PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 4, April 2015, Pages 1501–1514
S 0002-9939(2014)12219-2
Article electronically published on December 9, 2014

LOCAL L^1 ESTIMATES FOR ELLIPTIC SYSTEMS OF COMPLEX VECTOR FIELDS

J. HOUNIE AND T. PICON

(Communicated by James E. Colliander)

ABSTRACT. In this paper we present a strong local version of the Gagliardo-Nirenberg estimate that holds for elliptic systems of vector fields with smooth complex coefficients. We also consider L^1 estimates on forms analogous to those known in the case of the de Rham complex on $\mathbb{R}^N$.

1. INTRODUCTION

In a series of works ([BB1], [BB2], [BB3]) Bourgain and Brezis undertook the extension of classical L^p a priori estimates, $p > 1$, related to certain classes of elliptic linear equations and systems to the limiting case $p = 1$. These equations, involving the gradient, the divergence, the rotational or, more generally, the de Rham complex chain operators on $\mathbb{R}^N$, are closely associated to a system of N linearly independent real constant vector fields, namely, the partial derivatives ∂_{x_j}, $1 \leq j \leq N$. One of their results is a version of the Gagliardo-Nirenberg inequality for the critical exponent which was shown for the de Rham complex:

Theorem 1.1. *Assume $N \geq 4$ and $0 \leq k \leq N$ for $k \neq 1, N-1$. Then for every $u \in W^{1,1}(\Lambda^k \mathbb{R}^N)$ we have*

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|du\|_{L^1} + \|d^*u\|_{L^1}\right), \tag{1.1}$$

where d is the exterior differential and d^* its dual operator.

This estimate cannot be proved by taking the limit in the standard estimate obtained by combining the Calderón-Zygmund theory with Sobolev's inequalities

$$\|u\|_{L^{p^*}} \leq C_p\left(\|du\|_{L^p} + \|d^*u\|_{L^p}\right), \quad u \in C_c^\infty(\Lambda^k \mathbb{R}^N), \tag{1.2}$$

for $1 < p < N$ and $p^* = pN/(N-p)$, since the constant C_p obtained by this method blows up as $p \searrow 1$. The a priori estimate (1.1) was also proved by Lanzani and Stein [LS] using an elementary lemma of Van Schaftingen's [VS1]. They also showed that (1.1) still holds for $k = 1$ and $k = N-1$ if one replaces the L^1 norm with the Hardy H^1 norm in the appropriate term on the right hand side of (1.1).

Estimates of this type have been extended in several directions and have been applied to problems in different fields: within the framework of the div-curl estimates ([AH], [BOS], [M2], [MP]), improved Strichartz estimates for systems of inhomogeneous wave equations and Schrödinger equations [CP], and in the context of estimates for measures and divergence-free vector fields ([BS1], [S]). Estimates

Received by the editors August 11, 2012.
2010 *Mathematics Subject Classification.* Primary 35J46, 46E35; Secondary 35F05, 35N10.
This work was supported in part by CNPq and FAPESP.

in the setup of constant vector fields ([BS2], [M1], [MM], [O], [VS1], [VS2], [VS3], [VS4]), nilpotent groups ([CVS]) and CR complexes ([Y]) also have been considered recently.

Notice that if u is a 0-form, i.e., a function, (1.1) may be written as $\|u\|_{L^{N/(N-1)}} \leq C\|\nabla u\|_{L^1}$, so we may regard (1.1) as a generalization of the Sobolev-Gagliardo-Nirenberg inequality

$$\|u\|_{L^{N/(N-1)}} \leq C \sum_{j=1}^{n} \|L_j u\|_{L^1}, \quad u \in C_c^\infty(\mathbb{R}^N), \tag{1.3}$$

where $L_j = \partial_{x_j}$ and $n = N$. For elliptic systems of complex vector fields with smooth coefficients defined on a domain $\Omega \subset \mathbb{R}^N$ the following local version of the Gagliardo-Nirenberg theorem was proved in [HP]:

Theorem A. *If the system of complex vector fields with smooth coefficients $L_1, \dots, L_n$, $n \geq 2$, is elliptic, then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$*

$$\|u\|_{L^{N/(N-1)}} \leq C \sum_{j=1}^{n} \|L_j u\|_{L^1}, \quad u \in C_c^\infty(U), \tag{1.4}$$

holds. Conversely, if (1.4) *holds, the system must be elliptic on U.*

A standard duality consequence of estimate (1.4) is the following local solvability result.

Corollary 1.1. *Let $L_1, \dots, L_n$, $n \geq 2$, as before. Given $x_0 \in \Omega$, there exists a neighborhood U of x_0 such that for every $f \in L^N(U)$, the underdetermined equation*

$$A\vec{u} \doteq L_1 u_1 + \cdots + L_n u_n = f \tag{1.5}$$

can be solved in U with $\vec{u} = (u_1, \dots, u_n) \in L^\infty(U)^N$.

Note that the assumption $n \geq 2$ in Theorem A cannot be weakened (it fails for the Cauchy-Riemann vector field on $U \subset \mathbb{R}^2$). When $L_j = \partial_{x_j}$ and $n = N$ then equation (1.5) may be interpreted as

$$d^* u = f,$$

where $u \in \Lambda^1(\mathbb{R}^N)$ and $d^* u$ denotes its co-exterior derivative. However, sharper solvability results were obtained by Bourgain and Brezis [BB3] for the de Rham complex, namely

Theorem 1.2 (Theorem 5 in [BB3])**.** *If $N \geq 2$ and $1 \leq k \leq N-1$, then*

$$d^*[W^{1,N}(\Lambda^k)] = d^*[(W^{1,N} \cap L^\infty)(\Lambda^k)].$$

More precisely , given $X \in W^{1,N}(\Lambda^k \mathbb{R}^N)$ there exist some $Y \in (W^{1,N} \cap L^\infty)(\Lambda^k \mathbb{R}^N)$ such that

$$d^* Y = d^* X \quad \textit{and} \quad \|Y\|_\infty + \|\nabla Y\|_N \leq c\|d^* X\|_N.$$

Here $W^{1,N}(\Lambda^k)$ denotes the space of k-forms on $\mathbb{R}^N$ with coefficients in the Sobolev space $W^{1,N}(\mathbb{R}^N)$, defined as the closure $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\nabla u\|_{L^N}$. The conclusion of the theorem does not hold for $k = N$. One should also mention that if d^* is replaced by the exterior derivative d, results analogous to those of Theorem 1.2 hold. It is also an interesting feature that the correspondence $X \mapsto Y$ cannot be linear (Proposition 9 in [BB3]). An equivalent formulation of Theorem 1.2 may be expressed in terms of a priori estimates.

Theorem 1.3 (Corollary 17 in [BB3]). *Assume $N \geq 4$ and $2 \leq k \leq N-2$. Then for every $X \in W^{1,1}(\Lambda^k)$ we have*

$$\|X\|_{L^{\frac{N}{N-1}}} \leq C\left(\|dX\|_{L^1+W^{-1,\frac{N}{N-1}}} + \|d^*X\|_{L^1+W^{-1,\frac{N}{N-1}}}\right). \tag{1.6}$$

In particular,

$$\|X\|_{L^{\frac{N}{N-1}}} \leq C\left(\|dX\|_{L^1} + \|d^*X\|_{L^1}\right), \quad X \in W^{1,1}(\Lambda^k). \tag{1.7}$$

While (1.7) follows trivially from the difficult estimate (1.6), estimate (1.7) can also be proved directly by using an elementary lemma of Van Schaftingen's [VS1] (*cf.* [LS]).

