AN APPLICATION OF MACAULAY'S ESTIMATE TO SUMS OF SQUARES PROBLEMS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. Several questions in complex analysis lead naturally to the study of bihomogeneous polynomials $r(z, \bar{z})$ on $\mathbb{C}^n \times \mathbb{C}^n$ for which $r(z, \bar{z}) \|z\|^{2d} = \|h(z)\|^2$ for some natural number d and a holomorphic polynomial mapping $h = (h_1, \ldots, h_K)$ from $\mathbb{C}^n$ to $\mathbb{C}^K$. When r has this property for some d, one seeks relationships between d, K, and the signature and rank of the coefficient matrix of r. In this paper, we reformulate this basic question as a question about the growth of the Hilbert function of a homogeneous ideal in $\mathbb{C}[z_1, \ldots, z_n]$ and apply a well-known result of Macaulay to estimate some natural quantities.

1. INTRODUCTION

This paper explores the use of commutative algebra in the study of Hermitian sums of squares problems in several complex variables. Suppose $r(z, \bar{z})$ is a bihomogeneous polynomial of bi-degree (m, m) on $\mathbb{C}^n \times \mathbb{C}^n$. It is a well-known result of Quillen [Qui68], proved independently by Catlin and D'Angelo [CD97], that if r is positive on the unit sphere S^{2n-1}, then there exists a natural number d such that $r(z, \bar{z}) \|z\|^{2d}$ can be written as a sum of squares; that is,

$$(1) \qquad r(z, \bar{z}) \|z\|^{2d} = \sum_{k=1}^{K} |h_k(z)|^2 = \|h(z)\|^2$$

for holomorphic polynomials $h_k \colon \mathbb{C}^n \to \mathbb{C}$. See [CD99] for an interpretation of this result in terms of holomorphic line bundles and see [D'A11] for an interpretation as a Hermitian analogue of Hilbert's 17th problem. Recent investigations ([DZ13], [TY06]) have sought results linking the minimum d for which $r(z, \bar{z}) \|z\|^{2d}$ is a squared norm to properties of r. This minimum d must depend on the size of r on the sphere and not just on the dimension n and the bi-degree (m, m), as the example

$$(|z_1|^2 - |z_2|^2)^2 + \varepsilon |z_1 z_2|^2$$

shows; in this example, the minimum d tends to infinity as ε tends to 0.

Our approach is algebraic. Given a bihomogeneous r with signature (P, N) positive on the sphere, there is a d for which (1) holds. We seek to understand the relationship among the dimension n, the bi-degree (m, m), the natural number d, and the signature (P, N). The inequalities we obtain can be viewed in several ways;

they say, for a fixed r and d for which (1) holds, what inequality the signature (P, N) must satisfy. Alternatively, the inequalities give, for a bihomogeneous r positive on the sphere with signature (P, N), a lower bound on D_d, and hence on the d for which (1) holds.

We reformulate our basic question as a question about the growth of the Hilbert function of a homogeneous ideal in the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$, and we use a well-known estimate of Macaulay (Theorem 1 below). Ours is not the first paper to apply such results from commutative algebra to questions in several complex variables; Grundmeier, Lebl, and Vivas use a similar set of ideas in [GLV11] to prove a rigidity theorem for CR mappings of hyperquadrics. Our goal is not primarily to obtain specific inequalities but rather to illustrate how a set of ideas from this area of algebra can be brought to bear on questions arising in several complex variables. We end the paper with a section exploring more fully these connections with several complex variables and CR geometry.

