

## MINIMALLY ALMOST PERIODIC GROUP TOPOLOGIES ON COUNTABLY INFINITE ABELIAN GROUPS

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ABSTRACT. We obtain a positive answer to a question by Comfort: Every countable Abelian group  $G$  of infinite exponent admits a complete Hausdorff minimally almost periodic group topology.

### 1. INTRODUCTION

For an Abelian topological group  $G$ ,  $G^\wedge$  denotes the group of all continuous characters on  $G$  endowed with the compact-open topology. Denote by  $\mathbf{n}(G) = \bigcap_{\chi \in G^\wedge} \ker \chi$  the von Neumann radical of  $G$ . Following von Neumann [21], the group  $G$  is called *minimally almost periodic* (MinAP) if  $\mathbf{n}(G) = G$ , and  $G$  is called *maximally almost periodic* (MAP) if  $\mathbf{n}(G) = 0$ .

Let  $G$  be an infinite Abelian group. Denote by  $\mathcal{NR}(G)$  the set of all subgroups  $H$  of  $G$  for which there exists a non-discrete Hausdorff group topology  $\tau$  on  $G$  such that the von Neumann radical of  $(G, \tau)$  is  $H$ , i.e.,  $\mathbf{n}(G, \tau) = H$ . Let  $\mathcal{NRC}(G) \subseteq \mathcal{NR}(G)$  be the set of all subgroups  $H$  of  $G$  for which there exists a topology as above which is also complete.

Let  $G$  be an Abelian topological group. The richness of its dual group  $G^\wedge$  is one of the most important properties of  $G$  which plays a crucial role in Harmonic Analysis (see [19, 20]). The von Neumann radical measures this richness, thus the following general question is important:

**Question 1.1** ([12, Problem 2]). Describe the sets  $\mathcal{NR}(G)$  and  $\mathcal{NRC}(G)$ .

One can show (see [14, Theorem 1]) that every infinite Abelian group admits a complete non-trivial Hausdorff group topology with trivial von Neumann radical. Thus, the trivial group  $\{0\}$  belongs to  $\mathcal{NRC}(G)$  for every infinite Abelian group  $G$ , i.e.,  $\mathcal{NRC}(G)$  is never empty. A much deeper question is whether any infinite Abelian group admits a Hausdorff group topology with *non-zero* von Neumann radical. The positive answer was given by Ajtai, Havas and Komlós [1]. Using the method of  $T$ -sequences, Protasov and Zelenyuk [26] strengthened their result as follows: every infinite Abelian group admits a *complete sequential* Hausdorff group topology for which characters do not separate points. So  $\mathcal{NRC}(G) \neq \{\{0\}\}$  for every infinite Abelian group  $G$ .

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Question 1.1 has two interesting extreme cases. Denote by  $\mathcal{S}(G)$  the set of all subgroups of an infinite Abelian group  $G$ .

**Question 1.2** ([12, Problem 4]). Describe all infinite Abelian groups  $G$  such that  $\mathcal{NR}(G) = \mathcal{S}(G)$  (respectively,  $\mathcal{NRC}(G) = \mathcal{S}(G)$ ).

The second and, perhaps, the most interesting special case of Question 1.1 is the following:

**Question 1.3** ([12, Problem 5]). Describe all infinite Abelian groups  $G$  such that  $G \in \mathcal{NR}(G)$  (or  $G \in \mathcal{NRC}(G)$ ).

Note that the existence of Abelian groups admitting no non-zero continuous characters has been known for a long time. The classic example of linear topological spaces that (when viewed as Abelian topological groups) have no non-trivial continuous characters was given by Day in [7]. There are other examples in [19, 23, 32]. Nienhuys [22] showed the existence of a metric solenoidal monothetic MinAP group (see also [23]). Prodanov [24] gave an elementary example of a MinAP group.

Recall that an Abelian group  $G$  is of *finite exponent* or *bounded* if there exists a positive integer  $n$  such that  $ng = 0$  for every  $g \in G$ . The minimal integer  $n$  with this property is called the *exponent* of  $G$  and is denoted by  $\exp(G)$ . When  $G$  is not bounded, we write  $\exp(G) = \infty$  and say that  $G$  is of *infinite exponent* or *unbounded*.

