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THE WEAK BOUNDED APPROXIMATION PROPERTY OF PAIRS

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Abstract. We study the weak bounded approximation property (weak BAP) of pairs and show that each of the spaces c_0 and ℓ_1 has a subspace having the approximation property but failing the weak BAP.

1. Introduction

A Banach space X is said to have the *approximation property* (AP) if $id_X \in \overline{\mathcal{F}(X)}^{\tau_c}$, where id_X is the identity map on X, $\mathcal{F}(X)$ is the space of finite rank operators on X and τ_c is the topology of uniform convergence on each compact subset of X. If $id_X \in \overline{\{S \in \mathcal{F}(X) : \|S\| \leq \lambda\}}^{\tau_c}$ for some $\lambda \geq 1$, then we say that X has the λ-*bounded approximation property* (λ-BAP). If $\lambda = 1$, X is said to have the *metric approximation property* (MAP). Grothendieck [5] systematically investigated the AP and the BAP in various points of view. One of the main goals of Banach space theory is to discover the relationship between the AP and the BAP. It was shown that the AP and the BAP are equivalent to the MAP for separable dual spaces [5] (cf. [13, Theorem 1.e.15]). But Figiel and Johnson [3] showed that the AP does not imply the BAP in general. To be more precise, they constructed a Banach space with separable dual which has the AP but fails the BAP. Recently, Figiel, Johnson and Pelczyński [4, Corollary 1.13] introduced the notion of BAP for pairs and showed that each of the spaces c_0 and ℓ_1 has a subspace having the AP but failing the BAP.

In [12] the notion of *weak bounded approximation property* (weak BAP) was introduced by Lima and Oja, in an attempt to cope with the old open problem whether for a dual Banach space the AP implies the MAP. For $\lambda \geq 1$, a Banach space X is said to have the *weak* λ-BAP if for every Banach space Z and every

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$R \in \mathcal{W}(X, Z)$, the space of all weakly compact operators from X into Z, we have

$$id_X \in \overline{\{S \in \mathcal{F}(X) : \|RS\| \leq \lambda \|R\|\}}^{\tau_c}.$$

In [12, Proposition 2.2 and Theorem 3.6], it was shown that the AP is strictly weaker than the weak BAP, but they are both equivalent to the weak MAP for dual spaces. The BAP implies the weak BAP, but it is not known whether the converse holds. Lima and Oja [12] conjectured that they are different properties. The main result of this paper is the following theorem.

Theorem 1.1. *If X has the AP but X^* fails to have the AP, then there exists a subspace Y of $c_0(X)$ (respectively, $\ell_p(X)$ $(1 \leq p < \infty)$) such that Y has the AP but fails the weak BAP.*

Here $c_0(X)$ (resp. $\ell_p(X)$) is the Banach space of all null sequences (resp. absolutely p-summable sequences) in X. The proof of Theorem 1.1 uses the techniques in [4] and some characterizations of the weak BAP of pairs. We therefore prove Theorem 1.1 in Section 3 after we obtain the results about the weak BAP of pairs in Section 2.

Corollary 1.2. *Each of the spaces c_0 and ℓ_1 has a subspace which has the AP but fails the weak BAP.*

Proof. The case c_0: There exists a subspace X of c_0 which has a basis so that X^* fails the AP [7, Corollary JS]. It follows from Theorem 1.1 that there exists a subspace Y of $c_0(X) \subset c_0$ such that Y has the AP but fails the weak BAP.

The case ℓ_1: Let X be a subspace of ℓ_1 which fails the AP [15]. Let $(G_n)_n$ be a sequence of finite-dimensional subspaces of X which is dense in the class of all finite-dimensional subspaces of X. Let $C_1 = (\sum_n G_n)_{\ell_1}$. Then by [6, Theorem 3] C_1^* fails the AP. Obviously, C_1 has the AP. It follows from Theorem 1.1 that there exists a subspace Y of $\ell_1(C_1) \subset \ell_1$ such that Y has the AP but fails the weak BAP. □

Remark 1.3. The case c_0 in Corollary 1.2 can also be deduced from [4, Corollary 1.13] because the weak λ-BAP and the λ-BAP are equivalent for Banach spaces with separable dual (see [14, Corollary 1]).