2. DEFINITIONS AND A KEY LEMMA

Let r be a real-valued bihomogeneous polynomial of bi-degree (m, m) on $\mathbb{C}^n \times \mathbb{C}^n$; thus

$$r(z, \bar{z}) = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^\alpha \bar{z}^\beta$$

for some Hermitian matrix $(c_{\alpha\beta})$. Each such polynomial has a holomorphic decomposition

$$(2) \qquad r(z, \bar{z}) = \|f(z)\|^2 - \|g(z)\|^2 = \sum_{j=1}^{P} |f_j(z)|^2 - \sum_{k=1}^{N} |g_k(z)|^2$$

where the f_j, g_k, are holomorphic homogeneous polynomials of degree m and the set $\{ f_j, g_k : 1 \le j \le P,\ 1 \le k \le N \}$ is linearly independent. This decomposition is obtained by diagonalizing the coefficient matrix $(c_{\alpha\beta})$. Thus, although the polynomials appearing in the holomorphic decomposition of a given r are not uniquely determined, the signature pair (P, N) and the rank $R = P + N$ are. Observe that r has signature pair $(P, 0)$ if and only if it is itself a squared norm. The idea of studying real polynomials on $\mathbb{C}^n$ through Hermitian linear algebra has been used extensively by D'Angelo; see Chapter VI in [D'A02] and [D'A93].

Our results concern the dimensions of various vector spaces naturally associated with r. Let $\mathcal{P}_k$ be the space of homogeneous polynomials of degree k in $z_1, \ldots, z_n$. Let $D_k = \binom{k+n-1}{k}$, which is the dimension of this space. For any mapping $p = (p_1, \ldots, p_L) \colon \mathbb{C}^n \to \mathbb{C}^L$ with components in $\mathcal{P}_k$, let V_p be the subspace of $\mathcal{P}_k$ spanned by the components of p. If $q \colon \mathbb{C}^n \to \mathbb{C}^{L'}$ is a second mapping, then the mappings $p \oplus q$ and $p \otimes q$ satisfy $\|p \oplus q\|^2 = \|p\|^2 + \|q\|^2$ and $\|p \otimes q\|^2 = \|p\|^2 \|q\|^2$. We can think of z as the mapping whose components are the coordinate functions $z_1, \ldots, z_n$; Thus $z^{\otimes d}$ is the tensor product of this mapping with itself d times.

The following lemma, which appears in a more general form in [D'A05], is used heavily.

Lemma 1. *Let r be bihomogeneous of bi-degree (m, m) with $r(z, \bar{z}) \|z\|^{2d} = \|h(z)\|^2$. Let $F = f \otimes z^{\otimes d}$ and $G = g \otimes z^{\otimes d}$. Then $V_G \subseteq V_F$.*

Proof. Note that

$$(3) \qquad r(z, \bar{z}) \|z\|^{2d} = (\|f\|^2 - \|g\|^2) \|z\|^{2d} = \left\| f \otimes z^{\otimes d} \right\|^2 - \left\| g \otimes z^{\otimes d} \right\|^2.$$

Because $r(z, \bar{z}) \|z\|^{2d} = \|h(z)\|^2$,

$$(4) \qquad \left\| f \otimes z^{\otimes d} \right\|^2 = \left\| g \otimes z^{\otimes d} \right\|^2 + \|h\|^2 = \left\| (g \otimes z^{\otimes d}) \oplus h \right\|^2,$$

that is, $\|F(z)\|^2 = \|(G \oplus h)(z)\|^2$. Let L be a natural number such that the rank of F and the rank of $G \oplus h$ are at most L. By adding identically zero components as needed, we obtain two mappings $F \oplus 0$ and $G \oplus h \oplus 0$ from $\mathbb{C}^n$ to $\mathbb{C}^L$ with the same squared norm. Thus there exists a unitary transformation U on $\mathbb{C}^L$ such that $U(F \oplus 0) = (G \oplus h \oplus 0)$. It follows that the components of the mapping G are in the linear span of the components of the mapping F. $\qquad \square$

To this point, for a mapping h whose components are homogeneous of degree m, we have simply considered the vector subspace V_h of $\mathcal{P}_m$ spanned by the components of h. We change our perspective somewhat; we think of $S = \mathbb{C}[z_1, \ldots, z_n]$ as a graded ring, graded by degree. We let I_h as the homogeneous ideal generated by the components of h. I_h is a graded S-module, with its degree k component denoted by $(I_h)_k$. In particular, its first non-trivial component, $(I_h)_m$, is the vector space V_h discussed above. The *Hilbert function* of I_h is

$$H_{I_h}(k) = \dim_{\mathbb{C}}(I_h)_k.$$

We will make use of Macaulay's estimate on the growth of a homogeneous ideal. Macaulay's result gives an upper bound for the Hilbert function for S/I_h and hence a lower bound for the Hilbert function for I_h.

