

ANGLE GEOMETRY IN THE UNIVERSAL TEICHMÜLLER SPACE

JINHUA FAN AND YUNPING JIANG

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ABSTRACT. In this paper we investigate angle geometry in the universal Teichmüller space. We construct three examples of triangles bounded by geodesic segments such that the first example has the sum of the three inner angles less than π , the second example has the sum of the three angles equal to π , and the third example has the sum of the three angles greater than π . Our result gives a negative answer to a problem raised by Zhong Li and Yi Qi. Moreover, our results indicate that the universal Teichmüller space presents all hyperbolic, Euclidean, and spherical phenomena in angle geometry.

1. INTRODUCTION

First let us recall the definition of the universal Teichmüller space. Instead of using the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ in the complex plane in the definition, we use the strip

$$S = \{z = x + iy \in \mathbb{C} \mid -\infty < x < +\infty, 0 < y < 1\}.$$

Suppose $\mathcal{M}(S)$ is the open unit ball of $L^\infty(S)$. For every $\mu \in \mathcal{M}(S)$, we consider the Beltrami equation

$$(1) \quad f_{\bar{z}}(z) = \mu(z)f_z(z).$$

We use f^μ to denote the solution of the Beltrami equation which is a quasiconformal self-mapping of S and normalized by $0, i$ and ∞ fixed. Here $\mu \in \mathcal{M}(S)$ is called a Beltrami coefficient. Two Beltrami coefficients $\mu, \nu \in \mathcal{M}(S)$ are said to be equivalent, denoted by $\mu \sim \nu$, if $f^\mu|_{\partial S} = f^\nu|_{\partial S}$. We use $[\mu]$ or $[f^\mu]$ to denote the equivalence class of μ . Then the universal Teichmüller space

$$T(S) = \{[\mu] \mid \mu \in \mathcal{M}(S)\}$$

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is the space of all equivalence classes. We know that $T(S)$ is an infinite dimensional complex Banach manifold with Teichmüller distance

$$d_T([\mu], [\nu]) = \frac{1}{2} \inf_{\eta \in [f^\nu \circ (f^\mu)^{-1}]} \log \frac{1 + \|\eta\|_\infty}{1 - \|\eta\|_\infty}, \quad \forall [\mu], [\nu] \in T(S).$$

Definition 1. A Beltrami coefficient $\mu \in \mathcal{M}(S)$ is extremal if

$$\|\nu\|_\infty \geq \|\mu\|_\infty, \quad \forall \nu \in [\mu].$$

We say that μ is uniquely extremal if

$$\|\nu\|_\infty > \|\mu\|_\infty, \quad \forall \nu \neq \mu \in [\mu].$$

The corresponding map f^μ is called an extremal quasiconformal mapping if μ is extremal.

The reader can refer to [4, 5, 8] for further properties of the universal Teichmüller space, and to [1, 6, 15–17] for theories of extremal quasiconformal mappings.

Suppose $A = [\mu]$ and $B = [\nu]$ are two points in $T(S)$. Let $\eta \in [f^\nu \circ (f^\mu)^{-1}]$ be an extremal Beltrami coefficient. Then

$$(2) \quad \{[f^{t\eta} \circ f^\mu] \mid 0 \leq t \leq 1\}$$

is a geodesic connecting A and B . For any two points $A = [\mu]$ and $B = [\nu]$ in $T(S)$, we use γ_{AB} to denote a geodesic connecting A and B . For a given geodesic γ_{AB} , we use γ_{BA} to denote the geodesic segment which is same as γ_{AB} but with the starting point B and the endpoint A . Since there may be many geodesic segments between A and B in $T(S)$, it is worth noting that γ_{AB} and γ_{BA} may be not of the form (2) (refer to [3, 9]). Furthermore, that γ_{AB} is of the form (2) does not imply that γ_{BA} is of the form (2), and vice versa. However, in this paper, *when we talk about a geodesic between $A = [\mu]$ and $B = [\nu]$, we will mean γ_{AB} or γ_{BA} in the form of (2).*

For a given geodesic segment γ_{AB} , let $\gamma_{AB}(r)$ be the point in γ_{AB} such that the Teichmüller distance between A and $\gamma_{AB}(r)$ is

$$d_T(\gamma_{AB}(r), A) = \frac{1}{2} \log \frac{1+r}{1-r}.$$

For more properties of the geodesic geometry in the universal Teichmüller space, the reader can refer to [3, 9–13].

