MINIMALLY ALMOST PERIODIC GROUP TOPOLOGIES
ON COUNTABLY INFINITE ABELIAN GROUPS

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Abstract. We obtain a positive answer to a question by Comfort: Every countable Abelian group $G$ of infinite exponent admits a complete Hausdorff minimally almost periodic group topology.

1. Introduction

For an Abelian topological group $G$, $G^\wedge$ denotes the group of all continuous characters on $G$ endowed with the compact-open topology. Denote by $n(G) = \bigcap_{\chi \in G^\wedge} \ker \chi$ the von Neumann radical of $G$. Following von Neumann [21], the group $G$ is called minimally almost periodic (MinAP) if $n(G) = G$, and $G$ is called maximally almost periodic (MAP) if $n(G) = 0$.

Let $G$ be an infinite Abelian group. Denote by $\mathcal{N}\mathcal{R}(G)$ the set of all subgroups $H$ of $G$ for which there exists a non-discrete Hausdorff group topology $\tau$ on $G$ such that the von Neumann radical of $(G, \tau)$ is $H$, i.e., $n(G, \tau) = H$. Let $\mathcal{N}\mathcal{R}\mathcal{C}(G) \subseteq \mathcal{N}\mathcal{R}(G)$ be the set of all subgroups $H$ of $G$ for which there exists a topology as above which is also complete.

Let $G$ be an Abelian topological group. The richness of its dual group $G^\wedge$ is one of the most important properties of $G$ which plays a crucial role in Harmonic Analysis (see [19,20]). The von Neumann radical measures this richness, thus the following general question is important:

Question 1.1 ([12, Problem 2]). Describe the sets $\mathcal{N}\mathcal{R}(G)$ and $\mathcal{N}\mathcal{R}\mathcal{C}(G)$.

One can show (see [14, Theorem 1]) that every infinite Abelian group admits a complete non-trivial Hausdorff group topology with trivial von Neumann radical. Thus, the trivial group \{0\} belongs to $\mathcal{N}\mathcal{R}\mathcal{C}(G)$ for every infinite Abelian group $G$, i.e., $\mathcal{N}\mathcal{R}\mathcal{C}(G)$ is never empty. A much deeper question is whether any infinite Abelian group admits a Hausdorff group topology with non-zero von Neumann radical. The positive answer was given by Ajtai, Havas and Komlós [1]. Using the method of $T$-sequences, Protasov and Zelenyuk [26] strengthened their result as follows: every infinite Abelian group admits a complete sequential Hausdorff group topology for which characters do not separate points. So $\mathcal{N}\mathcal{R}\mathcal{C}(G) \neq \{\{0\}\}$ for every infinite Abelian group $G$.
Question 1.1 has two interesting extreme cases. Denote by $S(G)$ the set of all subgroups of an infinite Abelian group $G$.

**Question 1.2** ([12 Problem 4]). Describe all infinite Abelian groups $G$ such that $\mathcal{N}R(G) = S(G)$ (respectively, $\mathcal{N}RC(G) = S(G)$).

The second and, perhaps, the most interesting special case of Question 1.1 is the following:

**Question 1.3** ([12 Problem 5]). Describe all infinite Abelian groups $G$ such that $G \in \mathcal{N}R(G)$ (or $G \in \mathcal{N}RC(G)$).

Note that the existence of Abelian groups admitting no non-zero continuous characters has been known for a long time. The classic example of linear topological spaces that (when viewed as Abelian topological groups) have no non-trivial continuous characters was given by Day in [7]. There are other examples in [19, 23.32]. Nienhuys [22] showed the existence of a metric solenoidal monothetic MinAP group (see also [23]). Prodanov [24] gave an elementary example of a MinAP group.

Recall that an Abelian group $G$ is of finite exponent or bounded if there exists a positive integer $n$ such that $ng = 0$ for every $g \in G$. The minimal integer $n$ with this property is called the exponent of $G$ and is denoted by $\exp(G)$. When $G$ is not bounded, we write $\exp(G) = \infty$ and say that $G$ is of infinite exponent or unbounded.