Definition 1. Let c and ν be positive integers. The ν-th Macaulay representation of c is the (unique) way of writing

$$(5) \qquad c = \binom{k_\nu}{\nu} + \binom{k_{\nu-1}}{\nu-1} + \ldots + \binom{k_J}{J}$$

where $k_\nu > k_{\nu-1} > \ldots > k_J \geq J > 0$. We also write

$$(6) \qquad c^{<\nu>} = \binom{k_\nu + 1}{\nu + 1} + \binom{k_{\nu-1} + 1}{\nu} + \ldots + \binom{k_J + 1}{J + 1}.$$

See [Gre10] and [Gre89] for a more extensive discussion of these ideas, including proofs of the uniqueness of the ν-th Macaulay representation of a positive integer and of the elementary properties of the function $c \mapsto c^{<\nu>}$. We need only two such properties:

Lemma 2. *Let b, c, and ν be positive integers.*
  (1) *If $b < c$, then $b^{<\nu>} < c^{<\nu>}$.*
  (2) *For any $k \in \mathbb{N}$ with $k < c$, $(c - k)^{<\nu>} \leq c^{<\nu>} - k$.*

Theorem 1 (Macaulay's estimate on the growth of an ideal [Mac27])**.** *Let I be an ideal in $S = \mathbb{C}[z_1, \ldots, z_n]$ whose generators are homogeneous polynomials (not necessarily all of the same degree). Then*

$$(7) \qquad H_{S/I}(k + 1) \leq H_{S/I}(k)^{<k>}.$$

Elementary linear algebra shows that

$$(8) \qquad H_{S/I}(k) + H_I(k) = \binom{k+n-1}{k} = D_k.$$

3. The main theorem and its corollaries

In this section, we establish our main result concerning signatures of real polynomials on $\mathbb{C}^n$. See [HL13] for a discussion of connections of this problem to CR geometry and for constructions of families of polynomials illustrating the extent to which some such inequalities are sharp. Additional comments on the sharpness of these inequalities appear at the end of the section.

Theorem 2. *Let r be bihomogeneous of bi-degree (m, m) with signature (P, N) and rank R. Suppose $r(z, \bar{z}) \|z\|^{2d}$ is a squared norm. Then*

$$(9) \qquad P \geq \frac{D_{m+d} - [[H(m)^{<m>}]^{<m+1>} \cdots]^{<m+d-1>}}{D_d}$$

for $H(m) = D_m - R$. If the m-th Macaulay representation of $D_m - R$ is $\sum_{j=J}^{m} \binom{k_j}{j}$, this is equivalent to

$$(10) \qquad P \geq \frac{D_{m+d} - \sum_{j=J}^{m} \binom{k_j+d}{j+d}}{D_d}.$$

Proof. We obtain upper and lower bounds for $H_{I_f}(m + d)$, which equals $H_{I_{f \oplus g}}(m + d)$ since, by Lemma 1, the components of I_f and $I_{f \oplus g}$ in degree $m + d$ are the same. Since $V_f = (I_f)_m$ has as a basis the P generators f_j of the ideal I_f, $(I_f)_{m+d}$ is spanned by the polynomials $z^\alpha f_j$ for $|\alpha| = d$. We thus have the upper bound

$$H_{I_f}(m + d) \leq D_d P.$$

We use Theorem 1 to obtain an upper bound on $H_{S/I_{f \oplus g}}(m + d)$. To simplify notation, we drop the subscript on H when discussing the Hilbert function for $S/I_{f \oplus g}$. Since $H_{I_{f \oplus g}}(m) = R$,

$$(11) \qquad H(m) = D_m - R.$$

Suppose the m-th Macaulay representation of $D_m - R$ is $\sum_{j=J}^{m} \binom{k_j}{j}$. According to Macaulay's estimate,

$$(12) \qquad H(m + 1) \leq H(m)^{<m>} = \sum_{j=J}^{m} \binom{k_j + 1}{j + 1}.$$

