ON THE PRODUCT OF SMALL ELKIES PRIMES

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ABSTRACT. Given an elliptic curve $E$ over a finite field $\mathbb{F}_q$ of $q$ elements, we say that an odd prime $\ell \nmid q$ is an Elkies prime for $E$ if $t_E^2 - 4q$ is a quadratic residue modulo $\ell$, where $t_E = q + 1 - \#E(\mathbb{F}_q)$ and $\#E(\mathbb{F}_q)$ is the number of $\mathbb{F}_q$-rational points on $E$. The Elkies primes are used in the presently most efficient algorithm to compute $\#E(\mathbb{F}_q)$. In particular, the quantity $L_q(E)$ defined as the smallest $L$ such that the product of all Elkies primes for $E$ up to $L$ exceeds $4q^{1/2}$ is a crucial parameter of this algorithm. We show that there are infinitely many pairs $(p, E)$ of primes $p$ and curves $E$ over $\mathbb{F}_p$ with $L_p(E) \geq c \log p \log \log \log p$ for some absolute constant $c > 0$, while a naive heuristic estimate suggests that $L_p(E) \sim \log p$. This complements recent upper bounds on $L_q(E)$ proposed by Galbraith and Satoh in 2002, conditional under the Generalised Riemann Hypothesis, and by Shparlinski and Sutherland in 2011, unconditional for almost all pairs $(p, E)$.

1. Introduction

For an elliptic curve $E$ over a finite field $\mathbb{F}_q$ of $q$ elements we denote by $\#E(\mathbb{F}_q)$ the number of $\mathbb{F}_q$-rational points on $E$ and define the trace of Frobenius $t_E = q + 1 - \#E(\mathbb{F}_q)$; we refer to [1,13] for a background on elliptic curves.

We start with recalling that the first polynomial-time algorithm to compute $\#E(\mathbb{F}_q)$ is due to Schoof [11]. For a sufficiently large set of small primes $\ell$, Schoof [11] determines $t_E \pmod{\ell}$ by computing the action of Frobenius on the $\ell$-torsion subgroup $E[\ell] \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$ and then determines the unique integer $t_E$ that satisfies the Hasse bound $|t_E| \leq 2q^{1/2}$ via the Chinese remainder theorem. Elkies [5] has observed that when $t_E^2 - 4q$ is a quadratic residue modulo $\ell$, one can instead work in a cyclic subgroup of $E[\ell]$, which speeds up the computation considerably. One can also obtain partial information at other primes $\ell$, but this has no impact on the asymptotic performance of the algorithm.

We say that an odd prime $\ell \nmid q$ is an Elkies prime for $E$ if $t_E^2 - 4q$ is a quadratic residue modulo $\ell$; otherwise $\ell \nmid q$ is called an Atkin prime.

These primes play a key role in the Schoof-Elkies-Atkin (SEA) algorithm, see [11 Sections 17.2.2 and 17.2.5], and their distribution affects the performance of this algorithm in a rather dramatic way. Thus, for an elliptic curve $E$ over $\mathbb{F}_q$, we define $N_a(E; L)$ and $N_e(E; L)$ as the numbers of Atkin and Elkies primes $\ell \in [1, L]$, respectively. Obviously,

$$N_a(E; L) + N_e(E; L) = \pi(L) + O(1),$$

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where, as usual, \( \pi(L) \) denotes the number of primes \( \ell < L \). Furthermore, for any elliptic curve over a finite field, one expects about the same number of Atkin and Elkies primes \( \ell < L \) as \( L \to \infty \). That is, a naive heuristic argument suggests that

\[
N_a(E; L) \sim N_e(E; L) \sim \frac{1}{2} \pi(L),
\]

as \( L \to \infty \).

It has been noted by Galbraith and Satoh \([10, \text{Appendix A}]\), that under the Generalised Riemann Hypothesis (GRH), using the bound on sums of quadratic characters over primes, one can show that \((1)\) holds for \( L \geq (\log q)^{2+\varepsilon} \) for any fixed \( \varepsilon > 0 \) and a sufficiently large \( q \).

