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GRADED 3-CALABI-YAU ALGEBRAS AS ORE EXTENSIONS OF 2-CALABI-YAU ALGEBRAS

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Abstract. We study a class of graded algebras obtained from Ore extensions of graded Calabi-Yau algebras of dimension 2. It is proved that these algebras are graded Calabi-Yau and graded coherent. The superpotentials associated to these graded Calabi-Yau algebras are also constructed.

Introduction

Recently, Smith studied in [Sm2] a remarkable graded Calabi-Yau algebra B of dimension 3 constructed from the octonions. Amongst other things, Smith proved that B is a graded Ore extension of an Artin-Schelter regular algebra of global dimension 2 and uses that fact to show that B is graded 3-Calabi-Yau and graded coherent.

In this note, we show that the Calabi-Yau property and the coherence of B do not occur incidentally. A large class of graded algebras that are Ore extensions of graded Calabi-Yau algebras are themselves graded Calabi-Yau. The main result of this note is the following.

Theorem 0.1. *Let V be a finite dimension vector space with basis $\{x_1,\dots,x_n\}$, let M be an invertible $n\times n$ anti-symmetric matrix, and define*

$$r=(x_1,\dots,x_n)M\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in T(V).$$

Let $A=T(V)/\langle r\rangle$, where $\langle r\rangle$ is the ideal of $T(V)$ generated by r. Let δ be a degree-one graded derivation of $T(V)$ such that $\delta(r)=0$. Then δ induces a graded derivation $\bar\delta$ on A. Let $B=A[z;\bar\delta]$ be the Ore extension of A defined by $\bar\delta$. Then the following hold:

(i) *B is a graded 3-Calabi-Yau algebra.*

(ii) *Let $\widehat{V}=V\oplus\Bbbk z$, and $Q=\begin{pmatrix}-1&0\\0&M\end{pmatrix}$. Let $w=(z,x_1,\dots,x_n)Q\begin{pmatrix}r\\ r_1\\ \vdots\\ r_n\end{pmatrix}$,*

where $r_i=z\otimes x_i-x_i\otimes z-\delta(x_i)\in\widehat{V}\otimes\widehat{V}$ for all $i=1,\dots,n$. Then

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$(\alpha\otimes 1\otimes 1)(w) = (1\otimes 1\otimes\alpha)(w)$ for all $\alpha \in (\widehat{V})^*$, and $A[z;\bar{\delta}] \cong T(\widehat{V})/\langle \partial_{x_i}(w) : i = 0, \dots, n\rangle$, where we set $x_0 = z$ and $\partial_{x_i}(w)$ is the cyclic partial derivative of w with respect to x_i.

(iii) *Write $\delta(x_i) = \sum_{s,t=1}^n k_{st}^i x_i \otimes x_j$ for all $i = 1, \dots, n$. Assume there is an integer j such that $k_{jj}^i = 0$ for all $i = 1, \dots, n$, and M is a standard anti-symmetric matrix. Then B is graded coherent.*

Most of this note is devoted to the proof of Theorem 0.1. However, we will go a bit further to discuss the properties of the algebra B. Smith's algebra in [Sm2] is an example satisfying the conditions in the theorem. We will provide a few more examples. We remark that any quadratic algebra A defined by an invertible anti-symmetric matrix as in the above theorem is isomorphic to a quadratic algebra defined by a standard anti-symmetric matrix (see Convention 2.3 for the definition), because, for every invertible anti-symmetric matrix M, there is an invertible matrix P such that P^tMP is a standard anti-symmetric matrix.

Remark 0.2. Let V be a finite dimensional vector space with basis $\{x_1, \dots, x_n\}$. Take an element $r \in V\otimes V$. Since $V\otimes V \cong \mathrm{Hom}_{\Bbbk}(V^*, V)$, the element r corresponds to a linear map $f_r : V^* \to V$. The *rank* of r, denoted by rank(r), is defined to be the rank of f_r (cf. [Z2, Introduction]). One sees

$$\mathrm{rank}(r) = \min\{m | r = u_1 \otimes v_1 + \cdots + u_m \otimes v_m, \text{ for some } u_i, v_i \in V\}.$$

It has been shown that certain features of the algebra $T(V)/\langle r\rangle$ entirely depend on rank(r) (cf. [Z2, Theorem 0.1]). If M is an $n \times n$ matrix and

$$r = (x_1, \dots, x_n)M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in V \otimes V,$$

then rank(r)=rank(M). Therefore, the condition that M is invertible in Theorem 0.1 is equivalent to the condition that rank(r)=n.

Throughout $\Bbbk$ is a fixed field. The unadorned $\otimes$ means $\otimes_{\Bbbk}$. Let $U = \oplus_{n\in\mathbb{Z}} U_n$ be a graded vector space, and l an integer. We write $U(l)$ for the graded vector space with degree k component $U(l)_k = U_{k+l}$.

A connected graded algebra A is called a *graded Calabi-Yau algebra* of dimension d, or simply *graded d-CY algebra* (cf. [Gin]), if

(i) A is homologically smooth; that is, A has a finite resolution by finitely generated graded projective left A^e-modules, where $A^e = A \otimes A^{op}$ is the enveloping algebra of A;

(ii) the projective dimension of A as a left A^e-module is d, and $\mathrm{Ext}^i_{A^e}(A, A\otimes A) = 0$ if $i \neq d$ and $\mathrm{Ext}^d_{A^e}(A, A\otimes A) \cong A(l)$ for some integer l as a right A^e-module.