A dual consequence of Theorem 1.2 (specifically, the case $k = N-1$) is the a priori estimate

$$\|u\|_{L^{\frac{N}{N-1}}} \leq c\|\nabla u\|_{L^1+W^{-1,\frac{N}{N-1}}}, \quad u \in C_c^\infty(\mathbb{R}^N), \tag{1.8}$$

where

$$\|f\|_{W^{-1,\frac{N}{N-1}}} \doteq \sup_{\|\nabla\phi\|_{L^N}\leq 1} \{|\langle f, \phi\rangle|\}$$

and

$$\|f\|_{L^1+W^{-1,\frac{N}{N-1}}} \doteq \inf_{f=g+h}\left\{\|g\|_{L^1} + \|h\|_{W^{-1,\frac{N}{N-1}}}\right\}.$$

Although each of the priori estimates

$$\|u\|_{L^{\frac{N}{N-1}}} \leq c\|\nabla u\|_{L^1} \tag{1.9}$$

and

$$\|u\|_{L^{\frac{N}{N-1}}} \leq c\|\nabla u\|_{W^{-1,\frac{N}{N-1}}} \tag{1.10}$$

has elementary proofs, it does not follow that (1.9) and (1.10) could be combined to prove (1.8). In this work we improve Theorem A as follows:

Theorem 1.4. *If the system of vector fields $L_1, \ldots, L_n$, $n \geq 2$, is elliptic, then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$, the estimate*

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\sum_{j=1}^{n} \|L_j u\|_{L^1+W^{-1,\frac{N}{N-1}}}, \quad u \in C_c^\infty(U), \tag{1.11}$$

holds. Equivalently, if the system of vector fields $L_1, \ldots, L_n$, $n \geq 2$, is elliptic, then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$, so for all $f \in L^N(U)$ there exist $\vec{u} \in (W^{1,N} \cap L^\infty(U))$ such that

$$A\vec{u} = L_1u_1 + \cdots + L_nu_n = f \tag{1.12}$$

and

$$\|\nabla u\|_N + \|u\|_\infty \leq C\|f\|_N.$$

Suppose that the system $\mathcal{L} = \{L_1, \ldots, L_n\}$ is involutive; i.e., each commutator $[L_j, L_k]$, $1 \leq j, k \leq n$, is a linear combination of $L_1, \ldots, L_n$. The subbundle $\mathbb{C}T(\Omega)$ spanned by $\mathcal{L}$, denoted by $(\Omega, \mathcal{L})$, is called an involutive structure. In particular, there is a natural complex of differential operators $d_{\mathcal{L}}$ (check Section 3.3 or [BCH, Ch. 7] for more details) associated to the structure $\mathcal{L}$ (which is precisely the de Rham complex when $n = N$ and $L_j = \partial_{x_j}$). For this complex, the following extension of the a priori estimate (1.7) was proved in [HP].

Theorem B. *Assume that the system of vector fields $L_1, \ldots, L_n$, $n \geq 2$, is elliptic and involutive and that $0 \leq k \leq n$ is neither 1 nor $n-1$. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$,*

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|d_{\mathcal{L},k}u\|_{L^1} + \|d^*_{\mathcal{L},k}u\|_{L^1}\right), \quad u \in \tilde{E}^k_c(U). \tag{1.13}$$

In Theorem 3.2 below we show that the estimate (1.13) can be sharpened to

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|d_{\mathcal{L},k}u\|_{L^1+W^{-1,\frac{N}{N-1}}} + \|d^*_{\mathcal{L},k}u\|_{L^1+W^{-1,\frac{N}{N-1}}}\right), \quad u \in \tilde{E}^k_c(U). \tag{1.14}$$

Note that when $k = 0$, estimate (1.13) reduces to estimate (1.4), which does not require involutivity. In the noninvolutive situation, we can still define a chain of operators $d_{\mathcal{L},k}$ analogous to those associated to an involutive structure. However, we do not get a complex in general, as the fundamental complex property $d_{\mathcal{L},k+1} \circ d_{\mathcal{L},k} = 0$ might not hold. On the other hand, this chain still satisfies a "pseudo-complex" property in the sense that $d_{\mathcal{L},k+1} \circ d_{\mathcal{L},k} = 0$ is a differential of operator of order one rather than two as expected in general, which means that the complex property is satisfied at the level of principal symbols. This is the motivation to study estimates analogous to (1.13) for the "pseudo-complex" $\{d_{\mathcal{L}}\}$ and the result is:

Theorem 1.5. *Assume that the system of vector fields $L_1, \ldots, L_n$, $n \geq 2$, is elliptic and that k is neither 1 nor $n-1$. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$,*

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|d_{\mathcal{L},k}u\|_{L^1} + \|d^*_{\mathcal{L},k}u\|_{L^1}\right), \quad u \in C^\infty_c(U, \Lambda^k\mathbb{R}^n). \tag{1.15}$$

The organization of the paper is as follows. The proof of Theorem 1.4 is given in Section 2. Section 3 is concerned with chain operators that satisfy the pseudo-complex property; after defining them we prove an approximate Hodge decomposition that is instrumental in the proof of Theorem 1.5, which is then presented. Finally, we deal briefly with the involutive situation for which sharper estimates hold (Theorem 3.2).

2. Proof of Theorem 1.4

Consider n complex vector fields $L_1, \ldots, L_n$, $n \geq 1$, with smooth coefficients defined on a neighborhood Ω of the origin $0 \in \mathbb{R}^N$, $N \geq 2$, that may be viewed as sections of the vector bundle $\mathbb{C}T(\Omega)$ as well as first order differential operators. We will always assume that

(a) $L_1, \ldots, L_n$ are everywhere linearly independent,
(b) the system $\{L_1, \ldots, L_n\}$ is *elliptic*.

This means that for any *real* 1-form ω (i.e., any section of $T^*(\Omega)$) we have that $\langle \omega, L_j \rangle = 0 \implies \omega = 0$. This implies that the number n of vector fields must satisfy

$$\frac{N}{2} \leq n \leq N.$$

Alternatively, (b) is equivalent to saying that the second order operator

$$\Delta_L \doteq L_1^*L_1 + \cdots + L_n^*L_n \tag{2.1}$$

is elliptic. Here, $L_j^* = \overline{L_j^t}$, $j = 1, \ldots, n$, where $\bar{L}_j$ denotes the vector field obtained from L_j by conjugating its coefficients and L_j^t is the formal transpose of L_j.