Protasov [25] posed the question of whether every infinite Abelian group admits a minimally almost periodic group topology. Using a result of Graev [18], Remus [28] proved that for every natural number $n$ there exists a connected MinAP group which is algebraically generated by elements of order $n$. On the other hand, he gave (see [6]) a simple example of a group $G$ of finite exponent which does not admit any Hausdorff group topology $\tau$ such that $(G, \tau)$ is minimally almost periodic. So, for groups of finite exponent the answer to Protasov’s question is negative. This justifies the following problems:

**Question 1.4** ([6 Problem 521]). Does every Abelian group which is not of bounded exponent admit a minimally almost periodic topological group topology? What about the countable case?

**Question 1.5** ([27 Question 2.6.1]). Let $G$ be a torsion free countable Abelian group. Does there exist a Hausdorff group topology on $G$ with only zero character?

The main goal of the article is to give the positive answer to Comfort’s Question 1.4 for the countable case and, hence, to Protasov-Zelenyuk’s Question 1.5.

**Theorem 1.6.** Every countable Abelian group of infinite exponent admits a complete sequential Hausdorff minimally almost periodic group topology.

A complete characterization of bounded Abelian groups which admit Hausdorff MinAP group topologies is given in [15]: An infinite bounded Abelian group $G$ admits a MinAP group topology if and only if all its leading Ulm-Kaplansky invariants are infinite. This result with Theorem 1.6 gives a complete description of countably infinite Abelian groups admitting a Hausdorff MinAP group topology. This description has several applications (see [16, 17]).

2. **Proof of Theorem 1.6**

Let $G$ be an Abelian group. If $G$ is endowed with the discrete topology, we denote it by $G_d$. If $\kappa$ is a cardinal number, we denote by $G^{(\kappa)}$ the direct sum of $\kappa$
copies of the group $G$. The subgroup of $G$ generated by a subset $A$ is denoted by $\langle A \rangle$.

Let $X$ be an Abelian topological group. An element $x \in X$ is called a topological generator of $X$ if $\langle x \rangle$ is a dense subgroup of $X$.

Following [9], for a sequence $d = \{d_n\}_{n=0}^{\infty}$ of elements of an Abelian group $G$ we set

$$s_d((G_d)^\wedge) := \{x \in (G_d)^\wedge : (d_n, x) \to 1 \text{ in } T\}.$$ 

Following Protasov and Zelenyuk (see [20],[27]), we say that a sequence $d = \{d_n\}$ in an Abelian group $G$ is a $T$-sequence if there is a Hausdorff group topology on $G$ in which $d_n$ converges to zero. The group $G$ equipped with the finest group topology with this property is denoted by $(G, \tau_d)$.

**Theorem 2.1** ([27 Theorem 2.1.5]). Let $G$ be a subgroup of an Abelian Hausdorff topological group $S$. Suppose a sequence $\{a_n\}_{n=0}^{\infty}$ converges to zero and a sequence $\{b_n\}_{n=0}^{\infty}$ converges to an element $b \in S$ satisfying the condition $(b) \cap G = \{0\}$. For every $n \geq 0$, set

$$d_{2n} = a_n \quad \text{and} \quad d_{2n+1} = b_n.$$ 

Then $d = \{d_k\}_{k=0}^{\infty}$ is a $T$-sequence in $G$.

Recall (see [3]) that a sequence $d = \{d_n\}$ is called a $TB$-sequence in an Abelian group $G$ if there is a precompact Hausdorff group topology on $G$ in which $d_n \to 0$. The next lemma is a reformulation of Lemma 5.2 in [4]; for the sake of completeness we prove it.