We refer to [Z1] (also, cf. [Be] and [DV]) for the basic properties of a graded 2-CY algebra.

1. Ore extensions of graded Calabi-Yau algebras of dimension 2

Let V be a vector space with basis $\{x_1, \dots, x_n\}$. Let A be a graded quotient algebra of $T(V)$. If A is a graded 2-CY algebra, then it is defined by an $n \times n$ invertible anti-symmetric matrix M [Z1] (also, cf. [Be, Proposition 3.4]); that is,

$A \cong T(V)/\langle r \rangle$ with $r = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$. Henceforth, $A = T(V)/\langle r \rangle$ with $r = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$ for some fixed invertible anti-symmetric matrix M. Let $\pi : T(V) \to A$ be the natural projection map. Since degree(r)=2, we can, and we will, identify V with A_1 through the projection π.

Let $\delta : V \to V \otimes V$ be a linear map. Then δ extends in a unique way to a degree-one derivation (also denoted by δ) of $T(V)$. If $\delta(r) \in \langle r \rangle$, then δ induces a derivation $\bar{\delta}$ on A.

From now on, we assume that $\delta(r) \in \langle r \rangle$. Let $B = A[z; \bar{\delta}]$ be the graded Ore extension of A by the derivation $\bar{\delta}$; that is, we view z as an element of degree one, and $za = az + \bar{\delta}(a)$ for all $a \in A$.

Zhang proved in [Z1] that A is a Koszul algebra of global dimension 2, and the minimal projective resolution of ${}_A\Bbbk$ can be written as follows:

$$0 \longrightarrow A \otimes \Bbbk r \xrightarrow{\overline{d}^{-2}} A \otimes V \xrightarrow{\overline{d}^{-1}} A \xrightarrow{\varepsilon} {}_A\Bbbk \longrightarrow 0, \tag{1}$$

where ε is the augmentation map, $\overline{d}^{-1}(1 \otimes x) = \pi(x)$ for all $x \in V$, and $\overline{d}^{-2}(1 \otimes r) = r \in A_1 \otimes V$. Since B is an Ore extension of A, B is a Koszul algebra of global dimension 3. Note that B_A is a free A-module. Applying $B \otimes_A -$ to the sequence (1), we obtain the exact sequence

$$0 \longrightarrow B \otimes \Bbbk r \xrightarrow{d^{-2}} B \otimes V \xrightarrow{d^{-1}} B \longrightarrow B/BA_{\geq 1} \longrightarrow 0, \tag{2}$$

where the unlabeled map is the natural projection map, $d^{-1}(1 \otimes x) = \pi(x) \in B_1$ for all $x \in V$, and $d^{-2}(1 \otimes r) = r \in B_1 \otimes V$.

Lemma 1.1. *Suppose that $\delta(r) = 0$ and let $B = A[z; \bar{\delta}]$ be as above. We have the following morphism of cochain complexes:*

$$\begin{tikzcd} B \otimes \Bbbk r \arrow[r, "d^{-2}"] \arrow[d, "f^{-2}"'] & B \otimes V \arrow[r, "d^{-1}"] \arrow[d, "f^{-1}"'] & B \arrow[d, "f^{0}"] \\ B \otimes \Bbbk r \arrow[r, "d^{-2}"] & B \otimes V \arrow[r, "d^{-1}"] & B, \end{tikzcd}$$

where the vertical arrows are left B-module morphisms $f^{-2}(1 \otimes r) = z \otimes r$, $f^{-1}(1 \otimes x) = z \otimes x - \delta(x)$ for all $x \in V$, and $f^0(1) = z$.

Proof. We write $r = \sum_{i=1}^n u_i \otimes x_i$ with all $u_i \in V$, and assume $\delta(x_i) = \sum_{j=1}^n y_{ij} \otimes x_j$ for all $i = 1, \dots, n$ with all $y_{ij} \in V$. We prove the commutativity of the left square. The commutativity of the right one is easy. The identity $\delta(r) = 0$ is equivalent to $\sum_{i=1}^n \delta(u_i) \otimes x_i + \sum_{i=1}^n u_i \otimes \delta(x_i) = 0$, which is in turn equivalent to $\sum_{i=1}^n \delta(u_i) \otimes x_i + \sum_{i=1}^n \sum_{j=1}^n u_i \otimes y_{ij} \otimes x_j = 0$. Applying the map $\pi \otimes 1 : T(V) \otimes V \to A \otimes V$ to the last identity, we obtain $\sum_{i=1}^n \bar{\delta}(u_i) \otimes x_i + \sum_{i=1}^n \sum_{j=1}^n u_i y_{ij} \otimes x_j = 0$. Hence

$$\bar{\delta}(u_i) = -\sum_{j=1}^n u_j y_{ji} \tag{3}$$

for all $i = 1, \dots, n$. The following equations hold:

$$\begin{aligned} f^{-1} \circ d^{-2}(1 \otimes r) &= f^{-1}(\sum_{i=1}^{n} u_i \otimes x_i) \\ &= \sum_{i=1}^{n} u_i z \otimes x_i - \sum_{i=1}^{n}\sum_{j=1}^{n} u_i y_{ij} \otimes x_j \\ &= \sum_{i=1}^{n}(u_i z - \sum_{j=1}^{n} u_j y_{ji}) \otimes x_i, \end{aligned}$$

and

$$\begin{aligned} d^{-2} \circ f^{-2}(1 \otimes r) &= d^{-2}(z \otimes r) = \sum_{i=1}^{n} z u_i \otimes x_i \\ &= \sum_{i=1}^{n}(u_i z + \bar{\delta}(u_i)) \otimes x_i. \end{aligned}$$

By equation (3), $f^{-1} \circ d^{-2}(1 \otimes r) = d^{-2} \circ f^{-2}(1 \otimes r)$. Hence the left square of the diagram commutes. □

The mapping cone of the morphism in Lemma 1.1 provides a graded projective resolution of the trivial module ${}_B\mathbb{k}$ (see also, [GS, Ph]).