Applying the result again and using Lemma 2 gives
(13)

$$H(m + 2) \leq H(m + 1)^{<m+1>} \leq [H(m)^{<m>}]^{<m+1>} = \left(\sum_{j=J}^{m} \binom{k_j + 1}{j + 1} \right)^{<m+1>}.$$

Since the expression in parentheses is already in the form of the $(m+1)$st Macaulay representation of an integer, the full expression on the right can be easily evaluated and we obtain:

$$(14) \qquad H(m + 2) \leq \sum_{j=J}^{m} \binom{k_j + 2}{j + 2}.$$

Iterating gives

$$(15) \qquad H(m+d) \leq [[H(m)^{<m>}]^{<m+1>} \cdots]^{<m+d-1>} = \sum_{j=J}^{m} \binom{k_j + d}{j + d}$$

Since $H_{I_{f \oplus g}}(m+d) = D_{m+d} - H(m+d)$, the result follows. $\square$

We give two corollaries. In both cases, a simple direct proof also exists that does not require Macaulay's estimate. We sketch both arguments for Corollary 2. Furthermore, since we always use the upper bound

$$H_{I_f}(m+d) \leq D_d P$$

when f has P components, we need only describe how we obtain the lower bounds.

Corollary 1. *Let r be bihomogeneous of bi-degree (m, m) with signature pair (P, N) and rank R. If $r(z, \bar{z}) \|z\|^{2d}$ is a squared norm, then*

$$(16) \qquad\qquad N \leq (P-1)(D_d - 1).$$

Proof. First, observe that

$$(17) \qquad\qquad D_m = \sum_{k=0}^{m} \binom{k+n-2}{k}.$$

This follows by counting the monomials of degree m in n variables in two ways. On the one hand, this number is D_m. On the other hand, we may count the monomials of degree m involving x_1^{m-k} for $0 \leq k \leq m$, which gives $\binom{k+n-2}{k}$. Sum over k.

By Theorem 2,

$$D_d P \geq D_{m+d} - [[[D_m - R]^{<m>}]^{<m+1>} \cdots]^{<m+d-1>}.$$

By statement (2) in Lemma 2 and the above expression for D_m,

$$D_{m+d} - [[[D_m - R]^{<m>}]^{<m+1>} \cdots]^{<m+d-1>}$$
$$\geq \quad D_{m+d} - [[[D_m - 1]^{<m>}]^{<m+1>} \cdots]^{<m+d-1>} + R - 1$$
$$= \quad \sum_{k=0}^{m+d} \binom{k+n-2}{k} - \left[\left[\left[\sum_{k=1}^{m} \binom{k+n-2}{k} \right]^{<m>} \right]^{<m+1>} \cdots \right]^{<m+d-1>} + R - 1$$
$$= \quad \sum_{k=0}^{m+d} \binom{k+n-2}{k} - \sum_{k=1}^{m} \binom{k+n-2+d}{k+d} + R - 1$$
$$= \quad \sum_{k=0}^{d} \binom{k+n-2}{k} + R - 1$$
$$= \quad D_d + R - 1.$$

Thus

$$\frac{P}{R} \geq \frac{D_d + R - 1}{D_d R},$$

which is equivalent to

$$N \leq (P-1)(D_d - 1).$$

$\square$

Remark 1. The inequality in Corollary 1 immediately implies $N \leq P(D_d - 1)$. The latter appears in Theorem 1.1(i) in [HL13]. There, the proof given uses Hermitian linear algebra; although that proof is elementary, it is not straight-forward. On the other hand, the above proof uses a simple argument to prove a much stronger result, and hence illustrates the value of bringing techniques from commutative algebra to bear on such problems.