The vectors $L_1, \ldots, L_n$ span a subbundle of $\mathbb{C}T(\Omega)$ and the estimate (1.11) expresses a property of this bundle rather than a property of this specific set of generators. Hence, it is enough to prove (1.11) for a set of generators which have a simpler form. After modifying the vector fields outside a neighborhood of the origin, shrinking Ω, choosing appropriate generators and reordering the coordinates $x_1, \ldots, x_N$, we may assume without loss of generality that the vector fields $\{L_1, \ldots, L_n\}$ have the form

$$L_j = \frac{\partial}{\partial x_j} + \sum_{k=1}^{m} c_{jk}(x) \frac{\partial}{\partial x_{n+k}}, \quad j = 1, \ldots, n, \tag{a$'$}$$

with smooth coefficients globally defined on $\mathbb{R}^N$ that possess bounded derivatives of all orders and $m = N - n$. Similarly, the hypothesis (b) could have been replaced by the uniform ellipticity of Δ_L; i.e., for some constant $c > 0$ and all $x, \xi \in \mathbb{R}^N$,

$$\sum_{j=1}^{n} \left| \xi_j + \sum_{k=1}^{m} c_{jk} \xi_{n+k} \right|^2 \geq c|\xi|^2. \tag{b$'$}$$

Consider the operator A defined by (1.12). To prove Theorem 1.4 it is enough to solve locally the equation $A\vec{u} = f$, $f \in L^N$, with $\vec{u} \in W^{1,N} \cap L^\infty$. We will need the following weaker version of the result.

Lemma 2.1. *Every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$, so for all $f \in L^N(U)$ there exists $\vec{u} \in W^{1,N}(U)$ such that $A\vec{u} = f$. Thus,*

$$A : \bigoplus_{s=1}^{n} W^{1,N}(U) \longrightarrow L^N(U)$$

is surjective.

Proof. The result follows from a well known application of the Hahn-Banach theorem, once the a priori estimate

$$\|\phi\|_{L^{N/(N-1)}} \leq C \sum_{j=1}^{n} \|L_j \phi\|_{W^{-1,\frac{N}{N-1}}}, \quad \phi \in C_c^\infty(U), \tag{2.2}$$

has been established for a small ball U centered at x_0. This is a standard elliptic estimate, and we will omit its proof. $\square$

The fundamental ingredient in the proof of Theorem 1.4 is

Theorem 2.1 (Theorem 10 in [BB3])**.** *Let Q be a cube in $\mathbb{R}^N$ and let*

$$S : \bigoplus_{s=1}^{r} W^{1,N}(Q) \to Y$$

be a bounded operator into a Banach space Y with closed range. Assume further that for each $s \in \{1, \ldots, r\}$ there is an index $i_s \in \{1, \ldots, N\}$ such that

$$\left\| S\vec{f} \right\|_Y \leq c \max_s \max_{i \neq i_s} \|\partial_i f_s\|_N . \tag{2.3}$$

Then, for all $\vec{f} \in \bigoplus_{s=1}^{r} W^{1,N}(Q)$, *there is* $\vec{g} \in \bigoplus_{s=1}^{r}(W^{1,N} \cap L^{\infty})(Q)$ *satisfying*

$$S\vec{f} = S\vec{g} \quad \textit{and} \quad \|\nabla \vec{g}\|_N + \|\vec{g}\|_\infty \le c \left\| S\vec{f} \right\|_Y .$$

Thus, we need only show that the operator A satisfies the hypotheses of Theorem 2.1 for some small cube centered at x_0. If Q is contained in the neighborhood U granted by Lemma 2.1, it is clear that $A : \bigoplus_{s=1}^{r} W^{1,N}(Q) \to L^N(Q)$ is a surjective bounded operator, so it obviously has closed range. Recalling the special form (a$'$) of the vector fields L_j, $j = 1, \dots, n$, we have

$$\|A\vec{u}\|_{L^N} \le n^2 \max_j \left\{ 1, \max_k \|c_{jk}\|_{L^\infty} \right\} \max_{i \neq i_j} \|\partial_i u_j\|_{L^N}$$

for $i_j = j+1$, $j = 1, \dots, n-1$ and $i_n = 1$, so (2.3) holds with A in the place of S. Now Theorem 1.4 follows from a direct application of Theorem 2.1.

3. Complexes and pseudo-complexes

Let $L_1, \dots, L_n$ be a system of vector fields defined on open set $\Omega \subset \mathbb{R}^N$ satisfying the properties (a) and (b) of the previous section and let $C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$ denote the space of k-forms on $\mathbb{R}^n$, $0 \le k \le n$, with complex smooth coefficients defined on Ω. Each $f \in C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$ may be written as

$$f = \sum_{|I|=k} f_I dx_I, \quad dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $f_I \in C^\infty(\Omega)$ and $I = \{i_1, \dots, i_k\}$ is a set of strictly increasing indexes with $i_l \in \{1, \dots, n\}$, $l = 1, \dots, k$. Consider the differential operators

$$d_{\mathcal{L},k} : C^\infty(\Omega, \Lambda^k \mathbb{R}^n) \to C^\infty(\Omega, \Lambda^{k+1} \mathbb{R}^n)$$

given by

$$d_{\mathcal{L},0} f = \sum_{j=1}^{n} (L_j f) dx_j, \quad f \in C^\infty(\Omega),$$

and for $f = \sum_{|I|=k} f_I dx_I \in C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$, $1 \le k \le n-1$,

$$d_{\mathcal{L},k} f = \sum_{|I|=k} (d_{\mathcal{L},0} f_I) dx_I = \sum_{|I|=k} \sum_{j=1}^{n} (L_j f_I) dx_j \wedge dx_I. \tag{3.1}$$

Note that, in general, the complex property $d_{\mathcal{L},k+1} d_{\mathcal{L},k} = 0$ does not hold. We will refer to $(d_{\mathcal{L},k}, C^\infty(\Omega, \Lambda^k \mathbb{R}^n))$ as the pseudo-complex $\{d_{\mathcal{L}}\}$ associated with $\mathcal{L}$ on Ω.

We also define the dual pseudo-complex $d^*_{\mathcal{L},k} : C^\infty(\Omega, \Lambda^{k+1} \mathbb{R}^n) \to C^\infty(\Omega, \Lambda^k \mathbb{R}^n)$, $0 \le k \le n-1$, determined by

$$\int d_{\mathcal{L},k} u \cdot \overline{v}\, dx = \int u \cdot \overline{d^*_{\mathcal{L},k} v}\, dx, \quad u \in C^\infty_c(\Omega, \Lambda^k \mathbb{R}^n),\ v \in C^\infty_c(\Omega, \Lambda^{k+1} \mathbb{R}^n),$$

where the dot indicates the standard pairing on forms of the same degree. Thus, if $f = \sum_{|J|=k} f_J dx_J$, then $d^*_{\mathcal{L},k} f = \sum_{|J|=k} \sum_{j \in J} L^*_j f_J dx_j \vee dx_J$, where, for each $j_l \in J = \{j_1, \dots, j_k\}$ and $l \in \{1, \dots, k\}$, $dx_{j_l} \vee dx_J$ is given by

$$dx_{j_l} \vee dx_J = (-1)^{l+1} dx_1 \wedge \cdots \wedge dx_{j_{l-1}} \wedge dx_{j_{l+1}} \wedge \cdots \wedge dx_{j_k}.$$

3.1. **An approximate Hodge decomposition.** Consider the operator

$$\Delta_k : C^\infty(\Omega, \Lambda^k\mathbb{R}^n) \to C^\infty(\Omega, \Lambda^k\mathbb{R}^n)$$

given by

$$\Delta_k \doteq d_{\mathcal{L},k-1}d^*_{\mathcal{L},k-1} + d^*_{\mathcal{L},k}d_{\mathcal{L},k}, \quad 0 \le k \le n.$$

When $k = 0$ and $k = n$ the operators $d_{\mathcal{L},n}, d^*_{\mathcal{L},-1}$ must be understood as zero. If $f \in C^\infty(\Omega, \Lambda^k\mathbb{R}^n)$, then we write

$$\Delta_k f = \sum_{|J|=k} (\Delta_k f)_J dx_J.$$

It is convenient to express the principal part of Δ_k in terms of the elliptic second order operator Δ_L defined in (2.1).