Lemma 1.2. *Let r and B be the same as in Lemma 1.1. The minimal projective resolution of ${}_B\mathbb{k}$ is as follows:*

$$0 \longrightarrow B \otimes \mathbb{k}r \xrightarrow{\partial^{-3}} B \otimes \mathbb{k}r \oplus B \otimes V \xrightarrow{\partial^{-2}} B \otimes V \oplus B \xrightarrow{\partial^{-1}} B \longrightarrow \mathbb{k} \longrightarrow 0,$$

where $\partial^{-3} = \begin{pmatrix} f^{-2} \\ -d^{-2} \end{pmatrix}$, $\partial^{-2} = \begin{pmatrix} d^{-2} & f^{-1} \\ 0 & -d^{-1} \end{pmatrix}$, *and* $\partial^{-1} = \big(d^{-1}, f^{0}\big)$.

Let $A^!$ be the quadratic dual of A. As graded vector spaces $A^!_0 \cong \mathbb{k}$, $A^!_1 \cong V^*$ and $A^!_2 \cong \mathbb{k}r^*$, where $r^* \in (\mathbb{k}r)^*$ defined by $r^*(r) = 1$. The multiplication on $A^!$ is given by: $\alpha\beta = (a_1, \dots, b_n)M(b_1, \dots, b_n)^t r^*$, for $\alpha = a_1x_1^* + \cdots + a_nx_n^*$ and $\beta = b_1x_1^* + \cdots + b_nx_n^*$ in V^* (cf. [HVZ2, Section 3]), where $\{x_1^*, \dots, x_n^*\}$ is the basis of V^* dual to the basis $\{x_1, \dots, x_n\}$. Write $E^i(B) := \mathrm{Ext}^i_B({}_B\mathbb{k}, {}_B\mathbb{k})$ and $E(B) := \bigoplus_{i\geq 0} E^i(B)$. Then $E(B)$ is a graded algebra with the degree i component $E^i(B)$. The minimal projective resolution of ${}_B\mathbb{k}$ above implies that, as graded vector spaces,

$$E(B) \cong A^! \oplus A^!(-1). \tag{4}$$

We write an element in $E(B)$ as (α, β) for some $\alpha, \beta \in A^!$, and demote the Yoneda product on $E(B)$ by $(\alpha, \beta) * (\alpha', \beta')$.

Proposition 1.3. *Assume $\delta(r) = 0$. Then $A[z; \bar{\delta}]$ is a 3-CY algebra.*

Proof. By [HVZ1, Proposition 3.3]) in the Koszul case, $B = A[z; \bar{\delta}]$ is Calabi-Yau if and only if $E(B)$ is a graded symmetric algebra. Recall that a finite dimensional graded algebra $E = \bigoplus_{i\geq 0} E^i$ is graded symmetric if there is an integer d and a homogeneous nondegenerate bilinear form $\langle -, - \rangle : E \times E \longrightarrow \mathbb{k}(d)$ such that $\langle \alpha\beta, \gamma \rangle = \langle \alpha, \beta\gamma \rangle$ and $\langle \alpha, \beta \rangle = (-1)^{ij}\langle \beta, \alpha \rangle$ for all homogeneous elements $\alpha \in E^i, \beta \in E^j$ and $\gamma \in E^k$. Since the global dimension of B is 3 and $\dim E^3(B) = 1$,

$E(B)$ is graded symmetric if and only if, for all elements $\Phi \in E^1(B), \Theta \in E^2(B)$, $\Phi * \Theta = \Theta * \Phi$. Let $\Phi = (\alpha, k)$ with $\alpha \in A_1^! = V^*$ and $k \in \Bbbk$, and $\Theta = (r^*, \beta)$ with $\beta \in V^*$. The element Φ induces a B-module morphism $g : B \otimes V \oplus B \longrightarrow {}_B\Bbbk$ by $g(1 \otimes x, 1) = \alpha(x) + k$ for all $x \in V$, and the element Θ induces a B-module morphism $h : B \otimes \Bbbk r \oplus B \otimes V \longrightarrow {}_B\Bbbk$ by $h(1 \otimes r, 1 \otimes x) = 1 + \beta(x)$ for all $x \in V$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & B \otimes \Bbbk r \arrow[r, "\partial^{-3}"] \arrow[d, "g_2"] & B \otimes \Bbbk r \oplus B \otimes V \arrow[r, "\partial^{-2}"] \arrow[d, "g_1"] & B \otimes V \oplus B \arrow[r, "\partial^{-1}"] \arrow[d, "g_0"] \arrow[dr, "g"] & B \cdots \\
\cdots \arrow[r] & B \otimes \Bbbk r \oplus B \otimes V \arrow[r, "\partial^{-2}"] & B \otimes V \oplus B \arrow[r, "\partial^{-1}"] & B \arrow[r] & {}_B\Bbbk,
\end{tikzcd}$$

