapping $L : C(K) \to C(K)$ given by $L(h) = \frac{1-|f_0|^2}{(1-\overline{f_0}f)^2}h$ satisfies that, for $h \neq 0$ small enough,

$$\begin{aligned}\frac{T(f+h) - T(f) - L(h)}{||h||} &= \Big(\frac{f+h-f_0}{1-\overline{f_0}(f+h)} - \frac{f-f_0}{1-\overline{f_0}f} - \frac{1-|f_0|^2}{(1-\overline{f_0}f)^2}h\Big)/||h|| \\ &= \Big(\frac{1-|f_0|^2}{1-\overline{f_0}f}\cdot\frac{h}{1-\overline{f_0}(f+h)} - \frac{1-|f_0|^2}{(1-\overline{f_0}f)^2}h\Big)/||h|| \\ &= \frac{\overline{f_0}h}{(1-\overline{f_0})^2(1-\overline{f_0}(f+h))}(1-|f_0|^2)h/||h||,\end{aligned}$$

which goes to zero as $h \to 0$. Thus T is holomorphic.

Since T clearly has a necessarily holomorphic inverse ($S(f) = \frac{f+f_0}{1+\overline{f_0}\cdot f}$), we have that T is a biholomorphic function on B that sends f_0 to the function identically zero. $\square$

Lemma 3.3. *The biholomorphic function T from the previous lemma induces a mapping $\hat{T}$ on the spectrum $M_{H(B)}$, where H denotes either the algebra A_u or the algebra H^∞, that maps $M_{f_0}(B)$ onto $M_0(B)$.*

Proof. Note that T is a Lipschitz function. Indeed, if $f, g \in B$, then

$$||T(f) - T(g)|| = ||\frac{(1-|f_0|^2)(f-g)}{(1-\overline{f_0}f)(1-\overline{f_0}g)}|| \leq \frac{1}{(1-||f_0||)^2}||f-g||.$$

Thus for every $\psi \in H(B)$, $\psi \circ T \in H(B)$. So $\hat{T} : M_{H(B)} \to M_{H(B)}$, given by

$$\hat{T}(\tau)(\psi) = \tau(\psi \circ T), \ \ \forall \tau \in M_{H(B)}, \ \psi \in H(B),$$

is well defined. Moreover, given $\tau \in M_{f_0}(B)$ and $\psi \in A_u(B)$,

$$\hat{T}(\tau)(\psi) = \tau(\psi \circ T) = (\psi \circ T)(f_0) = \psi(0),$$

i.e., $\hat{T}(\tau) \in M_0(B)$, for every $\tau \in M_{f_0}(B)$.

Now, given $\tau \in M_0(B)$ it is clear that $\hat{\tau} : H(B) \to \mathbb{C}$ given by

$$\hat{\tau}(\psi) = \tau(\psi \circ T^{-1}), \ \ \forall \psi \in H(B),$$

is in $M_{H(B)}$, actually in $M_{f_0}(B)$, and $\forall\ \psi \in H(B)$,

$$\hat{T}(\hat{\tau})(\psi) = \hat{\tau}(\psi \circ T) = \tau(\psi),$$

i.e., $\hat{T}(\hat{\tau}) = \tau$. $\square$

The reader can easily check that the previous mapping $\hat{T}$ is actually a homeomorphism.

Corollary 3.4. *For $X = C(K)$, the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at 0 is equivalent to the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at every $f_0 \in B$.*

Proof. Let $f_0 \in B$ and set T as in Lemma 3.2. Then, $\forall\ \psi \in H^\infty(B)$,

$$\begin{aligned} \hat{\psi}(\mathcal{M}_0(B)) &= \hat{\psi} \circ \hat{T}(\mathcal{M}_{f_0}(B)) = \widehat{\psi \circ T}(\mathcal{M}_{f_0}(B)), \\ \mathsf{Cl}_B(\psi, 0) &= \mathsf{Cl}_B(\psi \circ T, f_0), \end{aligned}$$

because $\psi \circ T \in H^\infty(B)$ too, and $T^{-1}(f) = (f + f_0) \sum_{n=0}^{\infty} (-\overline{f_0} f)^n \ \ \forall f \in B_{C(K)}$ is polynomially-star continuous, because sums and norm limits of polynomially-star continuous maps are polynomially-star continuous, as well as multiplication by a fixed element of $C(K)$ (see p. 312 in [9]). $\square$

We now argue that we can extend the previous conclusions to the open unit ball of the second dual of $C(K)$:

In the statement of Lemma 3.2, we can rewrite $\frac{f - f_0}{1 - \overline{f_0} \cdot f}$ as $(f - f_0) \sum_{n=0}^{\infty} (\overline{f_0} f)^n$. Since it is known that $C(K)^{**}$ is a commutative C^*-algebra that extends the C^* structure of $C(K)$ (see pp. 310-311 in [9] and p. 43 in [13]), then Lemma 3.2 extends in the following manner.

Lemma 3.5. *Given $f_0^{**} \in B_{C(K)^{**}}$, let $T_{f_0^{**}} : B_{C(K)^{**}} \to B_{C(K)^{**}}$ be given by*

$$T_{f_0^{**}}(f^{**}) = (f^{**} - f_0^{**}) \sum_{n=0}^{\infty} (\overline{f_0^{**}} f^{**})^n \quad \forall f^{**} \in B_{C(K)^{**}}.$$

Then $T_{f_0^{**}}$ is biholomorphic.

Similarly, we clearly obtain the following analogues of Lemma 3.3 and Corollary 3.4.

Lemma 3.6. *For each $f_0^{**} \in B_{C(K)^{**}}$, the biholomorphic function $T_{f_0^{**}}$ from the previous lemma induces a mapping $\widehat{T_{f_0^{**}}}$ on the spectrum $M_{H(B^{**})}$, where H denotes either the algebra A_u or the algebra H^∞, that maps $\mathcal{M}_{f_0^{**}}(B^{**})$ onto $\mathcal{M}_0(B^{**})$.*

Corollary 3.7. *For $X = C(K)$, the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at* 0 *is equivalent to the cluster value theorem of $H^\infty(B)$ over $A_u(B)$ at every $f_0^{**} \in B_{C(K)^{**}}$.*

Note that this last result is actually a consequence of Corollary 3.4 since the double dual of a space of continuous functions is again a space of continuous functions.

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