Lemma 3.1. *Let* $u = \sum_{|J|=q} u_J dx_J$. *Then*

$$(\Delta_k u)_J = \Delta_L u_J + \sum_{j\in J}\sum_{k\in J} [L_j, L_k^*]\, u_{\{k\}\cup J\setminus\{j\}}.$$

Proof. Let $u = \sum_{|I|=q} u_I dx_I$. Computing $\Delta_q(u_I dx_I)$ we have

$$d^*_{\mathcal{L}}d_{\mathcal{L}}(u_I dx_I) = \sum_{j\notin I}\sum_{k\in I}(L_k^* L_j u_I) dx_k \vee dx_j \wedge dx_I + \sum_{j\notin I}(L_j^* L_j u_I) dx_I,$$

$$d_{\mathcal{L}}d^*_{\mathcal{L}}(u_I dx_I) = \sum_{s\in I}\sum_{t\notin I}(L_t L_s^* u_I) dx_t \wedge dx_s \vee dx_I + \sum_{s\in I}(L_s L_s^* u_I) dx_I.$$

Fix $|J| = q$ and let us write the coefficient of dx_J in $\Delta_q u$. We must compute the contributions of indexes which, considered as sets, satisfy $I \neq J$ and are such that

$$dx_k \vee dx_j \wedge dx_I = \pm dx_J,$$
$$dx_t \wedge dx_s \vee dx_I = \pm dx_J.$$

For example, a generic I in the first equation is given by $I = \{k\} \cup J \setminus \{j\}$ for $j \in J$ and $k \notin J$. Thus, the contribution of

$$\sum_{j\notin I}\sum_{k\in I}(L_k^* L_j u_I) dx_k \vee dx_j \wedge dx_I$$

to the coefficient of dx_J is given by the function $-L_k^* L_j u_{\{k\}\cup J\setminus\{j\}}$. In short, summing up all the contributions to the coefficient of dx_J we obtain

$$-\sum_{k\notin J}\sum_{j\in J} L_k^* L_j\ u_{\{k\}\cup J\setminus\{j\}} + \sum_{s\notin J}\sum_{t\in J} L_t L_s^*\ u_{\{s\}\cup J\setminus\{t\}} + \sum_{j\notin J}(L_j^* L_j u_J) + \sum_{s\in J}(L_s L_s^* u_J),$$

which may be written as

$$\Delta_L u_J + \sum_{j\in J}\sum_{k\in J} [L_j, L_k^*]\, u_{\{k\}\cup J\setminus\{j\}}. \qquad \square$$

Hence, we may write $\Delta_k f = \Delta_L f + \Gamma f$ with $\Delta_L f \doteq \sum_{|J|=k}(\Delta_L f_J) dx_J$, $\Gamma f = \sum_{|J|=k} \Gamma_J f dx_J$ and

$$\Gamma_J f = \sum_{j\in J}\sum_{k\in J} [L_j, L_k^*]\, f_{\{k\}\cup J\setminus\{j\}}.$$

We will make use of the theory of pseudo-differential operators with symbols in Hörmander's classes $S^m = S^m_{1,0}(\mathbb{R}^N)$ (since we will only work with symbols of type (1,0), the type will be omitted in the notation; on the subject of pseudo-differential operators we refer, for instance, to [H]). There exist scalar pseudo-differential operators $q(x,D)$ and $r(x,D)$ with symbols $q(x,\xi)\in S^{-2}$ and $r(x,\xi)\in S^{-\infty}$ such that

$$\phi = q(x,D)\Delta_L\phi + r(x,D)\phi, \quad \phi\in C^\infty(\Omega).$$

Thus, writing

$$q(x,D)f = q(x,D)\sum_{|J|=k} f_J dx_J \doteq \sum_{|J|=k} q(x,D)f_J dx_J$$

and $r(x,D)f \doteq \sum_{|J|=k} r(x,D)f_J dx_J$, we obtain

$$q(x,D)\Delta_k f = q(x,D)\Delta_L f + q(x,D)\Gamma f = f - r(x,D)f + q(x,D)\Gamma f$$

or

$$f = q(x,D)\Delta_k f + r(x,D)f - q(x,D)\Gamma f.$$

On the other hand,

$$\begin{aligned} q(x,D)\Delta_k f =& d_{\mathcal{L},k-1}q(x,D)d^*_{\mathcal{L},k-1}f + d^*_{\mathcal{L},k}q(x,D)d_{\mathcal{L},k}f \\ &+ [q(x,D),d_{\mathcal{L},k-1}]d^*_{\mathcal{L},k-1}f + [q(x,D),d^*_{\mathcal{L},k}]d_{\mathcal{L},k}f. \end{aligned}$$

Defining $f^a \doteq q(x,D)d_{\mathcal{L},k}f$, $f^b \doteq q(x,D)d^*_{\mathcal{L},k-1}f$ and

$$Q = [q(x,D),d_{\mathcal{L},k-1}]d^*_{\mathcal{L},k-1} + [q(x,D),d^*_{\mathcal{L},k}]d_{\mathcal{L},k} - q(x,D)\Gamma + r(x,D),$$

we obtain an approximate Hodge decomposition

$$f = d^*_{\mathcal{L},k}f^a + d_{\mathcal{L},k}f^b + Q(x,D)f, \qquad f\in L^N(\Omega,\Lambda^k\mathbb{R}^n), \tag{3.2}$$

with $f^a\in W^{1,N}(\Omega,\Lambda^{k+1}\mathbb{R}^n)$, $f^b\in W^{1,N}(\Omega,\Lambda^{k-1}\mathbb{R}^n)$ and $Q(x,D)$ of order -1. Note that the projections $L^N\ni f\mapsto f^a\in W^{1,N}$ and $L^N\ni f\mapsto f^b\in W^{1,N}$ are bounded.