where the vertical arrows are B-module morphisms defined as follows. As before, we write $r = \sum_{i=1}^n u_i \otimes x_i$ with all $u_i \in V$, and assume $\delta(x_i) = \sum_{j=1}^n y_{ij} \otimes x_j$ for all $i = 1, \ldots, n$ with all $y_{ij} \in V$. Then

$$\begin{aligned}
g_0(1 \otimes x_j, 1) &= \alpha(x_j)1 + k1; \\
g_1(1 \otimes r, 1 \otimes x_j) &= \left(\sum_{i=1}^n 1 \otimes u_i\alpha(x_i) - 1 \otimes kx_j - \sum_{i=1}^n 1 \otimes y_{ji}\alpha(x_j),\ \alpha(x_j)1 \right); \\
g_2(1 \otimes r) &= (1 \otimes kr,\ \sum_{i=1}^n 1 \otimes u_i\alpha(x_i)),
\end{aligned}$$

for all $j = 1, \ldots, n$. Since $\delta(r) = 0$, it follows that

$$\sum_{i=1}^n \delta(u_i) \otimes x_i + \sum_{i=1}^n \sum_{j=1}^n u_i \otimes y_{ij} \otimes x_j = 0.$$

Applying the linear map $1 \otimes 1 \otimes \alpha$ to this identity, one obtains:

$$\sum_{i=1}^n \delta(u_i)\alpha(x_i) + \sum_{i=1}^n \sum_{j=1}^n u_i \otimes y_{ij}\alpha(x_j) = 0. \tag{5}$$

Using equation (5) and the following computations:

$$\begin{aligned}
g_1 \circ \partial^{-3}(1 \otimes r) &= g_1(z \otimes r, -r) \\
&= \left(\sum_{i=1}^n z \otimes u_i\alpha(x_i) + \sum_{i=1}^n u_i \otimes kx_i + \sum_{i=1}^n \sum_{j=1}^n u_i \otimes y_{ij}\alpha(x_j), - \sum_{i=1}^n u_i\alpha(x_i) \right),
\end{aligned}$$

$$\begin{aligned}
\partial^{-2} \circ g_2(1 \otimes r) &= \partial^{-2} \left(1 \otimes kr, 1 \otimes \sum_{i=1}^n u_i\alpha(x_i) \right) \\
&= \left(kr + \sum_{i=1}^n z \otimes u_i\alpha(x_i) - \sum_{i=1}^n \delta(u_i)\alpha(x_i), - \sum_{i=1}^n u_i\alpha(x_i) \right),
\end{aligned}$$

we obtain the identity: $g_1 \circ \partial^{-3}(1 \otimes r) = \partial^{-2} \circ g_2(1 \otimes r)$. Hence $g_1 \circ \partial^{-3} = \partial^{-2} \circ g_2$.

Similar computations show that the second square in the diagram commutes. The commutativity of the triangle in the diagram is obvious. Thus, we have $h \circ g_2(1 \otimes r) = h(1 \otimes kr, \sum_{i=1}^n 1 \otimes u_i \alpha(x_i)) = k + \sum_{i=1}^n \beta(u_i)\alpha(x_i)$. By the definition of the Yoneda product, we have $\Theta * \Phi = (r^*, \beta) * (\alpha, k) = kr^* + \beta\alpha$, where $\beta\alpha$ is the product in $A^!$. Similarly, we can show that $\Phi * \Theta = kr^* - \alpha\beta$. Now A is Calabi-Yau, hence $A^!$ is graded symmetric; that is, $\alpha\beta = -\beta\alpha$ for all $\alpha, \beta \in A_1^!$. It follows that $\Phi * \Theta = \Theta * \Phi$. Therefore, $B = A[z; \bar{\delta}]$ is Calabi-Yau. □

The computation in the proof of Proposition 1.3 has given us the formulas of the Yoneda product of $E(B)$.

Corollary 1.4. *As vector spaces, $E(B) \cong A^! \oplus A^!(-1)$. The Yoneda product of $E(B)$ is given as follows: for $\alpha, \beta \in A_1^!$ and $k, k' \in \Bbbk$,*

$$(r^*, \beta) * (\alpha, k) = (\alpha, k) * (r^*, \beta) = kr^* + \beta\alpha$$

and

$$(\beta, k') * (\alpha, k) = (\beta\alpha, k'\alpha - k\beta - (\beta \otimes \alpha) \circ \delta),$$

where r^* is the basis of $A_2^!$ such that $r^*(r) = 1$.

Proof. The first identity is proved in the proof of Proposition 1.3. Keep the same notions as in the proof of Proposition 1.3. The element (β, k') induces a B-module morphism $g' : B \otimes V \oplus B \longrightarrow {}_B\Bbbk$ by $g'(1 \otimes x, 1) = \beta(x) + k'$ for all $x \in V$, and $(\beta\alpha, k'\alpha - k\beta - (\beta \otimes \alpha) \circ \delta)$ induces a B-module morphism $f : B \otimes \Bbbk r \oplus B \otimes V \longrightarrow {}_B\Bbbk$ by $f(1 \otimes r, 1 \otimes x_j) = \sum_{i=1}^n \beta(u_i)\alpha(x_i) + k'\alpha(x_j) - k\beta(x_j) - \sum_{i=1}^n \beta(y_{ji})\alpha(x_i)$ for all $j = 1, \ldots, n$. By the definition of Yoneda product, $(\beta, k') * (\alpha, k)$ is represented by $g' \circ g_1$. Now $g' \circ g_1(1 \otimes r, 1 \otimes x_j) = \sum_{i=1}^n \beta(u_i)\alpha(x_i) - k\beta(x_j) - \sum_{i=1}^n \beta(y_{ji})\alpha(x_i) + k'\alpha(x_j) = f(1 \otimes r, 1 \otimes x_j)$ for all $j = 1, \ldots, n$. Therefore the second identity follows. □

