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A COUNTEREXAMPLE TO A THEOREM OF BREMERMANN ON SHILOV BOUNDARIES

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Abstract. We give a counterexample to the following theorem of Bremermann on Shilov boundaries: if D is a bounded domain in $\mathbb{C}^n$ having a univalent envelope of holomorphy, say $\widetilde{D}$, then the Shilov boundary of D with respect to the algebra $\mathcal{A}(D)$, call it $\partial_S D$, coincides with the corresponding one for $\widetilde{D}$, called $\partial_S \widetilde{D}$.

Let us first repeat some basic notions: let D be a bounded domain in $\mathbb{C}^n$. Put $\mathcal{A}(D) := \mathcal{C}(\overline{D}) \cap \mathcal{O}(D)$. Then $\mathcal{A}(D)$ together with the supremum norm $\|\cdot\|_{\overline{D}}$ is a Banach algebra of functions on $\overline{D}$. Moreover, we set $\mathcal{B}(D)$ as the closure of $\mathcal{O}(\overline{D})$ in $\mathcal{C}(\overline{D})$ with the above norm. Again this is a Banach algebra. Then both of these algebras have a Shilov boundary, called $\partial_S D$, respectively, $\partial_B D$. Obviously, we have $\partial_B D \subset \partial_S D$.

Now we assume in addition that D has a univalent envelope of holomorphy $\widetilde{D}$. Then, we have

$$\partial_S \widetilde{D} \subset \partial_S D \subset \partial D \text{ and } \partial_B \widetilde{D} \subset \partial_B D \subset \partial D.$$

In [Bre 1959] (see Theorem in section 6.6), Bremermann claimed that equality $\partial_S \widetilde{D} = \partial_S D$ is also true. While he proved $\partial_S \widetilde{D} \subset \partial_S D$ in a detailed way he was saying that the inverse inclusion is an obvious fact. But as we will show (more than fifty years later) this claim is not true, already if D is a Hartogs domain over an annulus. For positive results regarding Reinhardt domains see [Kos-Zwo 2013].

Following the construction of some Hartogs domain with non univalent envelope of holomorphy (see [Jar-Pfl 2000], pages 1-2) we get the following result.

Theorem. *There exists a bounded Hartogs domain $D \subset \mathbb{C}^2$ with a univalent envelope of holomorphy $\widetilde{D}$ such that*

- $\partial_S D \neq \partial_S \widetilde{D}$,
- $\partial_B D \neq \partial_B \widetilde{D}$,
- *there exists a function $f \in \mathcal{O}(\overline{D})$ such that the holomorphic extension $\widetilde{f}$ of $f|_D$ to $\widetilde{D}$ has no continuous extension to $\overline{\widetilde{D}}$.*

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Proof. Let $A := \{z \in \mathbb{C} : 1/2 < |z| < 1\}$. Then we introduce

$$\begin{aligned} D :&= \{z \in A \times \mathbb{C} : \operatorname{Re} z_1 < 0,\ |z_2| < 3\} \\ &\cup \{z \in A \times \mathbb{C} : 0 \le \operatorname{Re} z_1,\ \operatorname{Im} z_1 > 0,\ |z_2| < 1\} \\ &\cup \{z \in A \times \mathbb{C} : 0 \le \operatorname{Re} z_1,\ \operatorname{Im} z_1 < 0,\ 2 < |z_2| < 3\}. \end{aligned}$$

Note that D is a Hartogs domain over the annulus A which cuts the base $A \times \{0\}$. Then Corollary 3.1.10(b) in [Jar-Pfl 2000] implies that D has a univalent envelope of holomorphy $\widetilde{D}$. Moreover, using the Cauchy integral formula shows that $\widetilde{D}$ contains the following domain

$$\{z \in A \times \mathbb{C} : \operatorname{Re} z_1 < 0 \text{ or (if } \operatorname{Re} z_1 \ge 0, \text{ then } \operatorname{Im} z_1 < 0),\ |z_2| < 3\}.$$

In particular, all the discs $\mathbb{D}_{z_1} := \{z_1\} \times 3\mathbb{D}$, $1/2 \le \operatorname{Re} z_1 = z_1 \le 1$, belong to $\overline{\widetilde{D}}$ (here $\mathbb{D}$ means the open unit disc in $\mathbb{C}$).

Therefore, if $f \in \mathcal{A}(\widetilde{D})$, then $f(z_1, \cdot) \in \mathcal{A}(3\mathbb{D})$ for all the above z_1 (use Weierstrass' theorem here). Hence, by the maximum principle, these discs don't contain any point of $\partial_S \widetilde{D}$.