Suppose $A = [\mu]$, $B = [\nu]$, and $C = [\eta]$ are three points in $T(S)$. Let γ_{AB} , γ_{AC} , and γ_{BC} be the geodesic segments connecting A and B , A and C , and B and C , respectively. In this paper, we say γ_{AB} , γ_{AC} and γ_{BC} form a triangle in $T(S)$ which we denote as

$$(3) \quad \triangle ABC,$$

if γ_{AB} or γ_{BA} , γ_{AC} or γ_{CA} , and γ_{BC} or γ_{CB} are in the form of (2).

Definition 2. We are given a triangle in the form of (3). The angle at A , denoted as $\angle A$, is defined as

$$(4) \quad \begin{aligned} \angle A &= 2 \arcsin \left\{ \frac{1}{2} \lim_{r \rightarrow 0} \frac{d_T(\gamma_{AB}(r), \gamma_{AC}(r))}{d_T(\gamma_{AB}(r), A)} \right\} \\ &= 2 \arcsin \left\{ \frac{1}{2} \lim_{r \rightarrow 0} \frac{d_T(\gamma_{AB}(r), \gamma_{AC}(r))}{r} \right\} \end{aligned}$$

if the limit exists. Similarly, we can define $\angle B$ and $\angle C$.

The definition of an angle was given in [14] when $A = [0]$. The above definition is more general. In the definition, the existence of the limit for a general triangle is still a problem. For a finite dimensional Teichmüller space, Yao has claimed that the limit always exists; see [19]. For a general Teichmüller space, Li and Qi in [14] used to give a sufficient condition for the existence of the limit. The following problem is raised from the paper [14].

Problem 1. Suppose the three inner angles $\angle A$, $\angle B$, and $\angle C$ of a triangle $\triangle ABC$ in the form of (3) exist. Is the sum $\angle A + \angle B + \angle C$ of the three inner angles always less than π ?

In this paper, we give a negative answer to this problem when the Teichmüller space is the universal Teichmüller space. We do not only give a negative answer to the problem but we also prove that all cases can happen in the universal Teichmüller space $T(S)$. More precisely, We prove the following three theorems.

Theorem 3. *There exists a triangle $\triangle ABC$ in the form of (3) in $T(S)$ such that*

$$\angle A + \angle B + \angle C < \pi.$$

Theorem 4. *There exists a triangle $\triangle ABC$ in the form of (3) in $T(S)$ such that*

$$\angle A + \angle B + \angle C = \pi.$$

Theorem 5. *There exists a triangle $\triangle ABC$ in the form of (3) in $T(S)$ such that*

$$\angle A + \angle B + \angle C > \pi.$$

We will give detailed proofs in the next three sections. We will keep the notation $z = x + iy$.

2. PROOF OF THEOREM 3

Suppose $\varphi(z)$ is a Strebel differential, which means that $[t\frac{\bar{\varphi}}{|\varphi|}]$ is a Strebel point for any $t \in \mathbb{D}$. For the definition and properties of a Strebel point, we refer to [2] and [7]. Consider the Beltrami coefficient

$$\mu_t = t \frac{\bar{\varphi}}{|\varphi|}.$$

Then $D_\varphi = \{[\mu_t] \mid t \in \mathbb{D}\}$ is a Teichmüller disk in $T(S)$. Consider the hyperbolic disk \mathbb{D} with the hyperbolic metric

$$d\rho = \frac{|dz|}{1 - |z|^2}.$$

Then the map $\alpha(t) = [\mu_t] : \mathbb{D} \rightarrow D_\varphi$ is an isometry.

Let $A = [0]$; $B = [\mu_{1/2}]$ and $C = [\mu_{i/2}]$. Also let $\triangle ABC$ be the geodesic triangle in the Teichmüller disk D_φ . We claim that $\triangle ABC$ is in the form of (3).