Let $\epsilon : A^! \to A^!$ be the automorphism of $A^!$ defined by $\epsilon(\alpha) = -\alpha$ for $\alpha \in A_1^!$ and $\epsilon(\beta) = \beta$ for all $\beta \in A_2^!$. Let ${}_\epsilon A^!$ be the graded $A^!$-bimodule whose right $A^!$-action is the regular action, and whose left $A^!$-action is twisted by the automorphism ϵ; that is, for all $\gamma, \theta \in A^!$, the left $A^!$-action $\gamma \cdot \theta = \epsilon(\gamma)\theta$. Let $I = {}_\epsilon A^!(-1)$, and let $E(A^!; I)$ be the trivial extension of $A^!$ by the $A^!$-bimodule I. By Corollary 1.4, $E(B)$ is isomorphic to $E(A^!; I)$.

Corollary 1.5. *The Yoneda algebra $E(B)$ is isomorphic to the trivial extension of $A^!$ by the $A^!$-bimodule I.*

Example 1.6. Consider the Calabi-Yau algebra studied by Smith in [Sm2]. Let $\Bbbk\langle x_1, \ldots, x_6 \rangle$ be the free algebra generated by six elements. Let $A = \Bbbk\langle x_1, \ldots, x_6 \rangle / \langle r \rangle$, where

$$r = (x_1, \ldots, x_6) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix}.$$

Define a derivation $\delta : \Bbbk\langle x_1, \ldots, x_6\rangle \to \Bbbk\langle x_1, \ldots, x_6\rangle$ by

$$\begin{aligned} \delta(x_1) &= x_4x_2 - x_2x_4 + x_3x_5 - x_5x_3 & \delta(x_2) &= x_1x_4 - x_4x_1 + x_3x_6 - x_6x_3 \\ \delta(x_3) &= x_5x_1 - x_1x_5 + x_6x_2 - x_2x_6 & \delta(x_4) &= x_2x_1 - x_1x_2 + x_5x_6 - x_6x_5 \\ \delta(x_5) &= x_1x_3 - x_3x_1 + x_6x_4 - x_4x_6 & \delta(x_6) &= x_2x_3 - x_3x_2 + x_4x_5 - x_5x_4. \end{aligned}$$

Then $\delta(r) = 0$, and $B = A[z; \bar{\delta}]$ is 3-CY [Sm2].

Keep the assumption that $\delta(r) = 0$. Let $\widehat{V} = V \oplus \Bbbk z$. Then $B = A[z; \bar{\delta}]$ is a quotient algebra of $T(\widehat{V})$. Since B is 3-CY, B is defined by a superpotential [Bo, Theorem 3.1]. Let $\{x_1^*, \ldots, x_n^*\}$ be the basis of V^* dual to $\{x_1, \ldots, x_n\}$. Recall that a *superpotential* is an element $w \in \widehat{V} \otimes \widehat{V} \otimes \widehat{V}$ such that $[\alpha w] = [w\alpha]$ for all $\alpha \in (\widehat{V})^*$, where $[\alpha w] = (\alpha \otimes 1 \otimes 1)(w)$ and $[w\alpha] = (1 \otimes 1 \otimes \alpha)(w)$. Given a superpotential w, the *partial derivative* of w by x_i is defined by $\partial_{x_i}(w) = [x_i^* w]$ (cf. [BSW]). By [Bo, Theorem 3.1], there is a superpotential $w \in \widehat{V} \otimes \widehat{V} \otimes \widehat{V}$ such that $B \cong T(\widehat{V})/\langle \partial_{x_i}(w) : i = 0, \ldots, x_n\rangle$ where $x_0 = z$. We next show that the superpotential w may be written out explicitly. For $i = 1, \ldots, n$, let $r_i = z \otimes x_i - x_i \otimes z - \delta(x_i) \in \widehat{V} \otimes \widehat{V}$. Clearly $r, r_1, \ldots, r_n$ are linearly independent in $\widehat{V} \otimes \widehat{V}$, moreover $B \cong T(\widehat{V})/\langle r, r_1, \ldots, r_n\rangle$. Before we construct the general form of the superpotentials, let us look at the following example.

Example 1.7. Let $\Bbbk\langle x, y\rangle$ be the free algebra generated by two elements. Let $\delta : \Bbbk\langle x, y\rangle \to \Bbbk\langle x, y\rangle$ be a derivation defined by $\delta(x) = bx^2 + cy^2$ and $\delta(y) = ax^2 - bxy - byx$, where $(a, b, c) \in \Bbbk^3$. Let $r = xy - yx$. Then it is easy to see that $\delta(r) = 0$. Therefore, δ induces a derivation $\bar{\delta}$ on $A = \Bbbk[x, y]$. Now $B = A[z; \bar{\delta}]$ is 3-CY. A straightforward verification shows that $w = yxz + zyx + xzy - xyz - zxy - yzx - ax^3 + cy^3 + bxyx + bx^2y + byx^2$ is a superpotential, and $B \cong \Bbbk\langle x, y, z\rangle/\langle \partial_x(w), \partial_y(w), \partial_z(w)\rangle$. Explicitly, the generating relations are $r_1 = zy - yz - ax^2 + byx + bxy, r_2 = xx - zx + cy^2 + bx^2$ and $r_3 = yx - xy$.