2. The Bounded approximation property of pairs

Figiel, Johnson and Pelczyński [4] introduced and investigated the BAP of pairs. Let Y be a closed subspace of a Banach space X. Let $\lambda \geq 1$. The pair (X, Y) is said to have the λ-BAP if for every finite-dimensional subspace F of X and for every $\varepsilon > 0$ there exists an $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda + \varepsilon$ such that $Sx = x$ for all $x \in F$ and $S(Y) \subset Y$. The pair (X, X) has the λ-BAP if and only if X has the λ-BAP (see [8, Proposition 1.1]). We proceed in a more general setting in this paper to include other approximation properties. We say that the pair (X, Y) has the λ-$\mathcal{A}$-BAP if the space $\mathcal{F}(X)$ is replaced by a linear subspace $\mathcal{A}(X)$ of $\mathcal{L}(X)$. In [2], the λ-$\mathcal{K}$-BAP of pairs was studied, where $\mathcal{K}$ is the space of all compact operators between Banach spaces.

The pair (X, Y) is said to have the *weak* λ-$\mathcal{A}$-BAP if for every Banach space Z and every $R \in \mathcal{W}(X, Z)$, for every finite-dimensional subspace F of X and for every $\varepsilon > 0$, there exists an $S \in \mathcal{A}(X)$ with $\|RS\| \leq (\lambda + \varepsilon)\|R\|$ such that $Sx = x$ for all $x \in F$ and $S(Y) \subset Y$. The pair (X, Y) is said to have the *weak* λ-BAP if

$\mathcal{A} = \mathcal{F}$. In this section, we give some characterizations of the weak BAP of pairs. The following lemma [9, Proposition 3.1] is needed in our proof.

Lemma 2.1. *Let Y be a closed subspace of X. Let $T \in \mathcal{L}(X)$ and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$ and $\alpha > 0$. Then the following statements are equivalent:*

(a) *For every reflexive Banach space Z and every $R \in \mathcal{W}(X, Z)$, we have*

$$RT \in \overline{\{RS : S \in \mathcal{A}(X), \|RS\| \leq \alpha\|R\|, S(Y) \subset Y\}}^{\tau_c}.$$

(b) *For every Banach space Z and every $R \in \mathcal{W}(X, Z)$, we have*

$$T \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \leq \alpha\|R\|, S(Y) \subset Y\}}^{\tau_c}.$$

Proposition 2.2. *Let Y be a closed subspace of X and let $\lambda \geq 1$. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{L}(X)$, which contains $\mathcal{F}(X)$. Then the following statements are equivalent:*

(a) *For every Banach space Z and every $R \in \mathcal{W}(X, Z)$, for every $\delta > 0$ we have*

$$R \in \overline{\{RS : S \in \mathcal{A}(X), \|RS\| \leq (\lambda + \delta)\|R\|, S(Y) \subset Y\}}^{\tau_c}.$$

(b) *For every Banach space Z and every $R \in \mathcal{W}(X, Z)$, we have*

$$id_X \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \leq \lambda\|R\|, S(Y) \subset Y\}}^{\tau_c}.$$

(c) *For every Banach space Z and every $R \in \mathcal{W}(X, Z)$, for every finite-dimensional subspace F of X and for every $\varepsilon > 0$, there exists an $S \in \mathcal{A}(X)$ with $\|RS\| \leq \lambda\|R\|$ such that $\|Sx - x\| \leq \varepsilon\|x\|$ for all $x \in F$ and $S(Y) \subset Y$.*

(d) *The pair (X, Y) has the weak λ-$\mathcal{A}$-BAP.*

Proof. (b)⇒(c) is clear.