A second ingredient in the proof of Theorem 1.5 is a generalization of Lemma 2.1 in [HP] that we state below (the proof is similar and will be omitted). Assume that X_t, $1\le t\le\gamma$, are vector fields defined on $\mathbb{R}^N$ with complex smooth coefficients that have bounded derivatives of all orders and involve derivatives with respect to x_k, only for $k>n$, where $x_1,\dots,x_N$ is the system of coordinates chosen in Section 2, in which (a′) holds, i.e.,

$$X_t = \sum_{k=n+1}^{N}\alpha_{tk}(x)\frac{\partial}{\partial x_k}, \quad t=1,\dots,\gamma. \tag{c}$$

Lemma 3.2. *Let $L_1,\dots,L_n$, $n\ge 2$, be linearly independent vector fields with complex smooth coefficients defined on $\mathbb{R}^N$ satisfying* (a′) *and vector fields X_j, $1\le j\le\gamma$, as above. Consider test functions $f_1,\dots,f_\nu,u_1,\dots,u_\tau\in C_c^\infty(\mathbb{R}^N)$ and suppose that for any $1\le k\le\nu$ there exists $1\le j\le n$ such that*

$$L_j f_k = \sum_{j'\ne j}\sum_{l_1=1}^{\nu} c_{kj'l}L_{j'}f_{l_1} + \sum_{t=1}^{\gamma}\sum_{l_2=1}^{\tau} X_t\, u_{l_2}$$

where $c_{kj'\ell}(x)$ are bounded smooth functions with bounded derivatives. Then, there exist $C>0$ such that for any $\phi\in C_c^\infty(\mathbb{R}^N)$,

$$\left|\int f_k(x)\phi(x)\,dx\right| \le C\left(\sum_{l_1=1}^{\nu}\|f_{l_1}\|_{L^1} + \sum_{l_2=1}^{\tau}\|u_{l_2}\|_{L^1}\right)\|\phi\|_{W^{1,N}}. \tag{3.3}$$

3.2. **Proof of Theorem 1.5.** Let $L_1, \dots, L_n$, $n \geq 2$, be linearly independent vector fields with complex smooth coefficients defined on $\mathbb{R}^N$. Assume that the system $\{L_1, \dots, L_n\}$ is elliptic. We can assume without loss of generality, as we did in the proof of Theorem 1.4, that the vector fields L_j are of the form (a′). Note that the vector fields $[L_j, L_k]$ satisfy condition (c). Let $2 \leq k \leq n-2$ and $x_0 = 0$. To prove (1.15) it is enough to find a neighborhood $U \subset \Omega$ of the origin and $C = C(U) > 0$ such that

$$|\langle u, \phi\rangle| \leq C\left(\|d_{\mathcal{L},k}u\|_{L^1} + \|d^*_{\mathcal{L},k}u\|_{L^1}\right)\|\phi\|_{L^N}, \quad \forall\, \phi \in C_c^\infty(U, \Lambda^k\mathbb{R}^n).$$

We will take U as a ball $B(0,\rho)$ centered at the origin with small radius $0 < \rho < 1/2$ to be chosen later. We apply (3.2) to ϕ, to wit,

$$\phi = d^*_{\mathcal{L},k}\phi^a + d_{\mathcal{L},k}\phi^b + Q(x,D)\phi.$$

Clearly,

$$\langle u, \phi\rangle = \left\langle d^*_{\mathcal{L},k}u, \phi^b\right\rangle + \langle d_{\mathcal{L},k}u, \phi^a\rangle + \langle u, Q\phi\rangle\,.$$

Initially we estimate the term $\langle d_{\mathcal{L},k}u, \phi^a\rangle$. If $d_{\mathcal{L},k}u = f$, then

$$\langle f, \phi^a\rangle = \sum_{|I|=k+1}\langle f_I, \phi^a_I\rangle$$

for $\phi^a = \sum_{|I|=k+1}\phi^a_I$. As a consequence of the identity

$$\begin{aligned} d_{\mathcal{L},k+1}f \quad &= \sum_{|H|=k}\sum_{i=1}^n\sum_{j=1}^n (L_iL_jf_H)dx_i\wedge dx_j\wedge dx_H \\ &= \sum_{|J|=k+2}\sum_{i,j\in J}^n(-1)^{\sigma(J,i,j)}[L_i,L_j]f_{J\setminus\{i,j\}}dx_J \end{aligned}$$

it follows that for each $|J| = k+2$ the expression

$$\sum_{j\in J}(-1)^{\sigma(J,j)}L_jf_{J\setminus\{j\}} = \sum_{\substack{s_1,s_2\in J\\ s_1<s_2}}(-1)^{\sigma(J,s_1,s_2)}[L_{s_1},L_{s_2}]u_{J\setminus\{s_1,s_2\}}$$

holds where we have denoted by $\sigma(J,j)$ the sign of the permutation $J\setminus j \longrightarrow \{j_1,\dots,j_{i-1},j_{i+1},\dots,j_{k+2}\}$ for $J = \{j_1,\dots,j_{k+2}\}$ and $j = j_i$. Fix $i_0 \notin I$ and consider $J_{i_0} \doteq I\cup\{i_0\}$. Thus,

$$\begin{aligned} L_{i_0}f_I = (-1)^{\sigma(J_{i_0},i_0)}&\sum_{j\in I}(-1)^{\sigma(J_{i_0},j)+1}L_jf_{J_{i_0}\setminus\{j\}} \\ &+ \sum_{\substack{s_1,s_2\in J\\ s_1<s_2}}(-1)^{\sigma(J_{i_0},s_1,s_2)}[L_{s_1},L_{s_2}]u_{J_{i_0}\setminus\{s_1,s_2\}}. \end{aligned}$$

We may apply Lemma 3.2 to get

$$\begin{aligned} |\langle f_I, \phi^a_I\rangle| &\leq c_1\left(\|f\|_{L^1} + \|u\|_{L^1}\right)\|\nabla\phi^a_I\|_{L^N} \\ &\leq c_2\left(\|d_{\mathcal{L},k}u\|_{L^1} + \|u\|_{L^1}\right)\|\phi\|_{L^N}, \end{aligned}$$

and so

$$|\langle d_{\mathcal{L},k}u, \phi^a\rangle| \leq c_3(\|d_{\mathcal{L},k}u\|_{L^1} + \|u\|_{L^1})\|\phi\|_{L^N}. \tag{3.4}$$

Similar arguments give

$$\left|\left\langle d^*_{\mathcal{L},k}u, \phi^b\right\rangle\right| \leq c_4\left(\|d^*_{\mathcal{L},k}u\|_{L^1} + \|u\|_{L^1}\right)\|\phi\|_{L^N}. \tag{3.5}$$

To estimate the term $\langle u, Q\phi\rangle$, fix $1<\alpha<N$ and $1<\beta<\frac{N}{N-1}$ such that $\frac{1}{\alpha}+\frac{1}{\beta}=1+\frac{1}{N}$. Thus,

$$\begin{aligned}\langle u, Q\phi\rangle &\leq \|u\|_{L^\beta}\|Q\phi\|_{L^{\beta'}} \quad \text{(Holder's inequality)}\\ &\leq \|u\|_{L^\beta}\|\nabla Q\phi\|_{L^\alpha} \quad (\beta^{'}=\alpha^*\text{and Gagliardo-Nirenberg theorem})\\ &\leq \|u\|_{L^\beta}\|\phi\|_{L^\alpha},\end{aligned}$$

where the last inequality follows from the fact that the pseudo-differential operator ∇Q has order zero and therefore is bounded in L^p, $1<p<\infty$. The local estimates

$$\begin{aligned}\|u\|_{L^\beta} &\leq c_5(\rho)\|u\|_{L^{\frac{N}{N-1}}},\\ \|\phi\|_{L^\alpha} &\leq c_6(\rho)\|\phi\|_{L^N}\end{aligned}$$

then give

$$|\langle u, Q\phi\rangle| \leq c_7(\rho)\|u\|_{L^{\frac{N}{N-1}}}\|\phi\|_{L^N}. \tag{3.6}$$