The unconditional results are much weaker and essentially rely on our knowledge of the distribution of primes in arithmetic progressions; see \([6, \text{Section 5.9}]\) or \([9, \text{Chapters 4 and 11}]\). However, for almost all pairs \((p, E)\) of primes \( p \) and elliptic curves \( E \) over \( \mathbb{F}_p \), Shparlinski and Sutherland \([12]\) have established the asymptotic formula \((1)\) for \( L \geq (\log p)^{\varepsilon} \) for any fixed \( \varepsilon > 0 \), that is, starting from much smaller values of \( L \) than those implied by the GRH. In particular, let \( \mathcal{L}_E(p) \) be the set of all Elkies primes for an elliptic curve \( E \) over \( \mathbb{F}_p \). We see that the prime number theorem and the result of \([12]\) implies that for some function \( L(p) \sim \log p \) for almost all pairs \((p, E)\) we have

\[
\prod_{3 \leq \ell \in \mathcal{L}_E(p)} \ell > 4p^{1/2}.
\]

Note that this condition is crucial to the performance of the SEA point counting algorithm, see \([1, \text{Sections 17.2.2 and 17.2.5}]\).

Here we show that this “almost all” result cannot be extended to all primes and curves even for a slightly larger values of \( L(p) \). More precisely, we show that there is an absolute constant \( c > 0 \) such that for any function \( L(p) \leq c \log p \log \log \log p \) the inequality \((2)\) fails in a very strong sense for infinitely many pairs \((p, E)\).

**Theorem 1.** There is a constant \( c > 0 \) such that for infinitely many pairs \((p, E)\) of primes \( p \) and curves \( E \) over \( \mathbb{F}_p \), and \( L \leq c \log p \log \log \log p \) we have

\[
\prod_{3 \leq \ell \in \mathcal{L}_E(p)} \ell = p^{o(1)}.
\]

We note that Galbraith and Satoh \([10, \text{Appendix A}]\) have conjectured and actually presented some arguments supporting a result of this kind. Moreover, under both the GRH and the conjecture that every positive integer \( n \equiv 1 \pmod{4} \) can be represented as \( n = 4p - t^2 \), the argument of Galbraith and Satoh \([10, \text{Appendix A}]\) can be made rigorous, and in fact under these assumptions it allows one to replace \( \log p \log \log \log p \) with \( \log p \log \log p \) in Theorem \([1]\). Unfortunately, the required representation \( n = 4p - t^2 \) is presently known to exist only for almost all \( n \) (see \([2,7]\)), which is not enough to complete the argument (even under the GRH).

## 2. Preparations

We recall the notations \( U = O(V) \), \( V = \Omega(U) \), \( U \ll V \) and \( V \gg U \), which are all equivalent to the statement that the inequality \(|U| \leq cV\) holds asymptotically, with some constant \( c > 0 \).
We always assume that $\ell$ and $p$ run through the prime values.

For integers $a$ and $m \geq 2$, we use $(a/m)$ to denote a Jacobi symbol of $a$ modulo $m$, see [6 Section 3.5]. We also use $\tau(k)$ and $\mu(k)$ to denote the number of positive integer divisors and the Möbius function of $k \geq 1$. It is easy to see that for a square-free $k$ we have

$$\tau(k) = 2^{\omega(k)},$$

where $\omega(k)$ is the number of prime divisors of $k$.

Our main tools are bounds of multiplicative character sums.

The following estimate is a slight generalisation of [8, Lemma 2.2] and is also given in [12].

**Lemma 2.** For any integers $a$ and $T \geq 1$ and a product $m = \ell_1 \ldots \ell_s$ of $s \geq 0$ distinct odd primes $\ell_1, \ldots, \ell_s$ with $\gcd(a, m) = 1$ we have

$$\sum_{|t| \leq T} \left( \frac{t^2 - a}{m} \right) \ll T/m + C^s m^{1/2} \log m,$$

for some absolute constant $C \geq 1$.