Since $\mu_{1/2}$ is uniquely extremal with constant modular $|\mu_{1/2}| = 1/2$, by the result in [3], the geodesic segment γ_{AB} connecting A and B in $T(S)$ is unique. For the same reason, the geodesic segment γ_{AC} connecting A and C in $T(S)$ is unique too. The Beltrami coefficient for $(f^{\mu_{1/2}})^{-1}$ can be calculated as

$$\mu_{(f^{\mu_{1/2}})^{-1}} = -\frac{1}{2} \frac{\bar{\varphi}}{|\varphi|} \frac{f_z^{\mu_{1/2}}}{f_z^{\mu_{1/2}}} \circ (f^{\mu_{1/2}})^{-1}.$$

Similarly, the Beltrami coefficient for $f^{\mu_{i/2}} \circ (f^{\mu_{1/2}})^{-1}$ can be calculated as

$$\mu_{f^{\mu_{i/2}} \circ (f^{\mu_{1/2}})^{-1}} = \frac{\mu_{i/2} - \mu_{1/2}}{1 - \mu_{i/2}\mu_{1/2}} \frac{f_z^{\mu_{1/2}}}{f_z^{\mu_{i/2}}} \circ (f^{\mu_{1/2}})^{-1} = \frac{2(i-1)}{4-i} \frac{\bar{\varphi}}{|\varphi|} \frac{f_z^{\mu_{1/2}}}{f_z^{\mu_{i/2}}} \circ (f^{\mu_{1/2}})^{-1}.$$

Since $[f^{\mu_{1/2}}]$ is a Strebel point, its inverse $(f^{\mu_{1/2}})^{-1}$ is also a Strebel point. This implies that there is another holomorphic function φ^* on $S = f^{\mu_{1/2}}(S)$ with $\|\varphi^*\| = 1$ such that

$$\mu_{(f^{\mu_{1/2}})^{-1}} = \frac{1}{2} \frac{\bar{\varphi}^*}{|\varphi^*|}.$$

This implies that

$$\mu_{f^{\mu_{i/2}} \circ (f^{\mu_{1/2}})^{-1}} = \frac{-2(i-1)}{4-i} \frac{\bar{\varphi}^*}{|\varphi^*|} = \nu.$$

Since $\|\varphi^*\| = 1$, ν is uniquely extremal with constant modular. So the geodesic segment γ_{BC} connecting B and C is unique. We concluded that the geodesic triangle $\triangle ABC$ in the Teichmüller disk D_φ is in the form of (3).

Remember that $\alpha(t)$ is an isometry between the hyperbolic disk \mathbb{D} and the Teichmüller disk D_φ . Consider the hyperbolic triangle $\triangle abc$ in \mathbb{D} for $a = 0, b = 1/2$ and $c = i/2$. Then $\alpha(\triangle abc) = \triangle ABC$. Thus $\angle A, \angle B$ and $\angle C$ in $\triangle ABC$ in $T(S)$ is equal to $\angle a, \angle b$ and $\angle c$ in $\triangle abc$ in \mathbb{D} respectively. Since the angle sum of any triangle in the hyperbolic plane \mathbb{D} is less than π . So

$$\angle a + \angle b + \angle c < \pi$$

in the triangle $\triangle abc$. This implies that

$$\angle A + \angle B + \angle C < \pi$$

in the triangle $\triangle ABC$. This completes the proof.

Remark 6. From the proof of this theorem, we actually proved that Theorem 3 holds for any Teichmüller space.

3. PROOF OF THEOREM 4

Let us recall a lemma from [18].

Lemma 7. *Suppose E is a subset in ∂S . Let $F_K(x, y) = Kx + iy$, where K is a positive constant. Consider*

$$Q(F_K, E) = \{f \mid f \text{ is a quasiconformal mapping of } S \text{ onto } F_K(S), f|_E = F_K|_E\}.$$

Then F_K is extremal in $Q(F_K, E)$, that is,

$$\|\mu_{F_K}\|_\infty \leq \|\mu_f\|_\infty, \quad \forall f \in Q(F_K, E),$$

if at least one of the points $\pm\infty$ is an accumulation point of E .

From Lemma 7, we have the following lemma.