Proposition 1.8. *Assume $\delta(r) = 0$. Let $Q = \begin{pmatrix} -1 & 0 \\ 0 & M \end{pmatrix}$, and let*

$$w = (z, x_1, \ldots, x_n)Q \begin{pmatrix} r \\ r_1 \\ \vdots \\ r_n \end{pmatrix},$$

where M is an invertible $n \times n$ anti-symmetric matrix, and $r_i = z \otimes x_i - x_i \otimes z - \delta(x_i) \in \widehat{V} \otimes \widehat{V}$ for all $i = 1, \ldots, n$. Then

(i) *w is a superpotential;*

(ii) *$A[z; \bar{\delta}] \cong T(\widehat{V})/\langle \partial_{x_i}(w) : i = 0, \ldots, n\rangle$, where we set $x_0 = z$.*

Proof. Let $\{m_{ij} | i, j = 1, \ldots, n\}$ be the entries of M. Then $r = \sum_{i,j=1}^n m_{ij} x_i \otimes x_j$. Since $\delta(r) = 0$, we have $\sum_{i,j=1}^n m_{ij} \delta(x_i) \otimes x_j = -\sum_{i,j=1}^n m_{ij} x_i \otimes \delta(x_j)$. Let us compute the element w.

$$
\begin{aligned}
w &= -z\otimes r + (x_1,\dots,x_n)M\begin{pmatrix} r_1\\ \vdots\\ r_n\end{pmatrix}\\
&= -\sum_{i,j=1}^{n} m_{ij} z\otimes x_i\otimes x_j + \sum_{i,j=1}^{n} m_{ij} x_i\otimes r_j\\
&= -\sum_{i,j=1}^{n} m_{ij} z\otimes x_i\otimes x_j + \sum_{i,j=1}^{n} m_{ij} x_i\otimes z\otimes x_j\\
&\quad -\sum_{i,j=1}^{n} m_{ij} x_i\otimes x_j\otimes z - \sum_{i,j=1}^{n} m_{ij} x_i\otimes \delta(x_j)\\
&= -\sum_{i,j=1}^{n} m_{ij}(z\otimes x_i - x_i\otimes z)\otimes x_j - \sum_{i,j=1}^{n} m_{ij} x_i\otimes x_j\otimes z + \sum_{i,j=1}^{n} m_{ij}\delta(x_i)\otimes x_j\\
&= -\sum_{i,j=1}^{n} m_{ij}(z\otimes x_i - x_i\otimes z - \delta(x_i))\otimes x_j - \sum_{i,j=1}^{n} m_{ij} x_i\otimes x_j\otimes z\\
&= -r\otimes z + (r_1,\dots,r_n)M^t\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix}.
\end{aligned}
$$

Now it is clear that $[x_i^* w] = [w x_i^*]$, and $\partial_{x_i}(w) = r_i$ for all $i = 0, 1, \dots, n$, where $r_0 = r$. □

2. Coherence of $A[z;\bar{\delta}]$

Notation and notions are as in the previous section. By [Z1, Theorem 0.2], A is Noetherian if and only if $\dim(V) = 2$. Since B is an Ore extension of A in variable z, B/Bz is isomorphic to A as a graded left B-module. Since A is not left Noetherian when $\dim(V) > 2$, neither is B. Similarly, B is not right Noetherian when $\dim(V) > 2$. Summarizing the foregoing argument, we obtain the following property.

Lemma 2.1. *$B = A[z;\bar{\delta}]$ is Noetherian if and only if $\dim(V) = 2$.*

Piontkovski showed in [Pi, Theorem 4.1] that any connected graded algebra with a single quadratic relation is graded coherent. Hence A is a graded coherent algebra. So, it is natural to ask whether B is a graded coherent algebra. The answer is affirmative. However, the proof of this property is not trivial because an Ore extension of a coherent ring need not be coherent. In fact, there is a commutative coherent ring R such that the polynomial extension $R[z]$ is not coherent [So]. Some other results about the coherence of polynomial rings may be found in [GV]. Let us recall the definition of a graded coherent algebra.

A graded algebra D is called a *graded left coherent* algebra if one of the following equivalent conditions is satisfied:

(i) every finitely generated graded left ideal of D is finitely presented; that is, if I is a graded left ideal of D, then there is a finitely graded free D-module

F and a surjective morphism $g : F \to I$ of graded modules such that $\ker g$ is also a finitely generated D-module;

(ii) every finitely generated graded submodule of a finitely presented graded module is finitely presented;

(iii) the category of all finitely presented graded left D-modules is an abelian category.

Similarly we can define a *graded right coherent* algebra. If a graded algebra is both graded left and right coherent, then it is called a *graded coherent* algebra.

Let $W = \bigoplus_{i\geq 0} W_i$ be a graded vector space with $\dim(W_i) < \infty$ for all i. Recall that the Hilbert series of W is defined to be the power series $H_W(t) = \sum_{i\geq 0} \dim(W_i)t^i$.