Combining the estimates (3.4), (3.5) and (3.6) we obtain

$$|\langle u,\phi\rangle| \leq C\left(\|d_{\mathcal{L},k}u\|_{L^1}+\|d^*_{\mathcal{L},k}u\|_{L^1}+\|u\|_{L^1}+\|u\|_{L^{\frac{N}{N-1}}}\right)\|\phi\|_{L^N},$$

for all $\phi\in C^\infty_c(U,\Lambda^k\mathbb{R}^n)$, which implies

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|d_{\mathcal{L},k}u\|_{L^1}+\|d^*_{\mathcal{L},k}u\|_{L^1}+\|u\|_{L^1}+\|u\|_{L^{\frac{N}{N-1}}}\right).$$

Choosing ρ small enough, the term $\|u\|_{L^1}+\|u\|_{L^{\frac{N}{N-1}}}$ can be absorbed, and one gets

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|d_{\mathcal{L},k}u\|_{L^1}+\|d^*_{\mathcal{L},k}u\|_{L^1}\right) \quad \forall\, u\in C^\infty(U,\Lambda^k\mathbb{R}^n).$$

For the special values $l=1,n-1$ we have estimates involving the norm of the localizable Hardy space $h^1(\mathbb{R}^N)$ of Goldberg [G].

Theorem 3.1. *Assume the same hypothesis from the last theorem. Then*

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|d_{\mathcal{L},1}u\|_{L^1}+\|d^*_{\mathcal{L},1}u\|_{h^1}\right), \quad u\in C^\infty_c(U,\Lambda^1\mathbb{R}^n),$$

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C\left(\|d_{\mathcal{L},n-1}u\|_{h^1}+\|d^*_{\mathcal{L},n-1}u\|_{L^1}\right), \quad u\in C^\infty_c(U,\Lambda^{n-1}\mathbb{R}^n).$$

Theorem 3.1 can be proved by adapting the arguments in [HP, Theorem 4.2], which do not require involutivity.

3.3. **The involutive case.** Let $L_1,\dots,L_n$ be a set of smooth complex vector fields defined on a neighborhood $\Omega\subset\mathbb{R}^N$ of the origin satisfying (a) and (b). Assume that the system $\{L_1,\dots,L_n\}$ is involutive, i.e.,

(d) $[L_j,L_k]=\sum_{l=1}^{n}c_{jkl}L_l, 1\leq j,k\leq n$ for some complex-valued functions $c_{jkl}\in C^\infty(\Omega)$,

so $L_1,\dots,L_n$ are generators of an involutive subbundle $\mathcal{L}$ of $\mathbb{C}T(\Omega)$. Denote by $E^k(\Omega)$ the space of k-forms, $0\leq k\leq N$, with complex coefficients, i.e., the smooth sections of the vector bundle $\bigwedge^k\mathbb{C}T^*(\Omega)$, and let $\mathcal{L}^\perp(\Omega)$ be the subbundle of $\mathbb{C}T^*(\Omega)$ of all $\omega\in E^1(\Omega)$ such that $\langle\omega,L\rangle=0$ for all sections of $\mathcal{L}$. Denote by $\mathcal{I}$ the ideal generated by $\mathcal{L}^\perp(\Omega)$ in $\bigotimes_{k=0}^N\bigwedge^k\mathbb{C}T^*(\Omega)$. Hence, if $\omega_1,\dots,\omega_m$, $n+m=N$, is a set of local generators of $\mathcal{L}^{\perp}(\Omega)$, then $\mathcal{I}^k(\Omega) \doteq \mathcal{I} \cap \bigwedge^k \mathbb{C}T^*(\Omega)$, $1 \leq k \leq n$, is spanned by the k-forms

$$\omega_j \wedge \omega', \quad j = 1 \ldots, m, \quad \omega' \in E^{k-1}(\Omega).$$

Write $\mathfrak{N}^k(\Omega) = \bigwedge^k \mathbb{C}T^*(\Omega)/\mathcal{I}^k(\Omega)$, $0 \leq k \leq n$, and denote by $\tilde{E}^k(\Omega)$ the space of smooth sections of the vector bundle $\mathfrak{N}^k(\Omega)$. The de Rham complex ([dR]) $d : E^k(\Omega) \to E^{k+1}(\Omega)$, given by the exterior derivative on complex-valued forms, gives rise to a new complex $d_{\mathcal{L}}$ associated to the structure $\mathcal{L}$,

$$d_{\mathcal{L},k} : \tilde{E}^k(\Omega) \longrightarrow \tilde{E}^{k+1}(\Omega), \quad 0 \leq k \leq n,$$

defined as follows: if $u \in E^k(\Omega)$, then $d_{\mathcal{L},k}(u + \mathcal{I}^k) \doteq d_k u + \mathcal{I}^{k+1}$. This is well defined because the involutivity of $\mathcal{L}$ implies that $d_k \mathcal{I}^k \subset \mathcal{I}^{k+1}$. In particular, we have the basic complex property $d_{\mathcal{L},k+1} d_{\mathcal{L},k} = 0$ (on the subject of the differential complex associated to an involutive (or formally integrable) structure; we refer to [BCH, Ch. 7] and [Tr]).

Let $p \in \Omega$ and consider an open neighborhood $\mathcal{W}$ of p and $\omega_1, \ldots, \omega_m$, $n + m = N$, and a set of local generators of $\mathcal{L}^{\perp}(\mathcal{W})$ at every point. After contracting $\mathcal{W}$ around p and a linear change on $\omega_1, \ldots, \omega_m$ we can obtain a coordinate system $(x_1, \ldots, x_m, t_1, \ldots, t_n)$ defined on $\mathcal{W}$ and centered at p in such a way that

$$\omega_k = dx_k - \sum_{j=1}^{n} b_{jk}(x,t) \frac{\partial}{\partial x_k}, \quad k = 1, \ldots, m,$$

with $b_{jk} \in C^\infty(\mathcal{W})$. Thus, we define the linearly independent vector fields over $\mathcal{W}$,

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} b_{jk}(x,t) \frac{\partial}{\partial x_k},$$

which satisfy $\omega_k(L_j) = 0$ for all $j = 1, \ldots, n$ and $k = 1, \ldots, m$. Note that $L_1, \ldots, L_n$ span $\mathcal{L}$ on $\mathcal{W}$ at each point. Thus, if $f \in \tilde{E}^0(\mathcal{W}) = C^\infty(\mathcal{W})$ it is plain that

$$d_{\mathcal{L},0} f = \sum_{j=1}^{n} L_j f \; dt_j,$$

and, more generally, for $f = \sum_{|J|=k} f_J dt_J \in \tilde{E}^k(\mathcal{W})$ we have

$$d_{\mathcal{L},k} f = \sum_{|J|=k} \sum_{j=1}^{n} (L_j f_J) dt_j \wedge dt_J, \quad k = 0, \ldots, n-1.$$

In local coordinates, the latter formulas have precisely the form of the operators $d_{\mathcal{L},k}$ associated to a pseudo-complex induced by a system of vector fields $L_1, \ldots, L_n$ (with the additional advantage that in this situation the complex property $d_{\mathcal{L},k+1} d_{\mathcal{L},k} = 0$ holds).