Protasov [25] posed the question of whether every infinite Abelian group admits a minimally almost periodic group topology. Using a result of Graev [18], Remus [28] proved that for every natural number  $n$  there exists a connected MinAP group which is algebraically generated by elements of order  $n$ . On the other hand, he gave (see [6]) a simple example of a group  $G$  of finite exponent which does not admit any Hausdorff group topology  $\tau$  such that  $(G, \tau)$  is minimally almost periodic. So, for groups of finite exponent the answer to Protasov's question is negative. This justifies the following problems:

**Question 1.4** ([6, Problem 521]). Does every Abelian group which is not of bounded exponent admit a minimally almost periodic topological group topology? What about the countable case?

**Question 1.5** ([27, Question 2.6.1]). Let  $G$  be a torsion free countable Abelian group. Does there exist a Hausdorff group topology on  $G$  with only zero character?

The main goal of the article is to give the positive answer to Comfort's Question 1.4 for the countable case and, hence, to Protasov-Zelenyuk's Question 1.5.

**Theorem 1.6.** *Every countable Abelian group of infinite exponent admits a complete sequential Hausdorff minimally almost periodic group topology.*

A complete characterization of bounded Abelian groups which admit Hausdorff MinAP group topologies is given in [15]: An infinite bounded Abelian group  $G$  admits a MinAP group topology if and only if all its leading Ulm-Kaplansky invariants are infinite. This result with Theorem 1.6 gives a complete description of countably infinite Abelian groups admitting a Hausdorff MinAP group topology. This description has several applications (see [16, 17]).

## 2. PROOF OF THEOREM 1.6

Let  $G$  be an Abelian group. If  $G$  is endowed with the discrete topology, we denote it by  $G_d$ . If  $\kappa$  is a cardinal number, we denote by  $G^{(\kappa)}$  the direct sum of  $\kappa$

copies of the group  $G$ . The subgroup of  $G$  generated by a subset  $A$  is denoted by  $\langle A \rangle$ .

Let  $X$  be an Abelian topological group. An element  $x \in X$  is called a *topological generator* of  $X$  if  $\langle x \rangle$  is a dense subgroup of  $X$ .

Following [9], for a sequence  $\mathbf{d} = \{d_n\}_{n=0}^\infty$  of elements of an Abelian group  $G$  we set

$$s_{\mathbf{d}}((G_d)^\wedge) := \{x \in (G_d)^\wedge : (d_n, x) \rightarrow 1 \text{ in } \mathbb{T}\}.$$

Following Protasov and Zelenyuk (see [26, 27]), we say that a sequence  $\mathbf{d} = \{d_n\}$  in an Abelian group  $G$  is a *T-sequence* if there is a Hausdorff group topology on  $G$  in which  $d_n$  converges to zero. The group  $G$  equipped with the finest group topology with this property is denoted by  $(G, \tau_{\mathbf{d}})$ .

**Theorem 2.1** ([27, Theorem 2.1.5]). *Let  $G$  be a subgroup of an Abelian Hausdorff topological group  $S$ . Suppose a sequence  $\{a_n\}_{n=0}^\infty$  converges to zero and a sequence  $\{b_n\}_{n=0}^\infty$  converges to an element  $b \in S$  satisfying the condition  $\langle b \rangle \cap G = \{0\}$ . For every  $n \geq 0$ , set*

$$d_{2n} = a_n \quad \text{and} \quad d_{2n+1} = b_n.$$

*Then  $\mathbf{d} = \{d_k\}_{k=0}^\infty$  is a T-sequence in  $G$ .*

Recall (see [3]) that a sequence  $\mathbf{d} = \{d_n\}$  is called a *TB-sequence* in an Abelian group  $G$  if there is a precompact Hausdorff group topology on  $G$  in which  $d_n \rightarrow 0$ . The next lemma is a reformulation of Lemma 5.2 in [4]; for the sake of completeness we prove it.