Lemma 2.2. *Let V be a vector space of dimension $n \geq 4$ with basis $\{x_1, \dots, x_n\}$, and let*

$$M = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \cdots & -1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -1 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{6}$$

be the invertible $n \times n$ anti-symmetric matrix with entries in the anti-diagonal line 1 or -1 and others 0. Let $r = (x_1, \dots, x_n)M(x_1, \dots, x_n)^t$, and $A = T(V)/\langle r\rangle$. Let δ be a derivation on $T(V)$ of degree one. We write $\delta(x_i) = \sum_{s,t=1}^{n} k^i_{st} x_s \otimes x_t$ for all $i = 1, \dots, n$. Assume that $k^i_{nn} = 0$ for all $i = 1, \dots, n$ and $\delta(r) = 0$. Let $\bar\delta$ be the derivation on A induced by δ. Write $B = A[z; \bar\delta]$. Then the following hold:

(i) *Let I be the ideal of B generated by the elements $x_1, \dots, x_{n-1}$. Then $B/I \cong \mathbb{k}[X, Z]$, where $\mathbb{k}[X, Z]$ is the commutative polynomial algebra in variables X and Z;*

(ii) *Let $L = \mathbb{k}x_1 \oplus \cdots \oplus \mathbb{k}x_{n-1}$ and $L' = \mathbb{k}x_2 \oplus \cdots \oplus \mathbb{k}x_{n-1}$. Then, as left B-modules, $I \cong B \otimes (L \oplus L'x_n \oplus L'x_n^2 \oplus \cdots)$, where $L'x_n^k$ $(k \geq 1)$ is the vector space spanned by the elements $x_2x_n^k, \dots, x_{n-1}x_n^k$.*

Convention 2.3. *We call an $n\times n$ $(n \geq 2)$ invertible anti-symmetric matrix of the form* (6) *a standard anti-symmetric matrix. If M is an invertible anti-symmetric matrix, there is an invertible matrix P such that P^tMP is standard.*

Proof of Lemma 2.2. (i) By assumption, $\delta(x_n) = \sum_{s,t=1}^{n} k^n_{st} x_s \otimes x_t$ and $k^n_{nn} = 0$. Therefore $\bar\delta(x_n) \in I$ and B/I is a commutative algebra. There is an algebra morphism $g : k[X, Z] \longrightarrow B/I$ defined by $g(X) = x_n$ and $g(Z) = z$. Next, we want to construct an algebra morphism from B/I to $\mathbb{k}[X, Z]$. As before, write $\widehat{V} = V \oplus \mathbb{k}z$. First, we define $f : T(\widehat{V}) \longrightarrow \mathbb{k}[X, Z]$ by letting $f(x_i) = 0$ for all $i = 1, \dots, n-1$, $f(x_n) = X$ and $f(z) = Z$. Denote by $\langle x_1, \dots, x_{n-1}\rangle$ and by $\langle z \otimes x_n - x_n \otimes z\rangle$ the ideals of $T(\widehat{V})$ respectively generated by $x_1, \dots, x_{n-1}$ and by $z\otimes x_n - x_n \otimes z$. Obviously, $\langle x_1, \dots, x_{n-1}\rangle + \langle z \otimes x_n - x_n \otimes z\rangle \subseteq \ker f$. Recall that B is a Koszul algebra and $B = T(\widehat{V})/J$ where $J = \langle r, z \otimes x_1 - x_1 \otimes z - \delta(x_1), \dots, z \otimes x_n - x_n \otimes z - \delta(x_n)\rangle$. Since $\delta(x_i) = \sum_{s,t=1}^{n} k^i_{st} x_s \otimes x_t$ such that $k^i_{nn} = 0$ for all $i = 1, \dots, n$, it follows that $\delta(x_i) \in \langle x_1, \dots, x_{n-1}\rangle$ for all $i = 1, \dots, n$. Hence

$r, z\otimes x_1 - x_1\otimes z - \delta(x_1), \dots, z\otimes x_{n-1} - x_{n-1}\otimes z - \delta(x_{n-1}) \in \langle x_1, \dots, x_{n-1}\rangle \subseteq \ker f$. Now $z\otimes x_n - x_n\otimes z - \delta(x_n) \in \langle z\otimes x_n - x_n\otimes z\rangle + \langle x_1, \dots, x_{n-1}\rangle \subseteq \ker f$. Hence $J \subseteq \ker f$. Therefore, f induces an algebra morphism $\overline{f} : B \longrightarrow \Bbbk[X, Z]$. Obviously, $\ker \overline{f} \supseteq I$. Hence $\overline{f}$ in turn induces an algebra morphism $\hat{f} : B/I \longrightarrow \Bbbk[X, Z]$. Now it is easy to see that $\hat{f} \circ g = id = g \circ \hat{f}$. The statement (i) follows.

(ii) Here we make use of the technique from [Sm2, Proposition 7.3]. Let $\mu : B\otimes B \to B$ be the multiplication of B. Then the restriction of μ defines a left B-module morphism (also denoted by μ):

$$\mu : B\otimes (L \oplus L'x_n \oplus L'x_n^2 \oplus \cdots) \longrightarrow I.$$