We may now state a sharp version of Theorem B.

Theorem 3.2. *Assume that the system of vector fields $L_1, \ldots, L_n$, $n \geq 2$, is elliptic and involutive and that $0 \leq k \leq n$ is neither 1 nor $n-1$. Then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$ such that for some $C > 0$*

$$\|u\|_{L^{\frac{N}{N-1}}} \leq C \left(\|d_{\mathcal{L},k} u\|_{L^1 + W^{-1,\frac{N}{N-1}}} + \|d^*_{\mathcal{L},k} u\|_{L^1 + W^{-1,\frac{N}{N-1}}} \right), \quad u \in \tilde{E}^k_c(U), \tag{3.7}$$

where $\tilde{E}^k_c(U)$ denotes the compactly supported elements of $\tilde{E}^k(U)$.

An equivalent formulation of Theorem 3.2 is

Theorem 3.3. *If the system of vector fields $L_1, \dots, L_n$, $n \geq 2$, is elliptic and involutive, then every point $x_0 \in \Omega$ is contained in an open neighborhood $U \subset \Omega$, so for all $X \in W^{1,N}\Lambda^k(U)$, $1 \leq k \leq n-1$, there exist $Y \in (W^{1,N} \cap L^\infty)\Lambda^k(U)$, $C > 0$, such that*

$$d_{\mathcal{L}}X = d_{\mathcal{L}}Y \quad \textit{and} \quad \|\nabla Y\|_N + \|Y\|_\infty \leq C\|d_{\mathcal{L}}X\|_N.$$

The same result holds with $d^*_{\mathcal{L}}$ in the place of $d_{\mathcal{L}}$.

Theorem 3.3 $\Rightarrow$ Theorem 3.2. Let $2 \leq k \leq n-2$ and $x_0 = 0$. To estimate the right hand side of (3.7) it is sufficient to show, as in the proof of Theorem 3.1, a neighborhood at the origin $U \doteq B(0,\rho) \subset \Omega$ and $C = C(U) > 0$ such that, for any $\phi \in \tilde{E}^k_c(U)$,

$$|\langle u, \phi\rangle| \leq C\left(\|d_{\mathcal{L},k}u\|_{L^1+W^{-1,\frac{N}{N-1}}} + \|d^*_{\mathcal{L},k}u\|_{L^1+W^{-1,\frac{N}{N-1}}} + \|u\|_{L^1}\right)\|\phi\|_{L^N}. \tag{3.8}$$

Consider the decomposition (3.2) for the involutive and elliptic complex $d_{\mathcal{L}}$,

$$\langle u, \phi\rangle = \left\langle u, d_{\mathcal{L},k}\phi^b\right\rangle + \left\langle u, d^*_{\mathcal{L},k}\phi^a\right\rangle + \langle u, Q\phi\rangle.$$

By Theorem 3.3, shrinking ρ if necessary, there is $\psi \in (W^{1,N} \cap L^\infty)\Lambda^k(U)$, $U = B(0,\rho)$, such that $d_{\mathcal{L},k}\phi^b = d_{\mathcal{L},k}\psi$ and $\|\nabla\psi\|_N + \|\psi\|_\infty \leq c_1\|d_{\mathcal{L},k}\phi^b\|_N$. Thus,

$$\begin{aligned} \left|\left\langle u, d_{\mathcal{L},k}\phi^b\right\rangle\right| &= |\langle u, d_{\mathcal{L},k}\psi\rangle| \\ &\leq \|d^*_{\mathcal{L},k}u\|_{L^1+W^{-1,\frac{N}{N-1}}}\left(\|\nabla\psi\|_N + \|\psi\|_\infty\right) \\ &\leq \tilde{c}_1\|d^*_{\mathcal{L},k}u\|_{L^1+W^{-1,\frac{N}{N-1}}}\ \|\phi\|_{L^N}, \end{aligned} \tag{3.9}$$

where in the last inequality we used the fact that the pseudo-differential operator $d_{\mathcal{L},k}\phi^b = d_{\mathcal{L},k}\phi^b q(x,D)d^*_{\mathcal{L},k}\phi$ of order zero is bounded in $L^N(\mathbb{R}^N)$. Similarly, if $\rho > 0$ is small enough we have

$$\left|\left\langle u, d^*_{\mathcal{L},k}\phi^a\right\rangle\right| \lesssim c_2\|d^*_{\mathcal{L},k}u\|_{L^1+W^{-1,\frac{N}{N-1}}}\ \|\phi\|_{L^N}.$$

Finally, we also have the estimate $|\langle u, Q\phi\rangle| \leq c\|u\|_{L^1}\|\phi\|_{L^N}$ for some $c = c(\rho) > 0$, proving that (3.8) holds for small ρ. Since $\|u\|_{L^1} \leq C\rho^{\frac{1}{N}}\|u\|_{L^{\frac{N}{N-1}}}$, the term $C\|u\|_{L^1}$ may be absorbed for small ρ, and this proves (3.7). $\square$

The proof of Theorem 3.3 can be carried out by adapting the arguments that were used in Section 2 to prove Theorem 1.4. One needs to check that the hypotheses in Theorem 2.1 are satisfied, so the main point is to verify that the range of

$$S = d_{\mathcal{L},k} : W^{1,N}\Lambda^k(U) \to W^{1,N}\Lambda^{k+1}(U), \quad 0 < k \leq n-1,$$

is closed. However, S does have closed range by an extension of the inequality (2.2) to the dual complex $d^*_{\mathcal{L}}$ valid for $d^*_{\mathcal{L},k+1}$-closed forms. More precisely, for each $x_0 \in \Omega$ there exist an open neighborhood $U \subset \Omega$ and a constant $C = C(U) > 0$ such that if $u \in \tilde{E}^{k+1}_c(U)$ and $d_{\mathcal{L},k+1}u = 0$, then $\|u\|_{L^{\frac{N}{N-1}}} \leq C\|d^*_{\mathcal{L},k}u\|_{W^{-1,\frac{N}{N-1}}}$. This estimate may be used in a standard way to show that $S(W^{1,N}\Lambda^k(U))$ is equal to the kernel of $d_{\mathcal{L},k+1} : L^N\Lambda^{k+1}(U) \to \mathcal{D}'\Lambda^{k+2}(U)$ and is thus a closed subspace.

Note that in the case of a general pseudo-complex $S = d_{\mathcal{L},k}$, $0 \leq k \leq n$, we lack the information that any $v = d_{\mathcal{L},k}u$ satisfies $d_{\mathcal{L},k+1}v = 0$ and the previous argument breaks down.