**Lemma 2.2.** Let $\pi : G \to X$ be a continuous monomorphism from a discrete countably infinite Abelian group $G$ to a compact metrizable Abelian group $X$ with dense image. For a sequence $d = \{d_n\}$ in $G$ the following assertions are equivalent:

(i) $\pi^\wedge(X^\wedge) \subseteq s_d(G^\wedge)$, where $\pi^\wedge$ is the adjoint homomorphism of $\pi$.

(ii) $\pi(d_n)$ converges to zero in $X$.

In particular, if (i) and (ii) hold, then $d$ is a $TB$-sequence in $G$.

**Proof.** (i)$\Rightarrow$(ii) By definition and assumption, we have $(\pi(d_n), y) = (d_n, \pi^\wedge(y)) \to 1$ for every $y \in X^\wedge$. But this is possible only if $\pi(d_n) \to 0$, since the topology of $X$ is determined by its continuous characters $y \in X^\wedge$.

(ii)$\Rightarrow$(i) Let $y \in X^\wedge$. Then $(\pi^\wedge(y), d_n) = (y, \pi(d_n)) \to 1$. Thus $\pi^\wedge(y) \in s_d(G^\wedge)$, and hence $\pi^\wedge(X^\wedge) \subseteq s_d(G^\wedge)$.

If (i) and (ii) hold, then $s_d(G^\wedge)$ is dense in $X$ because $\pi^\wedge$ has dense image by [19, 24.41]. Thus $d$ is a $TB$-sequence in $G$ (see [9]).

We use this to prove the following theorem.

**Theorem 2.3.** Let a countably infinite Abelian group $G$ be isomorphic to a dense subgroup of a compact connected second countable Abelian group $X$. Then $G$ admits a complete sequential Hausdorff minimally almost periodic group topology generated by a $T$-sequence.

**Proof.** Let $\pi : G_d \to X$ be a continuous monomorphism with dense image. So $\pi^\wedge$ is injective and has dense image. Since $X^\wedge$ is countable [19, 24.15], Theorem 1.4 of [8] (or alternatively, Theorem 3.1 of [5]) implies that there exists a sequence $a = \{a_n\}_{n=0}^{\infty}$ in $G$ such that $s_a((G_d)^\wedge) = \pi^\wedge(X^\wedge)$. By Lemma 2.2, $\pi(a_n) \to 0$ in $X$. 
Denote by \( \Omega \) the set of all topological generators of \( X \). The set \( \Omega \) is a dense \( G_δ \)-subset of \( X \) as it coincides with the intersection \( \bigcap_{\chi \in X^\setminus\{0\}} \chi^{-1}(T \setminus (Q/Z)) \). So \( \Omega \) has size continuum. Hence there exists \( b \in \Omega \) such that \( \pi(G) \cap \langle b \rangle = \{0\} \).

Let a sequence \( b = \{b_n\}_{n=0}^\infty \) in \( G \) be such that \( \pi(b_n) \) converges to \( b \) in \( X \). Then, by Theorem 2.4, the sequence \( d = \{d_n\}_{n=0}^\infty \), where for every \( n \geq 0 \)
\[
d_{2n} = a_n \quad \text{and} \quad d_{2n+1} = b_n
\]
is a \( T \)-sequence in \( G \).

By [13, Theorem 4], to show that \( (G, \tau_\mathcal{D}) \) is MinAP, it is enough to prove that \( s_\mathcal{D} ((G_\mathcal{D})^\wedge) = \{0\} \). Note that
\[
s_\mathcal{D} ((G_\mathcal{D})^\wedge) = s_\mathcal{D} ((G_\mathcal{D})^\wedge) \cap s_\mathcal{D} ((G_\mathcal{D})^\wedge) = \pi^\wedge (X^\wedge) \cap s_\mathcal{D} ((G_\mathcal{D})^\wedge)
\]
by the choice of \( \mathcal{D} \). Now if \( x \in s_\mathcal{D} ((G_\mathcal{D})^\wedge) \), then \( x = \pi^\wedge(y) \) for some \( y \in X^\wedge \) and \( (b_n, x) \to 1 \). Since \( (b_n, x) = (b_n, \pi^\wedge(y)) = (\pi(b_n), y) \to (b, y) \),
we obtain \( (b, y) = 1 \). But since \( b \) is a topological generator of \( X \), this immediately yields \( y = 0 \). Thus \( x = 0 \) and \( s_\mathcal{D} ((G_\mathcal{D})^\wedge) \) is trivial.