Lemma 8. *Define*

$$f(x, y) = \begin{cases} K_1x + iy, & (x, y) \in S, x \geq 0, \\ K_2x + iy, & (x, y) \in S, x < 0, \end{cases}$$

where K_1 and K_2 are two positive constants. Then μ_f is extremal in $[f]$.

Proof. Let $k_1 = \frac{K_1-1}{K_1+1}$ and $k_2 = \frac{K_2-1}{K_2+1}$; without loss of generality, we assume that $|k_2| \leq |k_1|$. Let $E = \{(x, y) \in \partial S \mid x \geq 0\}$ and let $F_{K_1}(x, y) = K_1x + iy$. Then

$$Q(F_{K_1}, E) = \{g \mid g \text{ is a quasiconformal mapping from } S \text{ onto } F_{K_1}(S), g|_E = F_{K_1}|_E\}.$$

By Lemma 7, F_{K_1} is extremal in $Q(F_{K_1}, E)$. Since $f \in Q(F_{K_1}, E)$ and since $\|\mu_f\|_\infty = \|\mu_{F_{K_1}}\|_\infty = k_1$, f is extremal in $Q(F_{K_1}, E)$. For $Q(F_{K_1}, E) \supset [f]$, we have that μ_f is extremal in $[f]$. \square

For $0 < k < 1$, let $\mu(x, y) = k$ for $(x, y) \in S$ and let

$$\nu(x, y) = \begin{cases} k, & (x, y) \in S, x \geq 0, \\ 0, & (x, y) \in S, x < 0. \end{cases}$$

Then both μ and ν are extremal.

Let $A = [0]$, $B = [\mu]$, and $C = [\nu]$. We have two geodesic segments,

$$\gamma_{AB} = \{[t\mu] \mid 0 \leq t \leq 1\} \quad \text{and} \quad \gamma_{AC} = \{[t\nu] \mid 0 \leq t \leq 1\}.$$

Let

$$\eta = \mu_{f^\mu \circ (f^\nu)^{-1}} = \frac{\mu - \nu}{1 - \mu\bar{\nu}} \frac{f_z^\nu}{f_z^\mu} \circ (f^\nu)^{-1} = \begin{cases} 0, & (x, y) \in S, x \geq 0, \\ k, & (x, y) \in S, x < 0; \end{cases}$$

then η is extremal (in $[f^\mu \circ (f^\nu)^{-1}]$). So we have the geodesic

$$\gamma_{CB} = \{[f^{t\eta} \circ f^\nu] \mid 0 \leq t \leq 1\}.$$

Thus the geodesic segments γ_{AB} , γ_{AC} , and γ_{CB} form a triangle $\triangle ABC$ in the form of (3). We will show that the sum of the three inner angles of this triangle is π .

First, we calculate $\angle A$. Note that $\gamma_{AB}(r) = [\frac{r}{k}\mu]$ and $\gamma_{AC}(r) = [\frac{r}{k}\nu]$. We have

$$\mu_{f^{\frac{r}{k}\mu} \circ (f^{\frac{r}{k}\nu})^{-1}} = \frac{\frac{r}{k}\mu - \frac{r}{k}\nu}{1 - \frac{r}{k}\mu\bar{\nu}} \frac{f_z^{\frac{r}{k}\nu}}{f_z^{\frac{r}{k}\mu}} \circ (f^{\frac{r}{k}\nu})^{-1} = \begin{cases} 0, & (x, y) \in S, x \geq 0, \\ r, & (x, y) \in S, x < 0. \end{cases}$$

Following Lemma 8, $\mu_{f^{\frac{r}{k}\mu} \circ (f^{\frac{r}{k}\nu})^{-1}}$ is extremal in $[f^{\frac{r}{k}\mu} \circ (f^{\frac{r}{k}\nu})^{-1}]$. So we have that

$$d_T(\gamma_{AC}(r), \gamma_{AB}(r)) = \frac{1}{2} \log \frac{1+r}{1-r}.$$

This implies that

$$\begin{aligned} \angle A &= 2 \arcsin \left\{ \frac{1}{2} \lim_{r \rightarrow 0} \frac{d_T(\gamma_{AB}(r), \gamma_{AC}(r))}{r} \right\} \\ (5) \quad &= 2 \arcsin \left\{ \frac{1}{2} \lim_{r \rightarrow 0} \frac{\frac{1}{2} \log \frac{1+r}{1-r}}{r} \right\} = \frac{\pi}{3}. \end{aligned}$$