Corollary 2. *Let* r *be bihomogeneous of bi-degree* (m, m) *with* $r(z, \bar{z}) \, \|z\|^{2d}$ *a squared norm and* $R = D_m$ *(which is the largest rank possible for this* m *and* n*). Then*

$$(18) \qquad\qquad P \geq \frac{D_{m+d}}{D_d}.$$

Remark 2. Suppose $n = 2$. Then (18) gives $P/R \geq (m + d + 1)/(d + 1)(m + 1)$. See [HL13] for an example of a polynomial in two variables for which this ratio is achieved.

Proof. To obtain a proof using Theorem 2, we need only observe that $D_m - R = 0$ and so the numerator of (9) reduces to D_{m+d}.

Next we give a direct proof for the lower bound on $H_{I_f}(m + d) = H_{I_{f \oplus g}}(m + d)$. If the rank of r is $D_m = \dim \mathcal{P}_m$, then in fact the components of $f \oplus g$ span $\mathcal{P}_m$. Thus the components of $F \oplus G$ span $\mathcal{P}_{m+d}$, i.e.,

$$H_{I_{f \oplus g}}(m + d) = D_{m+d}.$$

Then

$$P \geq \frac{D_{m+d}}{D_d}.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We examine our method of proof more closely: We obtain the upper bound $H_{I_f}(m + d) \leq D_d P$ because the $D_d P$ functions $z^\alpha f_j$ for $1 \leq j \leq P$ and $|\alpha| = d$ span $(I_f)_{m+d}$. We thus have equality if and only if these functions are linearly independent. On the other hand, we obtain our lower bound for $H_{I_{f \oplus g}}(m+d)$ using Macaulay's theorem. It is considerably more difficult to say when equality holds here. (See Green [Gre10] for a more detailed discussion.) We make a definition:

Definition 2. Let I be a monomial ideal. I is a **lex segment ideal** in degree s if I_s is spanned by the first $\dim(I_s)$ monomials of degree s in lexicographic order.

The following results are known.

Proposition 1 (Macaulay). *If* I *is a lex segment ideal in degree* s *and has no generators in degree* $s + 1$, $H_{S/I}(s + 1) = H_{S/I}(s)^{<s>}$.

Theorem 3 (Gotzmann's Persistence Theorem). *Let* I *be a homogeneous ideal generated in degree* $\leq s$. *If* $H_{S/I}(s+1) = H_{S/I}(s)^{<s>}$, *then* $H_{S/I}(k+1) = H_{S/I}(k)^{<k>}$ *for all* $k \geq s$.

When the R generators of a monomial ideal I are all of the same degree m, as is the case with all of our ideals, I is a lex segment ideal in all degrees if and only if the generators are the first R monomials of degree m in lexicographic order. Thus we have already encountered a simple lex segment ideal in Corollary 2.

With lexicographic order on the monomials and for a homogeneous ideal I, let $in(I)$ be its initial ideal. This is a monomial ideal. Then I and $in(I)$ have the same

Hilbert function in all degrees. We immediately obtain the following proposition describing a hypothesis under which we obtain an equality upon application of Macaulay's theorem.

Proposition 2. *Let* $r = \|f\|^2 - \|g\|^2$ *be as in Theorem 2. Suppose* $\text{in}(I_{f\oplus g})$ *is a lex segment ideal. Then*

(19) $$H_{I_{f\oplus g}}(m + d) = D_{m+d} - [[H(m)^{<m>}]^{<m+1>} \ldots]^{<m+d-1>}.$$

We now consider whether Corollaries 1 and 2 are sharp for all n, d, and m. In Corollary 2, $I_{f\oplus g}$ is a lex segment ideal and $H_{I_{f\oplus g}}(m + d) = D_{m+d}$, i.e., we have an equality rather than merely a lower bound. Thus we ask whether we can have equality in the upper bound $H_{I_{f\oplus g}}(m + d) \leq D_d P$. In the monomial case the question is whether there are D_{m+d}/D_d monomials x^β such that the D_{m+d} monomials $x^{\alpha+\beta}$ for $|\alpha| = d$ are distinct. When $n = 2$, such a collection of monomials sometimes exists. Indeed, take d a natural number and consider $m = k(d + 1)$ for some natural number k. Then the $k + 1$ monomials $x^{m-j(d+1)}y^{j(d+1)}$, $0 \leq j \leq k$ have this property.