Finally, Theorems 2.3.1 and 2.3.11 of [27] implies that the group \( (G, \tau_\mathcal{D}) \) is sequential and complete.

We are now ready to prove Theorem 1.6

**Proof of Theorem 1.6** Corollary 9.4 of [10] implies that \( G \) is isomorphic to a dense subgroup of \( T^\mathbb{N} \). Now the assertion immediately follows from Theorem 2.3.

For the sake of completeness we prove the next folklore proposition:

**Proposition 2.4.** If an Abelian topological group \( G \) has a family of minimally almost periodic subgroups \( \{G_i : i \in I\} \), such that their (not necessarily direct) sum \( \sum_{i \in I} G_i \) is dense in \( G \), then \( G \) is MinAP.

**Proof.** Set \( H := \sum_{i \in I} G_i \). Let \( \chi \in G^\wedge \). By assumption, \( \chi|_{G_i} = 0 \) for every \( i \in I \).
Hence \( \chi|_H = 0 \). Since \( H \) is dense in \( G \), we obtain \( \chi = 0 \). Thus \( G \) is MinAP.

**Corollary 2.5.** Let \( \{G_i\}_{i \in I} \) be a family of Hausdorff MinAP Abelian groups. Then the direct sum \( \bigoplus_{i \in I} G_i \) endowed with the product topology is also MinAP.

**Corollary 2.6.** For every infinite cardinal \( \kappa \) and each natural number \( m \), the group \( G := \mathbb{Z}(m)^{(\kappa)} \) admits a Hausdorff MinAP group topology.

**Proof.** By [11,26], the group \( H := \mathbb{Z}(m)^{(\omega)} \) admits a Hausdorff MinAP group topology \( \tau \). Then \( (H, \tau)^{(\kappa)} \) endowed with the product topology is MinAP by Corollary 2.5. Note that \( G \cong H^{(\kappa)} \) algebraically. Now applying Corollary 2.5 we obtain that the group \( G \) admits a Hausdorff MinAP group topology.

The main open problem still remaining is:

**Question 2.7.** Let \( G \) be an uncountable Abelian group of infinite exponent (for example, \( G \) is an uncountable torsion free Abelian group). Does \( G \) admit a Hausdorff MinAP group topology?

However, we prove the following special case. The set of all prime numbers we denote by \( \mathbb{P} \).
Proposition 2.8. Let \( G = \bigoplus_{i \in I} G_i \), where \( I \) is a non-empty set of indices and \( G_i \) is a nonzero countable Abelian group for every \( i \in I \). If \( G \) is of infinite exponent, then \( G \) admits a Hausdorff MinAP group topology.

Proof. The idea of our proof is the following. Using purely combinatorial arguments we show first that \( G = G' \oplus G'' \), where \( G' \) is a direct sum of unbounded countable groups, while \( G'' \) is a direct sum of homogeneous bounded groups \( H_j \) (i.e., each \( H_j \) is a direct sum of infinitely many copies of some fixed cyclic \( p \)-group, where \( p \) may depend on \( j \)). Then we apply Corollary 2.5 (via Theorem 1.6) and Corollary 2.6.

Let \( I_1 \) be the set of all indices \( i \) such that \( G_i \) is of finite exponent. Set \( I_2 = I \setminus I_1 \).

We distinguish between two cases.

**Case 1.** \( I_1 = \emptyset \). By Theorem 1.6 for every \( i \in I \) the group \( G_i \) admits a Hausdorff MinAP group topology. Thus \( G \) also admits a Hausdorff MinAP group topology by Corollary 2.6.