On the other side we discuss the following domain

$$\begin{aligned} D' :&= \{z \in A' \times \mathbb{C} : \operatorname{Re} z_1 < \varepsilon |\operatorname{Im} z_1|,\ |z_2| < 3 + \varepsilon\} \\ &\cup \{z \in A' \times \mathbb{C} : 0 \le \operatorname{Re} z_1,\ \operatorname{Im} z_1 > -\varepsilon \operatorname{Re} z_1,\ |z_2| < 1 + \varepsilon\} \\ &\cup \{z \in A' \times \mathbb{C} : 0 \le \operatorname{Re} z_1,\ \operatorname{Im} z_1 < \varepsilon \operatorname{Re} z_1,\ 2 - \varepsilon < |z_2| < 3 + \varepsilon\}, \end{aligned}$$

where $0 < \varepsilon \ll 1/4$ and $A' := \{z \in \mathbb{C} : 1/2 - \varepsilon < |z| < 1 + \varepsilon\}$. Observe that $D \subset\subset D'$.

Now we define the following concrete holomorphic function g on D':

$$g(z) := \begin{cases} \log_1 z_1, & \text{if } z \in D',\ |z_2| > 1.4 \\ \log_2 z_1, & \text{if } z \in D',\ |z_2| < 1.6 \end{cases},$$

where $\log_1$, respectively $\log_2$, is the branch of the logarithm function on $\mathbb{C} \setminus \{w \in \mathbb{C} : \operatorname{Re} w \ge 0,\ \operatorname{Im} w = \operatorname{Re} w\}$, respectively on $\mathbb{C} \setminus \{w \in \mathbb{C} : \operatorname{Re} w \ge 0,\ \operatorname{Im} w = -\operatorname{Re} w\}$, with $\log_1(-1/2) = \log_2(-1/2) = \log 1/2 + i\pi$. Observe that g is well defined on D' and $g \in \mathcal{O}(D')$.

Define $f := g|_{\overline{D}}$. Then $f \in \mathcal{B}(D)$ and if $z \in \overline{D}$ with $z_1 = \operatorname{Re} z_1 > 0$, then

$$f(z) = \begin{cases} \log \operatorname{Re} z_1, & \text{if } |z_2| \le 1 \\ \log \operatorname{Re} z_1 + 2\pi i, & \text{if } |z_2| \ge 2 \end{cases}.$$

Finally, we observe that the function h defined as $h(z) := \exp(if(z) + 2\pi)$, $z \in \overline{D}$, belongs to $\mathcal{B}(D)$ and

$$|h(z)| = \begin{cases} 1, & \text{if } z \in \overline{D},\ \operatorname{Re} z_1 > 0,\ |z_2| \ge 2 \\ e^{2\pi}, & \text{if } z \in \overline{D},\ \operatorname{Re} z_1 > 0,\ |z_2| \le 1 \end{cases}$$

and $|h(z)| < e^{2\pi}$ on the remaining part of $\overline{D}$. Therefore, $\partial_B D$ contains points $z \in \overline{D}$ with $0 < \operatorname{Re} z_1 = z_1$ and $|z_2| \le 1$.

Combining this concrete information with the general one from the former discussion on the Shilov boundaries for $\widetilde{D}$ we conclude that $\partial_S \widetilde{D}$ and $\partial_B \widetilde{D}$ are strictly contained in $\partial_B D$. In particular, this shows that the claimed equality in Bremermann's paper does not hold.

Moreover, the function f is the one whose existence was claimed in the third claim in the theorem. $\square$

Remark. (a) Recall that the equality $\partial_S D = \partial_S \widetilde{D}$ is true for a Reinhardt domain D.

(b) Let D be a bounded balanced domain. Obviously, $\overline{D}$ has a neighborhood basis of balanced domains G. Moreover, D and each G have univalent envelopes of holomorphy with $\widetilde{D} \subset\subset \widetilde{G}$. Hence the equality $\partial_B D = \partial_B \widetilde{D}$ holds in an obvious way.

(c) What remains is to discuss the equality of the Shilov boundaries for $\mathcal{A}(D)$ and $\mathcal{A}(\widetilde{D})$ in the case when D is a balanced domain.

References

[Bre 1959] H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Šilov boundaries*, Trans. Amer. Math. Soc. **91** (1959), 246–276. MR0136766 (25 #227)

[Kos-Zwo 2013] Łukasz Kosiński and Włodzimierz Zwonek, *Proper holomorphic mappings vs. peak points and Shilov boundary*, Ann. Polon. Math. **107** (2013), no. 1, 97–108, DOI 10.4064/ap107-1-7. MR3001625

[Jar-Pfl 2000] Marek Jarnicki and Peter Pflug, *Extension of holomorphic functions*, de Gruyter Expositions in Mathematics, vol. 34, Walter de Gruyter & Co., Berlin, 2000. MR1797263 (2001k:32017)

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