On the other hand, for $n > 2$, even when $d = 1$, it is not possible to find a collection of monomials $A = \{x^\beta : \beta \in \mathcal{B}\}$ such that the $3|A|$ monomials $x^{\beta+\mathbf{e}^j}$ for $\beta \in \mathcal{B}$ and $1 \leq j \leq 3\}$ are independent and span the space of homogeneous polynomials of degree $m + 1$. Indeed, let $m \geq 2$ be arbitrary (the case $m = 1$ is trivial). Since x_1^{m+1} must be one of the $x^{\beta+\mathbf{e}^j}$, necessarily $x_1^m \in A$. Next, one of $x_1^{m-1}x_2$ and $x_1^{m-2}x_2^2$ is in A. If it were the former, $x_1^m x_2$ would arise from two different monomials in A. Thus we suppose it is $x_1^{m-2}x_2^2$. Similarly, suppose $x_1^{m-2}x_3^2$. None of these three monomials gives rise to $x_1^{m-1}x_2 x_3$. Thus A must contain one of $x_1^{m-1}x_2$, $x_1^{m-1}x_3$, or $x_1^{m-2}x_2, x_3$. The first two possibilities have already been eliminated. But if $x_1^{m-2}x_2 x_3$ is in A, $x_1^{m-2}x_2^2 x_3$ comes from both $x_1^{m-2}x_2 x_3$ and $x_1^{m-2}x_2^2$ in A.

To summarize the above discussion, we see that even in the case in which the components of f and g are monomials, if $n > 2$ the inequality in Corollary 2 is not sharp.

Next we consider Corollary 1, which implies

(20) $$\frac{N}{P} \leq D_d - 1.$$

In [HL13], it is shown that this inequality is essentially sharp for $d = 1$ for all n. This is not inconsistent with the above discussion; the family of polynomials constructed there has several interesting properties.

(1) $I_{f\oplus g}$ is a monomial ideal and includes all monomials of degree m in n variables. Thus it is a lex segment ideal and $H_{I_{f\oplus g}}(m + d) = D_{m+d}$.

(2) Consider the set B of all components x^β of f for which $\beta_j \geq 1$ for all $1 \leq j \leq n$. The monomials $x^{\beta+\mathbf{e}^k}$ for $x^\beta \in A$ and $1 \leq k \leq n$ are all distinct.

Thus although there are components of f giving rise to the same monomial of degree $m + 1$, the proportion of such components of f goes to zero as m increases.

4. Polynomials with small rank

Theorem 2 and its corollaries give lower bounds on P/R, and hence upper bounds on N/P. In a sense, these results say which signatures are possible if $r(z, \bar{z}) \|z\|^{2d}$ is

a squared norm and r has large rank; in [HL13], those polynomials for which P/R is close to $1/D_d$ have large degree and rank. By contrast, our next two propositions explore the situation in which $r(z, \bar{z}) \|z\|^{2d}$ is a squared norm but the rank of r and the ratio N/P are small.

Proposition 3. *If r is the squared norm of a holomorphic homogeneous polynomial mapping, then either the rank K of $r(z, \bar{z}) \|z\|^{2d}$ satisfies*

$$(21) \qquad nP - \frac{P(P-1)}{2} \le K \le nP$$

for some $P \in \mathbb{N}$ with $P < n$, or $K \ge n(n+1)/2$.

Remark 3. In other words, for a fixed dimension n, if r is a squared norm, not all ranks of $r(z, \bar{z}) \|z\|^{2d}$ are possible; the smallest possible (non-zero) rank is n, and the next smallest are $2n - 1$ and $2n$. For example, when $n = 3$, ranks 1, 2, and 4 are not possible.

Proof. If r is a squared norm, so is $\tilde{r}(z, \bar{z}) = r(z, \bar{z}) \|z\|^{2(d-1)}$. Thus by replacing r by $\tilde{r}$ if necessary, it suffices to prove the result for $d = 1$.