(c)⇒(d) This proof is essentially due to the proof of [4, Lemma 1.5]. Let Z be a Banach space and let $R \in \mathcal{W}(X, Z)$. Let F be a finite-dimensional subspace of X and let $\varepsilon > 0$. Then we can find a projection P from X onto F with $P(Y) \subset Y$. Choose $\delta > 0$ such that $\delta\|P\| < \varepsilon$. By (c) there exists an $S \in \mathcal{A}(X)$ with $\|RS\| \leq \lambda\|R\|$ such that $\|Sx - x\| \leq \delta\|x\|$ for all $x \in F$ and $S(Y) \subset Y$. Put $S_0 = S + (id_X - S)P \in \mathcal{A}(X)$. Then we have that $S_0 x = x$ for all $x \in F$, $S_0(Y) \subset Y$, and

$$\begin{aligned}\|RS_0\| &= \|RS + R(id_X - S)P\| \\ &\leq \|RS\| + \|R\|\|(id_X - S)P\| \leq \lambda\|R\| + \|R\|\delta\|P\| \leq (\lambda + \varepsilon)\|R\|.\end{aligned}$$

Hence the pair (X, Y) has the weak λ-$\mathcal{A}$-BAP.

(d)⇒(a) Let Z be a Banach space and let $R \in \mathcal{W}(X, Z)$. Let $\delta > 0$. Let K be a compact subset of X and let $\varepsilon > 0$. Choose a $\gamma > 0$ so that

$$(\lambda + \delta + 1)\|R\|\gamma \leq \varepsilon.$$

Let $\{x_i\}_{i=1}^n \subset K$ be a γ-net for K. Then by (d) there exists an $S \in \mathcal{A}(X)$ with $\|RS\| \leq (\lambda + \delta)\|R\|$ such that $Sx_i = x_i$ for all $1 \leq i \leq n$ and $S(Y) \subset Y$.

Now let $x \in K$. Then there exists an x_{i_0} for some $1 \leq i_0 \leq n$ such that $\|x - x_{i_0}\| \leq \gamma$. We have that

$$\begin{aligned}\|Rx - RSx\| &\leq \|Rx - Rx_{i_0}\| + \|Rx_{i_0} - RSx_{i_0}\| + \|RSx_{i_0} - RSx\| \\ &\leq \gamma\|R\| + (\lambda + \delta)\gamma\|R\| \leq \varepsilon.\end{aligned}$$

Hence we obtain assertion (a).

(a)⇒(b) Let Z be a Banach space and let $R \in \mathcal{W}(X, Z)$. Let K be a compact subset of X and let $\varepsilon > 0$. Choose a $\delta > 0$ so that $(\delta/(\lambda + \delta)) \sup_{x \in K} \|Rx\| \leq \varepsilon/2$. By (a) there exists an $S \in \mathcal{A}(X)$ with $\|RS\| \leq (\lambda + \delta)\|R\|$ such that

$$\sup_{x \in K} \|RSx - Rx\| \leq \varepsilon/2$$

and $S(Y) \subset Y$. Put $S_0 = (\lambda/(\lambda + \delta))S \in \mathcal{A}(X)$. Then we have that $S_0(Y) \subset Y$, $\|RS_0\| \leq \lambda\|R\|$, and

$$\sup_{x \in K} \|RS_0x - Rx\| \leq \frac{\lambda}{\lambda + \delta} \sup_{x \in K} \|RSx - Rx\| + \frac{\delta}{\lambda + \delta} \sup_{x \in K} \|Rx\| \leq \varepsilon.$$

Hence $R \in \overline{\{RS : S \in \mathcal{A}(X), \|RS\| \leq \lambda\|R\|, S(Y) \subset Y\}}^{\tau_c}$. By Lemma 2.1 we obtain assertion (b). □

We can also extend [4, Lemma 1.5] as the following proposition.

Proposition 2.3. *Let Y be a closed subspace of X and let $\lambda \geq 1$. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{L}(X)$, which contains $\mathcal{F}(X)$. Then the following statements are equivalent:*

(a) $id_X \in \overline{\{S \in \mathcal{A}(X) : \|S\| \leq \lambda, S(Y) \subset Y\}}^{\tau_c}$.
(b) *For every finite-dimensional subspace F of X and for every $\varepsilon > 0$, there exists an $S \in \mathcal{A}(X)$ with $\|S\| \leq \lambda$ such that $\|Sx - x\| \leq \varepsilon\|x\|$ for all $x \in F$ and $S(Y) \subset Y$.*
(c) *The pair (X, Y) has the λ-$\mathcal{A}$-BAP.*

Proof. (a)⇒(b) is clear and we use the proof of [4, Lemma 1.5] to show (b)⇒(c). As in the proof of Proposition 2.2 we can use a simple perturbation argument to show (c)⇒(a). □

The following theorem is the main result in this section. Here $\mathcal{A}^*(X) := \{S^* : S \in \mathcal{A}(X)\}$.