We also need a slight extension of [6, Corollary 12.14]. In fact, we present it in much wider generality and strength than is needed for our purpose. First we note that for a square-free integer $m$ and any integers $u$ and $v$, we have

$$(3) \quad \gcd((u - v)^2, m) = \gcd(u - v, m).$$

Hence, in the case of quadratic polynomials, the bound of [6, Theorem 12.10] implies the following result.

**Lemma 3.** Assume that a square-free odd integer $m \geq 3$ and an arbitrary integer $N \geq 1$ are such that all prime factors of $m$ are at most $N^{1/9}$. Then for any two integers $u, v$ we have

$$\left| \sum_{n=1}^{N} \left( \frac{(n-u)(n-v)}{m} \right) \right| \leq 4N \left( \gcd(u - v, m)m^{-1} \tau(m)^{r^2+2r} \right)^{1/(r^2r)},$$

where $r$ is any positive integer with $N^r > m^3$.

**Proof.** As in the proof of [6, Corollary 12.14], we note that there is a factorisation

$$m = m_1 \ldots m_r$$

with $m_j \leq N^{4/9}$, $j = 1, \ldots, r$. In particular, by [6, Theorem 12.10], recalling [9], we see that for any $j = 1, \ldots, r$ we have

$$\left| \sum_{n=1}^{N} \left( \frac{(n-u)(n-v)}{m_j} \right) \right| \leq 4N \left( \gcd(u - v, m_j)m_j^{-1} \tau(m_j)^{r^2+2r} \right)^{1/2r}.$$

Since $m$ is square-free, we see that $m_1, \ldots, m_r$ are relatively prime. Using the multiplicativity of the divisor function, we obtain

$$\prod_{j=1}^{r} \gcd(u - v, m_j)m_j^{-1} \tau(m_j)^{r^2+2r} = \gcd(u - v, m)m^{-1} \tau(m)^{r^2+2r}.$$
Therefore, for some $j \in \{1, \ldots, r\}$ we have
\[
gcd(u - v, m_j)m_j^{-1}\tau(m_j)r^{2r} \leq \left(\gcd(u - v, m)m^{-1}\tau(m)r^{2r}\right)^{1/r},
\]
and the result now follows. \qed

We remark that several stronger and more general results of this type have recently been given by Chang \cite{3}.

We also recall the following classical result of Deuring \cite{4}.

Lemma 4. For any prime $p$ and any integer $t$ with $|t| \leq 2^{p^{1/2}}$, there is an elliptic curve $E$ over $\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = p + 1 - t$.

3. PROOF OF THEOREM

Let $Q$ be a sufficiently large integer. We then set
\[ L = \lfloor 0.3 \log Q \log \log \log Q \rfloor, \quad M = \left\lfloor \frac{\log Q}{\log \log \log Q} \right\rfloor, \quad T = \left\lfloor Q^{1/2} \right\rfloor. \]

Since, by the prime number theorem
\[ \prod_{\ell \leq M} \ell = Q^{o(1)}, \]
we see from Lemma 4 that it is enough to show that for any sufficiently large $Q$, there is an integer $t \in [1, T]$ and a prime $p \in (Q/2, Q]$ such that
\[ \left( \frac{t^2 - 4p}{\ell} \right) \neq 1 \]
for all primes $\ell \in [M, L]$.

Clearly, if the condition (1) is violated, then
\[ \prod_{\ell \in [M, L]} \left( 1 - \left( \frac{t^2 - 4p}{\ell} \right) \right) = 0. \]

Thus it is enough to show that the sum
\[ W = \sum_{1 \leq t \leq T} \sum_{Q/2 < p \leq Q} \prod_{\ell \in [M, L]} \left( 1 - \left( \frac{t^2 - 4p}{\ell} \right) \right) \]

is positive, that is, that
\[ W > 0 \]
for the above choice of $L$, $M$ and $T$, provided that $Q$ is sufficiently large.