Suppose r is the squared norm of the homogeneous holomorphic mapping $f = (f_1, \ldots, f_P)$ so that

$$(22) \qquad H_{I_f}(m + 1) \le nP.$$

We use Macaulay's estimate to obtain a lower bound.

Suppose first that $P < n$. Then

$$
\begin{aligned}
(D_m - P)^{<m>} &= \left(\sum_{k=1}^{m} \binom{k+n-2}{k} + 1 - P \right)^{<m>} \\
&= \left(\sum_{k=2}^{m} \binom{k+n-2}{k} + \binom{n-P}{1} \right)^{<m>} \\
&= \sum_{k=2}^{m} \binom{k+n-1}{k+1} + \binom{n-P+1}{2} \\
&= \sum_{k=1}^{m+1} \binom{k+n-2}{k} + \binom{n-P+1}{2} - (n-1) - \binom{n}{2} \\
&= D_{m+1} - \left(\frac{n^2 + n}{2} - \frac{(n-P+1)(n-P)}{2} \right).
\end{aligned}
$$

We obtain the lower bound

$$(23) \qquad H_{I_f}(m+1) \ge \frac{n^2+n}{2} - \frac{(n-P+1)(n-P)}{2} = nP - \frac{P(P-1)}{2}.$$

Thus for $P < n$,

$$nP - \frac{P(P-1)}{2} \le K \le nP.$$

Next suppose $P \ge n$. Then

$$H_{S/I_f}(m+1) \le (H_{S/I_f}(m))^{<m>} \le (D_m - n)^{<m>} = D_{m+1} - \frac{n(n+1)}{2},$$

and $H_{I_f}(m+1) \ge n(n+1)/2$. $\qquad\square$

Finally, we consider r of signature $(P, 1)$. In [HL13], the authors prove that if, in the holomorphic decomposition of such an r, the components of f and g are monomials, then $P \geq n$. Furthermore, it is clear that signature pair $(n, 1)$ is possible.

We consider this question in the general situation in which the components of f and g are arbitrary homogeneous polynomials of degree m. We use Macaulay's estimate to show that certain values for $(I_f)_k$ and $(I_{f \oplus g})_k$ are not possible if these spaces are known to agree for $k \geq m + 1$.

Proposition 4. *Let r be bihomogeneous of bi-degree (m, m) with $r(z, \bar{z}) \|z\|^2$ a squared norm. If $N = 1$, then*

$$(24) \qquad\qquad P^2 + P \geq 2n.$$

Proof. Since $H_{I_f}(m) = P$ and $H_{I_{f \oplus g}}(m) = P + 1$, $H_{S/I_{f \oplus g}}(m) = D_m - (P + 1)$.

We know that it is possible to take $P = n$. Thus our concern in this proposition is whether the Macaulay estimates themselves rule out the possibility of smaller P. Thus we consider $P < n$ and compare upper and lower bounds for $H_{S/I_{f \oplus g}}(m + 1)$. We expect to see that if P is too small, the lower bound given by the Macaulay estimates will be larger than our known upper bound.

To begin, we need the m-th Macaulay representation of $D_m - (P + 1)$, where $P < n$. We begin as in the previous proposition, writing

$$(D_m - (P+1))^{<m>} = \left(\sum_{k=1}^{m} \binom{k+n-2}{k} - P \right)^{<m>}$$

$$(25) \qquad = \left(\sum_{k=2}^{m} \binom{k+n-2}{k} + \binom{n-1-P}{1} \right)^{<m>}.$$