Theorem 2.4. *Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$, which contains $\mathcal{F}(X)$. Then the following statements are equivalent:*

(a) $id_{X^*} \in \overline{\mathcal{A}^*(X)}^{\tau_c}$.
(b) *For every finite-codimensional subspace Y of X, the pair (X, Y) has the weak $\mathcal{A}$-MAP.*
(c) *There exists a $\lambda \geq 1$ satisfying the fact that for every finite-codimensional subspace W of X, there exists a finite-codimensional subspace Y of X with $Y \subset W$ such that the pair (X, Y) has the weak λ-$\mathcal{A}$-BAP.*

Since $\mathcal{F}(X^*) \subset \overline{\mathcal{F}^*(X)}^{\tau_c}$ (see [13, Lemma 1.e.17]), we have:

Corollary 2.5. *The following statements are equivalent:*

(a) *X^* has the AP.*
(b) *For every finite-codimensional subspace Y of X, the pair (X, Y) has the weak MAP.*
(c) *There exists a $\lambda \geq 1$ satisfying the fact that for every finite-codimensional subspace W of X, there exists a finite-codimensional subspace Y of X with $Y \subset W$ such that the pair (X, Y) has the weak λ-BAP.*

Theorem 2.4(b)⇒(c) is clear, and we use the argument in the proof [4, Proposition 1.6] to show the other parts. We need the following result to show Theorem 2.4(a)⇒(b).

Proposition 2.6. *Suppose that* $\mathcal{A}(X)$ *is a convex subset of* $\mathcal{K}(X)$ *with* $0 \in \mathcal{A}(X)$ *and let* $T \in \mathcal{L}(X)$. *If* $T^* \in \overline{\mathcal{A}^*(X)}^{\tau_c}$, *then for every Banach space* Z *and every* $R \in \mathcal{W}(X, Z)$ *there exists a net* (T_α) *in* $\mathcal{A}(X)$ *with* $\|RT_\alpha\| \le \|T\|\|R\|$ *for all* α *such that*

$$T_\alpha \xrightarrow{\tau_c} T \quad \textit{and} \quad T_\alpha^* \xrightarrow{\tau_c} T^*.$$

In order to show Proposition 2.6, we need the following lemma which is contained in [10, Theorem 5.4].

Lemma 2.7. *Suppose that* $\mathcal{A}(X)$ *is a convex subset of* $\mathcal{K}(X)$ *with* $0 \in \mathcal{A}(X)$ *and let* $T \in \mathcal{L}(X)$. *If* $T \in \overline{\mathcal{A}(X)}^{\tau_c}$, *then for every Banach space* Z *and every* $R \in \mathcal{W}(Z, X)$, *we have*

$$T \in \overline{\{S \in \mathcal{A}(X) : \|SR\| \le \|T\|\|R\|\}}^{\tau_c}.$$

Proof of Proposition 2.6. Let Z be a Banach space and let $R \in \mathcal{W}(X, Z)$. Since $T^* \in \overline{\mathcal{A}^*(X)}^{\tau_c}$, by Lemma 2.7 there exists a net (S_β) in $\mathcal{A}(X)$ with $\|S_\beta^* R^*\| \le \|T^*\|\|R^*\|$ for all β such that

$$S_\beta^* \xrightarrow{\tau_c} T^*.$$

Hence by a standard argument (cf. [11, Lemma 3.3]) there exists a net (T_α) in $co(\{S_\beta\}) \subset \mathcal{A}(X)$ such that

$$T_\alpha \xrightarrow{\tau_c} T \quad \text{and} \quad T_\alpha^* \xrightarrow{\tau_c} T^*,$$

and it is also clear that $\|RT_\alpha\| \le \|T\|\|R\|$ for all α. □

Proof of Theorem 2.4(a)⇒(b). Let Y be a finite-codimensional subspace of X. Let $\lambda > 1$ be arbitrary. We show Proposition 2.2(c) to obtain the assertion that the pair (X, Y) has the weak λ-$\mathcal{A}$-BAP. Then in view of the definition of the weak BAP of pairs, the pair (X, Y) also has the weak $\mathcal{A}$-MAP.