Question. Is the sharp inequality (3.7) in Theorem 3.2 still valid without the involutivity hypothesis?

References

[AH] Chérif Amrouche and Huy Hoang Nguyen, *New estimates for the div, curl, grad operators and elliptic problems with L^1-data in the half-space*, Appl. Math. Lett. **24** (2011), no. 5, 697–702, DOI 10.1016/j.aml.2010.12.008. MR2765145 (2012h:35064)

[BB1] Jean Bourgain and Haïm Brezis, *On the equation* $\operatorname{div} Y = f$ *and application to control of phases*, J. Amer. Math. Soc. **16** (2003), no. 2, 393–426 (electronic), DOI 10.1090/S0894-0347-02-00411-3. MR1949165 (2004d:35032)

[BB2] Jean Bourgain and Haïm Brezis, *New estimates for the Laplacian, the div-curl, and related Hodge systems* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **338** (2004), no. 7, 539–543, DOI 10.1016/j.crma.2003.12.031. MR2057026 (2004m:26018)

[BB3] Jean Bourgain and Haïm Brezis, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 2, 277–315, DOI 10.4171/JEMS/80. MR2293957 (2009h:35062)

[BCH] Shiferaw Berhanu, Paulo D. Cordaro, and Jorge Hounie, *An introduction to involutive structures*, New Mathematical Monographs, vol. 6, Cambridge University Press, Cambridge, 2008. MR2397326 (2009b:32048)

[BOS] F. Bethuel, G. Orlandi, and D. Smets, *Approximations with vorticity bounds for the Ginzburg-Landau functional*, Commun. Contemp. Math. **6** (2004), no. 5, 803–832, DOI 10.1142/S0219199704001537. MR2100765 (2005g:35051)

[BS1] Haïm Brezis and Jean Van Schaftingen, *Circulation integrals and critical Sobolev spaces: problems of optimal constants*, Perspectives in partial differential equations, harmonic analysis and applications, Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 33–47, DOI 10.1090/pspum/079/2500488. MR2500488 (2010g:35008)

[BS2] Haïm Brezis and Jean Van Schaftingen, *Boundary estimates for elliptic systems with L^1-data*, Calc. Var. Partial Differential Equations **30** (2007), no. 3, 369–388, DOI 10.1007/s00526-007-0094-9. MR2332419 (2008m:35063)

[CP] Sagun Chanillo and Po-Lam Yung, *An improved Strichartz estimate for systems with divergence free data*, Comm. Partial Differential Equations **37** (2012), no. 2, 225–233, DOI 10.1080/03605302.2011.594475. MR2876830

[CVS] Sagun Chanillo and Jean Van Schaftingen, *Subelliptic Bourgain-Brezis estimates on groups*, Math. Res. Lett. **16** (2009), no. 3, 487–501, DOI 10.4310/MRL.2009.v16.n3.a9. MR2511628 (2010f:35042)

[dR] Georges de Rham, *Variétés différentiables. Formes, courants, formes harmoniques* (French), Actualités Sci. Ind., no. 1222 = Publ. Inst. Math. Univ. Nancago III, Hermann et Cie, Paris, 1955. MR0068889 (16,957b)

[G] David Goldberg, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42. MR523600 (80h:46052)

[H] Lars Hörmander, *The analysis of linear partial differential operators. III*, Pseudodifferential operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 274, Springer-Verlag, Berlin, 1985. MR781536 (87d:35002a)

[HP] J. Hounie and T. Picon, *Local Gagliardo-Nirenberg estimates for elliptic systems of vector fields*, Math. Res. Lett. **18** (2011), no. 4, 791–804, DOI 10.4310/MRL.2011.v18.n4.a16. MR2831843 (2012k:35108)

[LS] Loredana Lanzani and Elias M. Stein, *A note on div curl inequalities*, Math. Res. Lett. **12** (2005), no. 1, 57–61, DOI 10.4310/MRL.2005.v12.n1.a6. MR2122730 (2005m:58001)

[M1] Vladimir Maz'ya, *Bourgain-Brezis type inequality with explicit constants*, Interpolation theory and applications, Contemp. Math., vol. 445, Amer. Math. Soc., Providence, RI, 2007, pp. 247–252, DOI 10.1090/conm/445/08605. MR2381898 (2009h:46074)

[M2] Vladimir Maz'ya, *Estimates for differential operators of vector analysis involving L^1-norm*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 1, 221–240, DOI 10.4171/JEMS/195. MR2578609 (2011b:42050)

[MM] Irina Mitrea and Marius Mitrea, *A remark on the regularity of the div-curl system*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1729–1733, DOI 10.1090/S0002-9939-08-09671-8. MR2470831 (2009j:58002)

[MP] Pierre Bousquet and Petru Mironescu, *An elementary proof of an inequality of Maz'ya involving L^1 vector fields*, Nonlinear elliptic partial differential equations, Contemp. Math., vol. 540, Amer. Math. Soc., Providence, RI, 2011, pp. 59–63, DOI 10.1090/conm/540/10659. MR2807409 (2012i:42010)

[NN] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), 391–404. MR0088770 (19,577a)

[O] Donald Ornstein, *A non-equality for differential operators in the L_1 norm*, Arch. Rational Mech. Anal. **11** (1962), 40–49. MR0149331 (26 #6821)

[S] S. K. Smirnov, *Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows* (Russian, with Russian summary), Algebra i Analiz **5** (1993), no. 4, 206–238; English transl., St. Petersburg Math. J. **5** (1994), no. 4, 841–867. MR1246427 (94i:49054)

[Tr] François Trèves, *Hypo-analytic structures*, Princeton Mathematical Series, vol. 40, Princeton University Press, Princeton, NJ, 1992. Local theory. MR1200459 (94e:35014)

[VS1] Jean Van Schaftingen, *Estimates for L^1-vector fields* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 181–186, DOI 10.1016/j.crma.2004.05.013. MR2078071 (2005b:35018)

[VS2] Jean Van Schaftingen, *Limiting fractional and Lorentz space estimates of differential forms*, Proc. Amer. Math. Soc. **138** (2010), no. 1, 235–240, DOI 10.1090/S0002-9939-09-10005-9. MR2550188 (2010i:46064)

[VS3] Jean Van Schaftingen, *Estimates for L^1 vector fields under higher-order differential conditions*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 4, 867–882, DOI 10.4171/JEMS/133. MR2443922 (2010i:46063)

[VS4] Jean Van Schaftingen, *Limiting Sobolev inequalities for vector fields and canceling linear differential operators*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 3, 877–921, DOI 10.4171/JEMS/380. MR3085095

[Y] Po-Lam Yung, *Sobolev inequalities for $(0, q)$ forms on CR manifolds of finite type*, Math. Res. Lett. **17** (2010), no. 1, 177–196, DOI 10.4310/MRL.2010.v17.n1.a14. MR2592736 (2011a:32055)

Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brazil

E-mail address: hounie@dm.ufscar.br

Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, 13565-905, Brazil

E-mail address: picon@dm.ufscar.br

Current address: Departamento de Computação e Matemática, Universidade de São Paulo, Ribeirão Preto, SP, 14040-901, Brazil

E-mail address: picon@ffclrp.usp.br