We claim that μ is surjective. In fact, if we can show that the image $I' = \mathrm{im}(\mu)$ is also an ideal of B, then $I = I'$. So, it suffices to show that $I'x_n \subseteq I'$ and $I'z \subseteq I'$. Following the generating relation of A, we have $x_1x_n = x_nx_1 + (x_2x_{n-1} - x_{n-1}x_2) + \cdots + (x_{\frac{n}{2}}x_{\frac{n}{2}+1} - x_{\frac{n}{2}+1}x_{\frac{n}{2}}) \in BL \subseteq I'$. Therefore $I'x_n \subseteq I'$. In particular, $\bar{\delta}(x_i) \in I'$ for all $i = 1, \dots, n$. On the other hand, since $x_iz = zx_i - \bar{\delta}(x_i)$, it follows that $x_iz \in I'$ for all $i = 1, \dots, n-1$. For $2 \le i \le n-1$, we have $x_ix_nz = x_i(zx_n - \bar{\delta}(x_n)) = x_izx_n - x_i\bar{\delta}(x_n) \in I'x_n + x_iI' \subseteq I'$. Now assume $x_ix_n^jz \in I'$ for all $j < k$ and $2 \le i \le n-1$. Then

$$x_ix_n^kz = x_ix_n^{k-1}(zx_n - \bar{\delta}(x_n)) = (x_ix_n^{k-1}z)x_n - x_ix_n^{k-1}\bar{\delta}(x_n) \in I'x_n + x_ix_n^{k-1}I' \subseteq I'.$$

Hence $I'z \subseteq I'$. The claim follows. To show that μ is injective, we only need to compare the Hilbert series of I and that of $F := B\otimes(L\oplus L'x_n \oplus L'x_n^2 \oplus \cdots)$. Write $W = L \oplus L'x_n \oplus L'x_n^2 \oplus \cdots$. Clearly $H_F(t) = H_B(t)\cdot H_W(t)$. We have

$$H_W(t) = (n-1)t + (n-2)t^2 + (n-2)t^3 + \cdots = ((n-1)t - t^2)(1-t)^{-1}.$$

The exact sequence $0 \to I \to B \to B/I \to 0$ implies $H_I(t) = H_B(t) - H_{B/I}(t)$. Since B is Koszul of global dimension 3, it follows that

$$H_B(t) = \left(1 - (n+1)t + (n+1)t^2 - t^3\right)^{-1}$$

by [Sm1, Theorem 5.9] and the isomorphism (4) of the previous section. By (i), $H_{B/I}(t) = (1-t)^{-2}$. Hence

$$\begin{aligned} H_I(t) &= \left(1 - (n+1)t + (n+1)t^2 - t^3\right)^{-1} - (1-t)^{-2} \\ &= \left(1 - (n+1)t + (n+1)t^2 - t^3\right)^{-1} \cdot \left((n-1)t - t^2\right)(1-t)^{-1} \\ &= H_B(t)\cdot H_W(t) \\ &= H_F(t). \end{aligned}$$

Therefore μ is injective. So, (ii) follows. □

Proof of the statement (iii) of Theorem 0.1. If $n = 2$, then $A = \Bbbk[x_1, x_2]$. We obtain that $B = A[z; \bar{\delta}]$ is Noetherian, and hence coherent. Now assume $n \ge 4$. We only prove the statement when $j = n$ in the assumption, that is, $k_{nn}^i = 0$ for all $i = 1, \dots, n$. When $j \ne n$, the statement can be proved similarly. By Lemma 2.2, there is an exact sequence $0 \longrightarrow I \longrightarrow B \longrightarrow B/I \longrightarrow 0$ such that B/I is a polynomial algebra in two variables and I is a free graded left B-module. By [Pi, Proposition 3.2], B is graded right coherent. Note that the left version of Lemma 2.2(ii) holds too. Hence B is also graded left coherent. □

As a special case of the statement (iii) of Theorem 0.1, we have the following result, which can be viewed as a noncommutative version of [GV, Theorem 4.3].

Proposition 2.4. *Let A be a connected graded 2-CY algebra. Then $A[z]$ is a graded coherent algebra.*

Proof. By [Z1, Theorem 0.1] (also, cf. [Be, Proposition 3.4]), A is defined by an invertible anti-symmetric matrix M, that is, $A = T(V)/\langle r\rangle$ with $r = (x_1, \dots, x_n) \times M(x_1, \dots, x_n)^t$. For an invertible anti-symmetric matrix M, there is an invertible matrix P such that P^tMP is a standard invertible anti-symmetric matrix. Then the algebras defined by M and P^tMP respectively are isomorphic to each other. Hence we may assume that the anti-symmetric matrix M itself is standard. Now by (iii) of Theorem 0.1, we see that $A[z]$ is graded coherent. □

Now assume that $B = A[z; \bar{\delta}]$ is graded coherent. We may form a noncommutative projective space from B. Following [Po], we denote by coh$B$ the category of all finitely presented graded left B-modules, and by fdim$B$ the category of all finite dimensional graded left B-modules. Since B is graded coherent, fdim$B$ is a Serre subcategory of coh$B$. Hence the quotient category

$$\mathrm{cohproj}B := \mathrm{coh}B/\mathrm{fdim}B$$

is also an abelian category. Since B is Koszul and 3-CY, B is Artin-Schelter regular with Gorenstein parameter -3. Hence the Beilinson algebra of B (for the terminology, see [MM, Definition 4.7]) is

$$\nabla B = \begin{pmatrix} \Bbbk & B_1 & B_2 \\ 0 & \Bbbk & B_1 \\ 0 & 0 & \Bbbk \end{pmatrix}.$$

Let mod∇B be the category of finite dimensional left ∇B-modules. Then by [MM, Theorem 4.14], we have the following corollary.

Corollary 2.5. *If the conditions of Theorem* 0.1 *are satisfied, then there is an equivalence of triangulated categories:*

$$D^b(\mathrm{cohproj}B) \cong D^b(\mathrm{mod}\nabla B),$$

where $D^b(-)$ is the bounded derived category of the corresponding abelian category.

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