**Lemma 2.2.** *Let  $\pi : G \rightarrow X$  be a continuous monomorphism from a discrete countably infinite Abelian group  $G$  to a compact metrizable Abelian group  $X$  with dense image. For a sequence  $\mathbf{d} = \{d_n\}$  in  $G$  the following assertions are equivalent:*

- (i)  $\pi^\wedge(X^\wedge) \subseteq s_{\mathbf{d}}(G^\wedge)$ , where  $\pi^\wedge$  is the adjoint homomorphism of  $\pi$ .
- (ii)  $\pi(d_n)$  converges to zero in  $X$ .

*In particular, if (i) and (ii) hold, then  $\mathbf{d}$  is a TB-sequence in  $G$ .*

*Proof.* (i) $\Rightarrow$ (ii) By definition and assumption, we have  $(\pi(d_n), y) = (d_n, \pi^\wedge(y)) \rightarrow 1$  for every  $y \in X^\wedge$ . But this is possible only if  $\pi(d_n) \rightarrow 0$ , since the topology of  $X$  is determined by its continuous characters  $y \in X^\wedge$ .

(ii) $\Rightarrow$ (i) Let  $y \in X^\wedge$ . Then  $(\pi^\wedge(y), d_n) = (y, \pi(d_n)) \rightarrow 1$ . Thus  $\pi^\wedge(y) \in s_{\mathbf{d}}(G^\wedge)$ , and hence  $\pi^\wedge(X^\wedge) \subseteq s_{\mathbf{d}}(G^\wedge)$ .

If (i) and (ii) hold, then  $s_{\mathbf{d}}(G^\wedge)$  is dense in  $X$  because  $\pi^\wedge$  has dense image by [19, 24.41]. Thus  $\mathbf{d}$  is a TB-sequence in  $G$  (see [9]).  $\square$

We use this to prove the following theorem.

**Theorem 2.3.** *Let a countably infinite Abelian group  $G$  be isomorphic to a dense subgroup of a compact connected second countable Abelian group  $X$ . Then  $G$  admits a complete sequential Hausdorff minimally almost periodic group topology generated by a T-sequence.*

*Proof.* Let  $\pi : G_d \rightarrow X$  be a continuous monomorphism with dense image. So  $\pi^\wedge$  is injective and has dense image. Since  $X^\wedge$  is countable [19, 24.15], Theorem 1.4 of [8] (or alternatively, Theorem 3.1 of [5]) implies that there exists a sequence  $\mathbf{a} = \{a_n\}_{n=0}^\infty$  in  $G$  such that  $s_{\mathbf{a}}((G_d)^\wedge) = \pi^\wedge(X^\wedge)$ . By Lemma 2.2,  $\pi(a_n) \rightarrow 0$  in  $X$ .

Denote by  $\Omega$  the set of all topological generators of  $X$ . The set  $\Omega$  is a dense  $G_\delta$ -subset of  $X$  as it coincides with the intersection  $\bigcap_{\chi \in X^\wedge \setminus \{0\}} \chi^{-1}(\mathbb{T} \setminus (\mathbb{Q}/\mathbb{Z}))$ . So  $\Omega$  has size continuum. Hence there exists  $b \in \Omega$  such that  $\pi(G) \cap \langle b \rangle = \{0\}$ .

Let a sequence  $\mathbf{b} = \{b_n\}_{n=0}^\infty$  in  $G$  be such that  $\pi(b_n)$  converges to  $b$  in  $X$ . Then, by Theorem 2.1, the sequence  $\mathbf{d} = \{d_n\}_{n=0}^\infty$ , where for every  $n \geq 0$

$$d_{2n} = a_n \quad \text{and} \quad d_{2n+1} = b_n$$

is a  $T$ -sequence in  $G$ .

By [13, Theorem 4], to show that  $(G, \tau_{\mathbf{d}})$  is MinAP, it is enough to prove that  $s_{\mathbf{d}}((G_{\mathbf{d}})^\wedge) = \{0\}$ . Note that

$$s_{\mathbf{d}}((G_{\mathbf{d}})^\wedge) = s_{\mathbf{a}}((G_{\mathbf{d}})^\wedge) \cap s_{\mathbf{b}}((G_{\mathbf{d}})^\wedge) = \pi^\wedge(X^\wedge) \cap s_{\mathbf{b}}((G_{\mathbf{d}})^\wedge)$$