We consider two cases. First, suppose $P = n - 1$. Then

$$(25) = \sum_{k=2}^{m} \binom{k+n-1}{k+1}$$

$$= \sum_{k=1}^{m+1} \binom{k+n-2}{k} - \binom{n}{2} - \binom{n-1}{1}$$

$$= D_{m+1} - \frac{n(n+1)}{2}$$

$$= D_{m+1} - \frac{(P+1)(P+2)}{2}.$$

For the second case, suppose $P < n - 1$. We obtain

$$(25) = \sum_{k=2}^{m} \binom{k+n-1}{k+1} + \binom{n-P}{2}$$

$$= D_{m+1} - \left(n + nP - \frac{P^2 + P}{2} \right).$$

Since for $P = n - 1$

$$n + nP - \frac{P^2 + P}{2} = \frac{(P+1)(P+2)}{2},$$

for any $P < n$ we obtain the lower bound

$$(26) \qquad H_{S/I_{f \oplus g}}(m+1) = n + nP - \frac{P^2 + P}{2}.$$

As always, we trivially have the upper bound

$$H_{I_{f \oplus g}}(m+1) \le nP.$$

Thus necessarily

$$(27) \qquad n + nP - \frac{P(P+1)}{2} \le nP.$$

Thus we require

$$2n \le P^2 + P.$$

$\square$

The two propositions of this section illustrate what information about bihomogeneous polynomials can be gained through the use of this algebraic tool; rather than establishing the existence of bihomogeneous polynomials with certain algebraic properties, Macaulay's estimate allows us to show that certain scenarios are impossible because of necessary relationships between the dimensions of various vector spaces naturally associated with a polynomial.

5. Connections with mapping problems in CR geometry

Hermitian sums of squares arise in many places in several complex variables. In this section, we describe connections with CR geometry.

Given two real hypersurfaces in $\mathbb{C}^n$, a fundamental problem is to find and classify all CR mappings between them. A natural starting point is to study CR mappings between spheres. Thus if $h \colon S^{2n-1} \to S^{2N-1}$ is a CR mapping,

$$(28) \qquad ||h(z)||^2 - 1 = S(z, \bar{z})(||z||^2 - 1),$$

and hence for CR mappings of spheres, Hermitian sums of squares naturally enter the problem. Huang [Hua99] showed that rank $(S(z, \bar{z})||z||^2) \ge n$. This result, known as "Huang's Lemma," is then used in many places in CR geometry (see for example [EHZ05], [HJY13], and [Ebe13] and the references therein). In particular, Huang [Hua99] used this lemma to prove that if $f \colon S^{2n-1} \to S^{2N-1}$ is a $\mathcal{C}^2$-smooth CR mapping and $N < 2n - 1$, $n > 1$, then f is equivalent to the linear map $z \mapsto (z, 0)$.

A natural generalization is to allow negative eigenvalues in the defining equations; namely, we consider the hyperquadrics given by

$$(29) \qquad Q(P, N) = \left\{ z \in \mathbb{C}^{P+N} : \sum_{j=1}^{P} |z_j|^2 - \sum_{j=P+1}^{P+N} |z_j|^2 = 1 \right\}.$$

Again many results are known in this situation. In particular, the first author [Gru11] studied the possible target hyperquadrics of CR mappings from a sphere invariant under a group action. In this setting, one again naturally studies Hermitian sums of squares; however, we keep track of the signature pair instead of just the rank. That is, one studies the signature pairs of $S(z, \bar{z})(||z||^2 - 1)$ for particular $S(z, \bar{z})$.

The work of Catlin and D'Angelo [CD96] gives another connection between CR geometry and the sorts of sums of squares problems considered here. Let B_n denote

the unit ball in $\mathbb{C}^n$, and suppose $f = \frac{p}{q} \colon \mathbb{C}^n \to \mathbb{C}^N$ is a rational mapping such that $f(\overline{B_n}) \subseteq B_N$. Catlin and D'Angelo show that it is a consequence of (1) that one can find a natural number K and a polynomial mapping $g \colon \mathbb{C}^n \to \mathbb{C}^K$ such that $\frac{p \oplus g}{q}$ is a CR mapping from S^{2n-1} to $S^{2(N+K)-1}$. Thus the result of Quillen-Catlin-D'Angelo gives a way to arbitrarily assign many of the components of CR mappings between spheres. In order to understand the relationships among the degree, the domain dimension, and the target dimension for CR mappings constructed in this manner, it is essential in (1) to understand the relationships among the bi-degree and signature of r, the natural number d, the dimension n, and the rank K of the Hermitian squared norm. Some of these quantities certainly depend on analytic properties of r, as we pointed out in the introduction. Our purpose here has been to understand which relationships are purely algebraic.

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