**Case 2.** \( I_1 \neq \emptyset \). For every \( i \in I_1 \), the group \( G_i \) is a direct sum of finite cyclic groups \([11, 11.2]:\)

\[
G_i = \bigoplus_{(p,n) \in A_i} \mathbb{Z}(p^n)^{k_i(p,n)},
\]

where \( A_i \subset \mathbb{P} \times \mathbb{N} \) is finite and \( k_i(p,n) \) are non-zero cardinal numbers.

Set \( C := \bigcup_{i \in I_1} A_i \) and \( k_{i(p,n)} := \sum_{i \in I_1} k_i(p,n) \). Put \( C_1 := \{(p,n) \in C : k_{i(p,n)} < \infty \} \) and \( C_2 = C \setminus C_1 \). Then we can represent the group \( G \) in the following form:

\[
G = \left( \bigoplus_{(p,n) \in C_1} \mathbb{Z}(p^n)^{k_i(p,n)} \right) \oplus \bigoplus_{(p,n) \in C_2} \mathbb{Z}(p^n)^{k_i(p,n)} \oplus \bigoplus_{i \in I_2} G_i.
\]

Set \( H := \bigoplus_{(p,n) \in C_1} \mathbb{Z}(p^n)^{k_{i(p,n)}} \) and \( H_{p,n} := \mathbb{Z}(p^n)^{k_{i(p,n)}} \) for \( (p,n) \in C_2 \). By Theorem 1.6 and Corollary 2.6, for each \( (p,n) \in C_2 \) and each \( i \in I_2 \), the groups \( H_{p,n} \) and \( G_i \) admit a Hausdorff MinAP group topology.

**Subcase 2(a).** Assume that \( H \) is either countably infinite (and hence \( \exp(H) = \infty \)) or trivial. By Theorem 1.6 the group \( H \) in (2.1) admits a Hausdorff MinAP group topology. Thus \( G \) also admits a Hausdorff MinAP group topology by Corollary 2.5.

**Subcase 2(b).** Let \( H \) be non-zero and finite. If \( I_2 \neq \emptyset \), take arbitrarily \( i_0 \in I_2 \). Then we have

\[
G = (H \oplus G_{i_0}) \oplus \bigoplus_{(p,n) \in C_2} H_{p,n} \oplus \bigoplus_{i \in I_2, i \neq i_0} G_i.
\]

By Theorem 1.6, the group \( H \oplus G_{i_0} \) in (2.2) admits a Hausdorff MinAP group topology. Hence \( G \) also admits a Hausdorff MinAP group topology by Corollary 2.5.

Assume that \( I_2 = \emptyset \). Then \( C_2 \) is infinite since \( \exp(G) = \infty \). Since \( H_{p,n} \cong \mathbb{Z}(p^n) \oplus H_{p,n} \) we have

\[
G \cong \left( H \oplus \bigoplus_{(p,n) \in C_2} \mathbb{Z}(p^n) \right) \oplus \bigoplus_{(p,n) \in C_2} H_{p,n}.
\]
Also in this case, by Theorem 1.6, the group \( H \oplus \bigoplus_{(p,n) \in C_2} \mathbb{Z}(p^n) \) in (2.3) admits a Hausdorff MinAP group topology. Therefore \( G \) admits a Hausdorff MinAP group topology by Corollary 2.5.

Since a divisible group is a direct sum of full rational groups \( \mathbb{Q} \) and groups of the form \( \mathbb{Z}(p^\infty) \) [11, Theorem 19.1], we obtain the following:

**Corollary 2.9.**
(i) \( \mathcal{B} \) Every free Abelian group admits a Hausdorff MinAP group topology.
(ii) \( \mathcal{B} \) Each divisible Abelian group admits a Hausdorff MinAP group topology.
(iii) Every linear space over \( \mathbb{R} \) or \( \mathbb{C} \) admits a Hausdorff MinAP group topology.
(iv) The circle group \( \mathbb{T} \) admits a Hausdorff MinAP group topology.

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**References**


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