Now, let Z be a Banach space and let $R \in \mathcal{W}(X, Z)$. Let F be a finite-dimensional subspace of X and let $\varepsilon > 0$. By (a) and Proposition 2.6, there exists a net (T_α) in $\mathcal{A}(X)$ with $\|RT_\alpha\| \le \|R\|$ for all α such that $T_\alpha \xrightarrow{\tau_c} id_X$ and $T_\alpha^* \xrightarrow{\tau_c} id_{X^*}$. We can find a finite rank projection $P : X \to X$ such that P^* maps from X^* onto the finite-dimensional space $Y^\perp := \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in Y\}$. Let $\delta > 0$ be such that $1 + \delta\|P\| < \lambda$ and $\delta(1 + \|P\|) < \varepsilon$. Since $T_\alpha \xrightarrow{\tau_c} id_X$ and $T_\alpha^* \xrightarrow{\tau_c} id_{X^*}$, there exists an α_0 such that

$$\|T_{\alpha_0}x - x\| \le \delta \text{ for all } x \in B_F \text{ and } \|T_{\alpha_0}^* x^* - x^*\| \le \delta \text{ for all } x^* \in B_{Y^\perp}.$$

Put $S = T_{\alpha_0} + P(id_X - T_{\alpha_0}) \in \mathcal{A}(X)$. Then we have that

$$\begin{aligned} \|RS\| &= \|RT_{\alpha_0} + RP(id_X - T_{\alpha_0})\| \\ &\le \|R\| + \|R\|\|P(id_X - T_{\alpha_0})\| \\ &= \|R\| + \|R\|\|(T_{\alpha_0}^* - id_{X^*})P^*\| \\ &\le \|R\|(1 + \delta\|P\|) \\ &\le \lambda\|R\|, \end{aligned}$$

and for every $x \in F$

$$\|Sx - x\| = \|T_{\alpha_0}x - x + P(x - T_{\alpha_0}x)\| \leq \delta\|x\| + \delta\|P\|\|x\| \leq \varepsilon\|x\|.$$

Moreover, since for all $x^* \in Y^\perp$

$$S^*x^* = T_{\alpha_0}^*x^* + (id_{X^*} - T_{\alpha_0}^*)P^*x^* = x^*,$$

we see that $S(Y) \subset Y$. □

We need the following lemma to show Theorem 2.4(c)⇒(a).

Lemma 2.8 ([10, Theorem 5.2]). *Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{L}(X)$ and let $T \in \mathcal{L}(X)$. Then $T \in \overline{\mathcal{A}(X)}^{\tau_c}$ if and only if for every separable reflexive Banach space Z and every $R \in \mathcal{K}(Z, X)$, we have $TR \in \overline{\{SR : S \in \mathcal{A}(X)\}}^{\tau_c}$.*

Proof of Theorem 2.4(c)⇒(a). First we assert that for every Banach space Z and every $R \in \mathcal{W}(X, Z)$, for every finite-dimensional subspace F of X^* and for every $\varepsilon > 0$, there exists an $S \in \mathcal{A}(X)$ with $\|RS\| \leq (\lambda + \varepsilon)\|R\|$ such that $S^*x^* = x^*$ for all $x^* \in F$.

In order to show the assertion, by (c), there exists a finite-codimensional subspace Y of X with $Y \subset F_\perp := \{x \in X : x^*(x) = 0 \text{ for all } x^* \in F\}$ such that the pair (X, Y) has the weak λ-$\mathcal{A}$-BAP. Then $Y^\perp$ is finite-dimensional, $F \subset Y^\perp$, and we can choose a finite-dimensional subspace E of X such that $E^\perp \cap Y^\perp = \{0\}$.