Secondly, we calculate $\angle C$. Let $r' = \frac{k-r}{1-kr}$ then

$$\begin{aligned} d_T(\gamma_{CA}(r), A) &= d_T(C, A) - d_T(\gamma_{CA}(r), C) \\ &= \frac{1}{2} \log \frac{1+k}{1-k} - \frac{1}{2} \log \frac{1+r}{1-r} \\ &= \log \frac{1+r'}{1-r'} \\ &= d_T(\gamma_{AC}(r'), A). \end{aligned}$$

Thus

$$\gamma_{CA}(r) = \gamma_{AC}(r') = \left[\frac{r'}{k}\nu \right].$$

By the definition of γ_{CB} , we have

$$\gamma_{CB}(r) = [f^{\frac{r}{k}}\eta \circ f^\nu] = [\eta'],$$

where

$$\eta' = \mu_{f^{\frac{r}{k}}\eta \circ f^\nu} = \frac{\frac{r}{k}\eta \circ f^\nu(z) \frac{\overline{f_z^\nu}}{f_z^\nu} + \nu}{1 + \overline{\nu} \frac{r}{k}\eta \circ f^\nu(z) \frac{\overline{f_z^\nu}}{f_z^\nu}} = \begin{cases} k, & (x, y) \in S, x \geq 0, \\ r, & (x, y) \in S, x < 0. \end{cases}$$

So we have

$$\mu_{f^{\frac{r'}{k}}\nu \circ (f^{\eta'})^{-1}} = \frac{\frac{r'}{k}\nu - \eta' \frac{f_z^{\eta'}}{f_z^{\eta'}}}{1 - \frac{r'}{k}\nu \overline{\eta'}} \circ (f^{\eta'})^{-1} = \begin{cases} \frac{r' - k}{1 - r'k}, & (x, y) \in S, x \geq 0, \\ -r, & (x, y) \in S, x < 0. \end{cases}$$

Since

$$\left| \frac{r' - k}{1 - r'k} \right| = \frac{k - \frac{k-r}{1-kr}}{1 - \frac{k-r}{1-kr}k} = \frac{k(1 - kr) - (k - r)}{(1 - kr) - k(k - r)} = r$$

and since $\mu_{f^{\frac{r'}{k}}\nu \circ (f^{\eta'})^{-1}}$ is extremal in $[f^{\frac{r'}{k}}\nu \circ (f^{\eta'})^{-1}]$, the Teichmüller distance

$$\begin{aligned} d_T(\gamma_{CB}(r), \gamma_{CA}(r)) &= \frac{1}{2} \inf_{\kappa \in [f^{\frac{r'}{k}}\nu \circ (f^{\eta'})^{-1}]} \log \frac{1 + \|\kappa\|_\infty}{1 - \|\kappa\|_\infty} \\ &= \frac{1}{2} \log \frac{1 + \|\mu_{f^{\frac{r'}{k}}\nu \circ (f^{\eta'})^{-1}}\|_\infty}{1 - \|\mu_{f^{\frac{r'}{k}}\nu \circ (f^{\eta'})^{-1}}\|_\infty} \\ &= \frac{1}{2} \log \frac{1 + r}{1 - r}. \end{aligned}$$

By the same computation as (5), we have

$$(6) \quad \angle C = \frac{\pi}{3}.$$

Finally, we calculate $\angle B$. Similarly, $\gamma_{BA}(r) = \gamma_{AB}(r') = [\frac{r'}{k}\mu]$. We also have

$$\gamma_{BC}(r) = \gamma_{CB}(r') = [f^{\frac{r'}{k}}\eta \circ f^\nu] = [\eta''],$$

where

$$\eta'' = \mu_{f^{\frac{r'}{k}}\eta \circ f^\nu} = \frac{\frac{r'}{k}\eta \circ f^\nu(z) \frac{\overline{f_z^\nu}}{f_z^\nu} + \nu}{1 + \overline{\nu} \frac{r'}{k}\eta \circ f^\nu(z) \frac{\overline{f_z^\nu}}{f_z^\nu}} = \begin{cases} k, & (x, y) \in S, x \geq 0, \\ r', & (x, y) \in S, x < 0. \end{cases}$$