Let $\mathcal{M}$ be the set of $2^{\pi(L) - \pi(M)}$ square-free products (including the empty product) composed of primes $\ell \in [M, L]$, and let $\mathcal{M}^* = \mathcal{M} \setminus \{1\}$. We have
\[ W = \sum_{1 \leq t \leq T} \sum_{Q/2 < p \leq Q} \sum_{m \in \mathcal{M}} \mu(m) \left( \frac{t^2 - 4p}{m} \right). \]

Changing the order of summation and separating the term $T(\pi(Q) - \pi(Q/2))$ corresponding to $m = 1$, we derive
\[ W = T(\pi(Q) - \pi(Q/2)) + \sum_{m \in \mathcal{M}^*} \mu(m)S(m), \]
where
\[ S(m) = \sum_{1 \leq t \leq T} \sum_{Q/2 < p \leq Q} \left( \frac{t^2 - 4p}{m} \right). \]

We have
\[ |S(m)| \leq \sum_{Q/2 < p \leq Q} \left| \sum_{1 \leq t \leq T} \left( \frac{t^2 - 4p}{m} \right) \right|. \]

For \( m \leq T^{1/4} \) we use Lemma 2 (clearly, we can assume that \( Q \) is large enough so that \( M > 2 \) and thus \( m \) is odd). We also note that
\[ C^{\omega(m)} = \tau(m)^{\log C/\log 2} = m^{o(1)}, \]
where \( C \) is the constant of Lemma 2 so we obtain
\[ S(m) \ll \pi(Q) \left( T/m + C^{\omega(m)} m^{1/2} \log m \right) \ll \pi(Q) T/m. \]

Thus for the contribution from all such sums we derive
\[ \sum_{m \in \mathbb{M}^* \atop m \leq T^{1/4}} |S(m)| \ll \pi(Q) T \sum_{m \in \mathbb{M}^* \atop m \leq T^{1/4}} 1/m \ll \pi(Q) T \left( \prod_{\ell \in [M, L]} \left( 1 + \frac{1}{\ell} \right) - 1 \right). \] (7)

Furthermore,
\[ \log \prod_{\ell \in [M, L]} \left( 1 + \frac{1}{\ell} \right) = \sum_{\ell \in [M, L]} \log \left( 1 + \frac{1}{\ell} \right) \ll \sum_{\ell \in [M, L]} \frac{1}{\ell}. \]

By the Mertens theorem, see [6, Equation (2.15)],
\[ \sum_{\ell \in [M, L]} \frac{1}{\ell} = \log \frac{\log L}{\log M} + O(1/\log M) \]
\[ = \log \frac{\log \log Q + \log \log \log \log Q + \log 0.3}{\log \log Q - \log \log \log \log Q} + O(1/\log M) \]
\[ = \log \left( 1 + O \left( \frac{\log \log \log \log Q}{\log \log Q} \right) \right) + O(1/\log M) \]
\[ \ll \frac{\log \log \log \log Q}{\log \log Q}. \]

Therefore,
\[ \prod_{\ell \in [M, L]} \left( 1 + \frac{1}{\ell} \right) = 1 + O \left( \frac{\log \log \log \log Q}{\log \log Q} \right). \]

Inserting this bound in (7), we obtain
\[ \sum_{m \in \mathbb{M}^* \atop m \leq T^{1/4}} |S(m)| \ll \pi(Q) T \frac{\log \log \log \log Q}{\log \log Q} = o(\pi(Q) T). \] (8)
To estimate the sums $S(m)$ for $m > T^{1/4}$, using the Cauchy inequality and then extending the summation range over all positive integers $n \leq 4Q$, we derive

$$|S(m)|^2 \leq (\pi(Q) - \pi(Q/2)) \sum_{Q/2 < r \leq Q} \left| \sum_{1 \leq t \leq T} \left( \frac{t^2 - 4p}{m} \right) \right|^2$$

$$\leq \pi(Q) \sum_{n \leq 4Q} \left| \sum_{1 \leq t \leq T} \left( \frac{t^2 - n}{m} \right) \right|^2$$

$$= \pi(Q) \sum_{1 \leq s, t \leq T} \sum_{n \leq 4Q} \left( \frac{(s^2 - n)(t^2 - n)}{m} \right).$$