by the choice of  $\mathbf{a}$ . Now if  $x \in s_{\mathbf{d}}((G_{\mathbf{d}})^\wedge)$ , then  $x = \pi^\wedge(y)$  for some  $y \in X^\wedge$  and  $(b_n, x) \rightarrow 1$ . Since

$$(b_n, x) = (b_n, \pi^\wedge(y)) = (\pi(b_n), y) \rightarrow (b, y),$$

we obtain  $(b, y) = 1$ . But since  $b$  is a topological generator of  $X$ , this immediately yields  $y = 0$ . Thus  $x = 0$  and  $s_{\mathbf{d}}((G_{\mathbf{d}})^\wedge)$  is trivial.

Finally, Theorems 2.3.1 and 2.3.11 of [27] implies that the group  $(G, \tau_{\mathbf{d}})$  is sequential and complete. □

We are now ready to prove Theorem 1.6.

*Proof of Theorem 1.6.* Corollary 9.4 of [10] implies that  $G$  is isomorphic to a dense subgroup of  $\mathbb{T}^\mathbb{N}$ . Now the assertion immediately follows from Theorem 2.3. □

For the sake of completeness we prove the next folklore proposition:

**Proposition 2.4.** *If an Abelian topological group  $G$  has a family of minimally almost periodic subgroups  $\{G_i : i \in I\}$ , such that their (not necessarily direct) sum  $\sum_{i \in I} G_i$  is dense in  $G$ , then  $G$  is MinAP.*

*Proof.* Set  $H := \sum_{i \in I} G_i$ . Let  $\chi \in G^\wedge$ . By assumption,  $\chi|_{G_i} = 0$  for every  $i \in I$ . Hence  $\chi|_H = 0$ . Since  $H$  is dense in  $G$ , we obtain  $\chi = 0$ . Thus  $G$  is MinAP. □

Proposition 2.4 immediately implies the next corollary:

**Corollary 2.5.** *Let  $\{G_i\}_{i \in I}$  be a family of Hausdorff MinAP Abelian groups. Then the direct sum  $\bigoplus_{i \in I} G_i$  endowed with the product topology is also MinAP.*

**Corollary 2.6.** *For every infinite cardinal  $\kappa$  and each natural number  $m$ , the group  $G := \mathbb{Z}(m)^{(\kappa)}$  admits a Hausdorff MinAP group topology.*

*Proof.* By [1, 26], the group  $H := \mathbb{Z}(m)^{(\omega)}$  admits a Hausdorff MinAP group topology  $\tau$ . Then  $(H, \tau)^{(\kappa)}$  endowed with the product topology is MinAP by Corollary 2.5. Note that  $G \cong H^{(\kappa)}$  algebraically. Now applying Corollary 2.5 we obtain that the group  $G$  admits a Hausdorff MinAP group topology. □

The main open problem still remaining is:

**Question 2.7.** Let  $G$  be an uncountable Abelian group of infinite exponent (for example,  $G$  is an uncountable torsion free Abelian group). Does  $G$  admit a Hausdorff MinAP group topology?

However, we prove the following special case. The set of all prime numbers we denote by  $\mathbb{P}$ .

**Proposition 2.8.** *Let  $G = \bigoplus_{i \in I} G_i$ , where  $I$  is a non-empty set of indices and  $G_i$  is a nonzero countable Abelian group for every  $i \in I$ . If  $G$  is of infinite exponent, then  $G$  admits a Hausdorff MinAP group topology.*

*Proof.* The idea of our proof is the following. Using purely combinatorial arguments we show first that  $G = G' \oplus G''$ , where  $G'$  is a direct sum of unbounded countable groups, while  $G''$  is a direct sum of homogeneous bounded groups  $H_j$  (i.e., each  $H_j$  is a direct sum of infinitely many copies of some fixed cyclic  $p$ -group, where  $p$  may depend on  $j$ ). Then we apply Corollary 2.5 (via Theorem 1.6) and Corollary 2.6.

Let  $I_1$  be the set of all indices  $i$  such that  $G_i$  is of finite exponent. Set  $I_2 = I \setminus I_1$ . We distinguish between two cases.