Thus

$$\tau = \mu_{f^{\frac{r'}{k}}\mu \circ (f^{\eta''})^{-1}} = \frac{\frac{r'}{k}\mu - \eta'' \frac{\partial_z f^{\eta''}}{\partial_z f^{\eta''}}}{1 - \frac{r'}{k}\mu \overline{\eta''}} \circ (f^{\eta''})^{-1} = \begin{cases} \frac{r' - k}{1 - r'k}, & (x, y) \in S, x \geq 0, \\ 0, & (x, y) \in S, x < 0. \end{cases}$$

So τ is extremal in $[f^{\frac{r'}{k}\mu} \circ (f^{\eta''})^{-1}]$ and the Teichmüller distance

$$\begin{aligned} d_T(\gamma_{BC}(r), \gamma_{BA}(r)) &= \frac{1}{2} \inf_{\kappa \in [f^{\frac{r'}{k}\mu} \circ (f^{\eta''})^{-1}]} \log \frac{1 + \|\kappa\|_\infty}{1 - \|\kappa\|_\infty} \\ &= \frac{1}{2} \log \frac{1 + \|\mu_{f^{\frac{r'}{k}\mu} \circ (f^{\eta''})^{-1}}\|_\infty}{1 - \|\mu_{f^{\frac{r'}{k}\mu} \circ (f^{\eta''})^{-1}}\|_\infty} \\ &= \frac{1}{2} \log \frac{1 + |\frac{r'-k}{1-r'k}|}{1 - |\frac{r'-k}{1-r'k}|} = \frac{1}{2} \log \frac{1+r}{1-r}. \end{aligned}$$

Using this and the same computation as in (5), we have

$$(7) \quad \angle C = \frac{\pi}{3}.$$

Put (5), (6), and (7) together, we get

$$\angle A + \angle B + \angle C = \pi.$$

This completes the proof of Theorem 4.

4. PROOF OF THEOREM 5

For $0 < k < 1$, let $\mu(x, y) = k$ for $(x, y) \in S$. Define

$$\nu(x, y) = \begin{cases} -k, & (x, y) \in S, x \geq 0, \\ 0, & (x, y) \in S, x < 0. \end{cases}$$

Then μ and ν are extremal.

Let $A = [0]$, $B = [\mu]$ and $C = [\nu]$. Then

$$\gamma_{AB} = \{[t\mu] \mid 0 \leq t \leq 1\} \quad \text{and} \quad \gamma_{AC} = \{[t\nu] \mid 0 \leq t \leq 1\}$$

are geodesic segments. Let

$$\eta = \mu_{f^\mu \circ (f^\nu)^{-1}} = \frac{\mu - \nu}{1 - \mu\bar{\nu}} \frac{f_z^\nu}{f_z^\mu} \circ (f^\nu)^{-1} = \begin{cases} \frac{2k}{1+k^2}, & (x, y) \in S, x \geq 0, \\ k, & (x, y) \in S, x < 0; \end{cases}$$

then η is extremal (in $[f^\mu \circ (f^\nu)^{-1}]$). We have the geodesic segment

$$\gamma_{CB} = \{[f^{t\eta} \circ f^\nu] \mid 0 \leq t \leq 1\}.$$

Three geodesic segments γ_{AB} , γ_{AC} , and γ_{CB} form a triangle $\triangle ABC$ in the form of (3). We will prove that the sum of inner angles of this triangle is greater than π .