If $\gcd(s^2 - t^2, m) > m^{1/2}$, we estimate the inner sum trivially as $O(Q)$. The total contribution from such pairs $(s, t)$ is at most

$$\sum_{d|m} \sum_{d > m^{1/2}} \frac{1}{s \equiv t^2 \pmod{d}} \leq \sum_{d|m} T(T/d + 1)2^{\omega(d)}$$

$$\leq T\left(T/m^{1/2} + 1\right)\tau(m)^2,$$

since for a square-free $d$, by the Chinese remainder theorem, any quadratic congruence of the form $s^2 \equiv a \pmod{d}$, $1 \leq s \leq d$, has at most $2^{\omega(d)}$ solutions.

If $\gcd(s^2 - t^2, m) \leq m^{1/2}$, we apply Lemma 3 to the inner sum, getting

$$\left| \sum_{n \leq 4Q} \left( \frac{(s^2 - n)(t^2 - n)}{m} \right) \right|$$

$$\leq 16Q \left( \gcd(s^2 - t^2, m)m^{-1}\tau(m)^{r^2 + 2r} \right)^{1/(r^2r)}$$

$$\leq 16Q \left( m^{-1/2}\tau(m)^{r^2 + 2r} \right)^{1/(r^2r)}$$

for any positive integer $r$ with

$$\begin{align*}
(4Q)^r &> m^3.
\end{align*}$$

Therefore, combining (9) and (10), we obtain

$$S(m)^2 \ll \pi(Q)Q T\left(T/m^{1/2} + 1\right)\tau(m)^2$$

$$\quad + \pi(Q)Q T^2 \left(m^{-1/2}\tau(m)^{r^2 + 2r} \right)^{1/(r^2r)}.$$
provided that \( Q \) is large enough. Hence,
\[
m^{-1/2} \tau(m) r^2 + 2r \leq m^{-1/3} \leq T^{-1/12}.
\]
Furthermore, since \([13]\) implies that \( \tau(m) = T^{o(1)} \) for \( m \in \mathcal{M} \), we see that \([12]\) implies that for \( m > T^{1/4} \), for any \( r \) satisfying \([11]\) and \([14]\), we have
\[
S(m) \ll QT^{1-1/(24r^2)}.
\]
Therefore,
\[
\sum_{\substack{m \in \mathcal{M}^* \\
m > T^{1/4}}} |S(m)| \ll 2^{\pi(L)} QT^{1-1/(24r^2)}
\leq QT^{1-1/(24r^2)} \exp \left( (\log 2 + o(1)) \frac{\log Q \log \log \log Q}{\log \log Q} \right).
\]
In particular, if we set
\[
r = \lfloor \log \log \log Q \rfloor
\]
then
\[
T^{1/(24r^2)} = \exp \left( \frac{\log Q}{(\log \log Q)^{\log 2 + o(1)}} \right).
\]
Therefore,
\[
(15) \quad \sum_{\substack{m \in \mathcal{M}^* \\
m > T^{1/4}}} |S(m)| \ll QT^{1-1/(25r^2)} = o(\pi(Q)T).
\]
It is also obvious that \([14]\) is satisfied for the above choice of \( r \). Furthermore, the condition \([11]\) is satisfied as well because
\[
(4Q)^r \geq \exp((1 + o(1)) \log Q \log \log Q)
\]
and
\[
\max_{m \in \mathcal{M}} m = \exp((1 + o(1))L) = \exp((0.3 + o(1)) \log Q \log \log Q).
\]
Substituting \([8]\) and \([15]\) in \([6]\), we see that \([5]\) holds, which concludes the proof.

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