Now there exists an $S \in \mathcal{A}(X)$ with $\|RS\| \leq (\lambda + \varepsilon)\|R\|$ such that $Sx = x$ for all $x \in E$ and $S(Y) \subset Y$. Then a simple verification shows that $S^*x^* - x^* \in E^\perp \cap Y^\perp = \{0\}$ for all $x^* \in Y^\perp$. Thus $S^*x^* = x^*$ for all $x^* \in F \subset Y^\perp$. This completes the assertion.

We now use Lemma 2.8 to show that $id_{X^*} \in \overline{\mathcal{A}^*(X)}^{\tau_c}$. Let Z be a reflexive Banach space and let $R \in \mathcal{K}(Z, X^*)$. Since Z is reflexive, there exists a $U \in \mathcal{K}(X, Z^*)$ such that $U^* = R$. Let K be a compact subset of Z and let $\varepsilon > 0$. Choose a $\delta > 0$ so that $(\delta + \delta(\lambda + \delta))\|U\| \leq \varepsilon$. Let $\{z_i\}_{i=1}^n$ be a δ-net for K. Applying the above assertion to $U \in \mathcal{K}(X, Z^*)$ and $F = \operatorname{span}\{Rz_i\}_{i=1}^n$, we obtain an $S \in \mathcal{A}(X)$ with $\|US\| \leq (\lambda + \delta)\|U\|$ such that $S^*Rz_i = Rz_i$ for all $i = 1, \ldots, n$. Then, for every $z \in K$, there exists a z_{i_0} such that $\|z - z_{i_0}\| \leq \delta$. Thus we have that

$$\begin{aligned}
\|Rz - S^*Rz\| &\leq \|Rz - Rz_{i_0}\| + \|Rz_{i_0} - S^*Rz\| \\
&\leq \|R\|\delta + \|S^*Rz_{i_0} - S^*Rz\| \\
&\leq \|R\|\delta + \delta\|US\| \\
&\leq \|U\|\delta + \delta(\lambda + \delta)\|U\| \leq \varepsilon.
\end{aligned}$$

Hence $R \in \overline{\{S^*R : S \in \mathcal{A}^*(X)\}}^{\tau_c}$. □

The proof of [4, Proposition 1.6] actually shows the following more general result.

Theorem 2.9. *Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{L}(X)$, which contains $\mathcal{F}(X)$. Then the following statements are equivalent:*

(a) $id_{X^*} \in \overline{\{S^* \in \mathcal{A}^*(X) : \|S\| \leq \lambda\}}^{\tau_c}$.

(b) *For every finite-codimensional subspace Y of X, the pair (X, Y) has the λ-$\mathcal{A}$-BAP.*

(c) *For every finite-codimensional subspace W of X, there exists a finite-codimensional subspace Y of X with $Y \subset W$ such that the pair (X, Y) has the λ-$\mathcal{A}$-BAP.*

3. A proof of Theorem 1.1

We say that a Banach space X has the weak λ-$\mathcal{A}$-BAP if the space $\mathcal{F}(X)$ in the definition of the weak λ-BAP can be replaced by a linear subspace $\mathcal{A}(X)$ of $\mathcal{L}(X)$. In view of Proposition 2.2, if $\mathcal{A}(X)$ contains $\mathcal{F}(X)$, then X has the weak λ-$\mathcal{A}$-BAP if and only if for every Banach space Z and every $R \in \mathcal{W}(X, Z)$, for every finite-dimensional subspace F of X and for every $\varepsilon > 0$, there exists an $S \in \mathcal{A}(X)$ with $\|RS\| \leq (\lambda + \varepsilon)\|R\|$ such that $Sx = x$ for all $x \in F$.

Proposition 3.1. *Suppose that $\mathcal{A} = \mathcal{F}$ or $\mathcal{K}$. Let Y be a finite-codimensional subspace of X. If Y has the weak λ-$\mathcal{A}$-BAP, then the pair (X, Y) has the weak 3λ-$\mathcal{A}$-BAP.*

We need the following lemma, which is contained in the proof of [4, Proposition 1.8], to show Proposition 3.1.