First, we calculate $\angle A$. It is easy to see that $\gamma_{AB}(r) = [\frac{r}{k}\mu]$ and $\gamma_{AC}(r) = [\frac{r}{k}\nu]$. Then

$$\tau = \mu_{f^{\frac{r}{k}\mu} \circ (f^{\frac{r}{k}\nu})^{-1}} = \frac{\frac{r}{k}\mu - \frac{r}{k}\nu}{1 - \frac{r}{k}\mu\bar{\frac{r}{k}\nu}} \frac{f_z^{\frac{r}{k}\nu}}{f_z^{\frac{r}{k}\mu}} \circ (f^{\frac{r}{k}\nu})^{-1} = \begin{cases} \frac{2r}{1+r^2}, & (x, y) \in S, x \geq 0, \\ r, & (x, y) \in S, x < 0. \end{cases}$$

Since $\frac{2r}{1+r^2} > r$ and since τ is extremal in $[f^{\frac{r}{k}\mu} \circ (f^{\frac{r}{k}\nu})^{-1}]$, we can calculate the Teichmüller distance as

$$d_T(\gamma_{AC}(r), \gamma_{AB}(r)) = \frac{1}{2} \log \frac{1 + \frac{2r}{1+r^2}}{1 - \frac{2r}{1+r^2}} = \log \frac{1+r}{1-r}.$$

Thus we have

$$\begin{aligned}
 \angle A &= 2 \arcsin \left\{ \frac{1}{2} \lim_{r \rightarrow 0} \frac{d_T(\gamma_{AB}(r), \gamma_{AC}(r))}{r} \right\} \\
 (8) \qquad &= 2 \arcsin \left\{ \frac{1}{2} \lim_{r \rightarrow 0} \frac{\log \frac{1+r}{1-r}}{r} \right\} \\
 &= 2 \arcsin 1 = \pi.
 \end{aligned}$$

Secondly, we calculate $\angle C$. Similarly to the proof of Theorem 4 we know that

$$\gamma_{CA}(r) = \gamma_{AC}(r') = \left[\frac{r'}{k} \nu \right]$$

and

$$\gamma_{CB}(r) = [f^{\frac{r(1+k^2)}{2k}} \eta \circ f^\nu] = [\eta'],$$

where

$$\eta' = \mu_{f^{\frac{r(1+k^2)}{2k}} \eta \circ f^\nu} = \frac{\frac{r(1+k^2)}{2k} \eta \circ f^\nu(z) \frac{\overline{f'_z}}{f'_z} + \nu}{1 + \overline{\nu} \frac{r(1+k^2)}{2k} \eta \circ f^\nu(z) \frac{\overline{f'_z}}{f'_z}} = \begin{cases} \frac{r-k}{1-rk}, & (x, y) \in S, x \geq 0, \\ r, & (x, y) \in S, x < 0. \end{cases}$$

Thus

$$\eta'' = \mu_{f^{\frac{r'}{k} \nu} \circ (f^{\eta'})^{-1}} = \frac{\frac{r'}{k} \nu - \eta' \frac{f''_z}{f'_z}}{1 - \frac{r'}{k} \nu \overline{\eta'} \frac{\overline{f''_z}}{f'_z}} \circ (f^{\eta'})^{-1} = \begin{cases} 0, & (x, y) \in S, x \geq 0, \\ -r, & (x, y) \in S, x < 0. \end{cases}$$

So η'' is extremal in $[f^{\frac{r'}{k} \nu} \circ (f^{\eta'})^{-1}]$ and the Teichmüller distance

$$d_T(\gamma_{CB}(r), \gamma_{CA}(r)) = \frac{1}{2} \log \frac{1+r}{1-r}.$$

Using this and the same computation as in (5), we have

$$(9) \qquad \qquad \qquad \angle C = \frac{\pi}{3}.$$

Then (8) and (9) already imply that

$$\angle A + \angle B + \angle C > \pi.$$

This completes the proof.

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DEPARTMENT OF APPLIED MATHEMATICS, NANJING UNIVERSITY OF SCIENCE AND TECHNOLOGY, NANJING, 210094, PEOPLE'S REPUBLIC OF CHINA
E-mail address: jinhuafan@hotmail.com

MATHEMATICS9 DEPARTMENT, QUEENS COLLEGE OF CUNY, 65-30 KISSENA BLVD, FLUSHING, NEW YORK 11367 — AND — PH.D. PROGRAM IN MATHEMATICS, GRADUATE CENTER OF CUNY, 365 FIFTH AVENUE, NEW YORK, NEW YORK 10016
E-mail address: yunping.jiang@qc.cuny.edu