*Case 1.*  $I_1 = \emptyset$ . By Theorem 1.6, for every  $i \in I$  the group  $G_i$  admits a Hausdorff MinAP group topology. Thus  $G$  also admits a Hausdorff MinAP group topology by Corollary 2.5.

*Case 2.*  $I_1 \neq \emptyset$ . For every  $i \in I_1$ , the group  $G_i$  is a direct sum of finite cyclic groups [11, 11.2]:

$$G_i = \bigoplus_{(p,n) \in A_i} \mathbb{Z}(p^n)^{(k_{(p,n)}^i)},$$

where  $A_i \subset \mathbb{P} \times \mathbb{N}$  is finite and  $k_{(p,n)}^i$  are non-zero cardinal numbers.

Set  $C := \bigcup_{i \in I_1} A_i$  and  $k_{(p,n)} := \sum_{i \in I_1} k_{(p,n)}^i$ . Put  $C_1 := \{(p, n) \in C : k_{(p,n)} < \infty\}$  and  $C_2 = C \setminus C_1$ . Then we can represent the group  $G$  in the following form:

$$(2.1) \quad G = \left( \bigoplus_{(p,n) \in C_1} \mathbb{Z}(p^n)^{(k_{(p,n)})} \right) \oplus \bigoplus_{(p,n) \in C_2} \mathbb{Z}(p^n)^{(k_{(p,n)})} \oplus \bigoplus_{i \in I_2} G_i.$$

Set  $H := \bigoplus_{(p,n) \in C_1} \mathbb{Z}(p^n)^{(k_{(p,n)})}$  and  $H_{p,n} := \mathbb{Z}(p^n)^{(k_{(p,n)})}$  for  $(p, n) \in C_2$ . By Theorem 1.6 and Corollary 2.6, for each  $(p, n) \in C_2$  and each  $i \in I_2$ , the groups  $H_{p,n}$  and  $G_i$  admit a Hausdorff MinAP group topology.

*Subcase 2(a).* Assume that  $H$  is either countably infinite (and hence  $\exp(H) = \infty$ ) or trivial. By Theorem 1.6, the group  $H$  in (2.1) admits a Hausdorff MinAP group topology. Thus  $G$  also admits a Hausdorff MinAP group topology by Corollary 2.5.

*Subcase 2(b).* Let  $H$  be non-zero and finite. If  $I_2 \neq \emptyset$ , take arbitrarily  $i_0 \in I_2$ . Then we have

$$(2.2) \quad G = (H \oplus G_{i_0}) \oplus \bigoplus_{(p,n) \in C_2} H_{p,n} \oplus \bigoplus_{i \in I_2, i \neq i_0} G_i.$$

By Theorem 1.6, the group  $H \oplus G_{i_0}$  in (2.2) admits a Hausdorff MinAP group topology. Hence  $G$  also admits a Hausdorff MinAP group topology by Corollary 2.5.

Assume that  $I_2 = \emptyset$ . Then  $C_2$  is infinite since  $\exp(G) = \infty$ . Since  $H_{p,n} \cong \mathbb{Z}(p^n) \oplus H_{p,n}$  we have

$$(2.3) \quad G \cong \left( H \oplus \bigoplus_{(p,n) \in C_2} \mathbb{Z}(p^n) \right) \oplus \bigoplus_{(p,n) \in C_2} H_{p,n}.$$

Also in this case, by Theorem 1.6, the group  $H \oplus \bigoplus_{(p,n) \in C_2} \mathbb{Z}(p^n)$  in (2.3) admits a Hausdorff MinAP group topology. Therefore  $G$  admits a Hausdorff MinAP group topology by Corollary 2.5.  $\square$

Since a divisible group is a direct sum of full rational groups  $\mathbb{Q}$  and groups of the form  $\mathbb{Z}(p^\infty)$  [11, Theorem 19.1], we obtain the following:

**Corollary 2.9.** (i) [28] *Every free Abelian group admits a Hausdorff MinAP group topology.*

(ii) [28] *Each divisible Abelian group admits a Hausdorff MinAP group topology.*

(iii) *Every linear space over  $\mathbb{R}$  or  $\mathbb{C}$  admits a Hausdorff MinAP group topology.*

(iv) *The circle group  $\mathbb{T}$  admits a Hausdorff MinAP group topology.*

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