Lemma 3.2. *Let Y be a finite-codimensional subspace of X and let $\varepsilon > 0$. Then there exists a finite-dimensional subspace E of X such that the map $Q : E \oplus_1 Y \to X$ defined by $Q(e, y) = e + y$ is a $3(1 + \varepsilon)$-quotient operator.*

Proof of Proposition 3.1. Let Z be a Banach space and let $R \in \mathcal{W}(X, Z)$. Let F be a finite-dimensional subspace of X and let $\varepsilon > 0$. Choose a $\delta > 0$ so that $(\lambda + \delta)(3 + 3\delta) < 3\lambda + \varepsilon$.

Now, let E and Q be the objects in Lemma 3.2 for the $\delta > 0$. We may assume that $F \supset E$ and $F \cap Y \neq \{0\}$. Since Y has the weak λ-$\mathcal{A}$-BAP, there exists a $T \in \mathcal{A}(Y)$ with $\|R|_Y T\| \leq (\lambda + \delta)\|R\|$ such that $Ty = y$ for all $y \in F \cap Y$. In view of the proof of [4, Proposition 1.8], the map $S : X \to X$ defined by $Sx = e + Ty$, where $Q(e, y) = x$, is well defined, $Sx = x$ for all $x \in F$, and $S(Y) \subset Y$. Also it is easily seen that $S \in \mathcal{A}(X)$.

Now, let $x \in X$. Then by Lemma 3.2 there exist $e \in E$ and $y \in Y$ such that $Q(e, y) = x$ and $\|e\| + \|y\| \leq 3(1 + \delta)\|x\|$. We now have that

$$\begin{aligned}
\|RSx\| &= \|Re + RTy\| \\
&\leq \|R\|\|e\| + (\lambda + \delta)\|R\|\|y\| \\
&\leq (\lambda + \delta)\|R\|(\|e\| + \|y\|) \\
&\leq (\lambda + \delta)\|R\| 3(1 + \delta)\|x\| \\
&\leq (3\lambda + \varepsilon)\|R\|\|x\|.
\end{aligned}$$

Hence the pair (X, Y) has the weak 3λ-$\mathcal{A}$-BAP. □

Corollary 3.3. *Suppose that $\mathcal{A} = \mathcal{F}$ or $\mathcal{K}$. If $id_{X^*} \notin \overline{\mathcal{A}^*(X)}^{\tau_c}$, then for any $\lambda \geq 1$, there exists a finite-codimensional subspace Y_λ of X such that Y_λ does not have the weak λ-$\mathcal{A}$-BAP.*

Proof. Suppose that there exists a $\lambda \geq 1$ such that for every finite-codimensional subspace Y of X, Y has the weak λ-$\mathcal{A}$-BAP, and so the pair (X, Y) has the weak 3λ-$\mathcal{A}$-BAP by Proposition 3.1. It follows from Theorem 2.4 that $id_{X^*} \in \overline{\mathcal{A}^*(X)}^{\tau_c}$, which gives a contradiction. □

Proof of Theorem 1.1. Since X^* fails the AP, by Corollary 3.3, for each n, there exists a finite-codimensional subspace Y_n of X such that Y_n fails to have the weak n-BAP. Since each Y_n is complemented in X, it has the AP, and hence the desired space $Y = (\sum_n Y_n)_{c_0}$ (resp. $(\sum_n Y_n)_{\ell_p}$ $(1 \le p < \infty)$) has the AP (cf. [1, Proposition 2.14]). But Y fails to have the weak BAP. □

A Banach space X is said to have the *compact approximation property* (CAP) if $id_X \in \overline{\mathcal{K}(X)}^{\tau_c}$. It is well known that the CAP is different from the AP (see [16]). The same proof of Theorem 1.1 yields the following:

Theorem 3.4. *If X has the CAP but $id_{X^*} \notin \overline{\mathcal{K}^*(X)}^{\tau_c}$, then there exists a subspace Y of $c_0(X)$ (respectively, $\ell_p(X)$ $(1 \le p < \infty)$) such that Y has the CAP but fails the weak $\mathcal{K}$-BAP.*

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