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ON THE CONCEPT OF ANALYTIC HARDNESS

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ABSTRACT. Let $H \subseteq Z \subseteq 2^\omega$. Using only classical descriptive set theory we prove that if Borel functions from 2^ω to Z give as preimages of H all analytic subsets of 2^ω, then so do continuous injections. This strengthens a theorem Kechris proved by means of effective descriptive set theory.

Let $H \subseteq Z$ be subsets of the Cantor space $\mathcal{C} = 2^\omega$. The pair (H, Z) is called $\mathbf{\Sigma}^1_1$-hard, resp. Borel $\mathbf{\Sigma}^1_1$-hard, if for any $\mathbf{\Sigma}^1_1$ set $Q \subseteq \mathcal{C}$ there is a continuous, resp. Borel, function $f\colon \mathcal{C} \to Z$ with $Q = f^{-1}[H]$. Using effective descriptive set theory Kechris [1] showed that (H, Z) is $\mathbf{\Sigma}^1_1$-hard iff it is Borel $\mathbf{\Sigma}^1_1$-hard. Since the statement of Kechris's theorem is purely classical, one would like to have a classical proof, and, in fact, Kechris asked about a possibility of such a proof.

Using only classical methods we prove the following:

Theorem. *Let $n \geq 1$ and $H \subseteq Z \subseteq \mathcal{C}$. If Borel functions from 2^ω to Z give as preimages of H all $\mathbf{\Sigma}^1_n(\mathcal{C})$ sets, then so do continuous injections.*

Note that for any separable metrizable space S there exists a Borel injection $e\colon S \to \mathcal{C}$ whose inverse is continuous (e.g., $e(s)(i) = 1 \Leftrightarrow s \in O_i$, where $\{O_i\}_{i\in\omega}$ is a basis of S). Moreover, e can be chosen to be continuous if S is zero-dimensional. It follows that we can change in the Theorem the range space Z to any separable metrizable space, and the domain space $\mathcal{C}$ to any zero-dimensional uncountable Polish space.

Let X be an arbitrary separable metrizable space. The projective classes $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$, and $\mathbf{\Delta}^1_n(X)$, $n \geq 1$, are defined in the same way they are defined for a Polish space (see [2, 25.A]). In particular, $Q \in \mathbf{\Delta}^1_1(X)$ iff $Q \in \mathbf{\Sigma}^1_1(X) \cap \mathbf{\Pi}^1_1(X)$, and if X is a subspace of a Polish space $\bar{X}$, then $Q \in \mathbf{\Sigma}^1_n(X)$ iff there is $\bar{Q} \in \mathbf{\Sigma}^1_n(\bar{X})$ with $Q = X \cap \bar{Q}$.

The $\mathbf{\Sigma}^1_1(X)$, $\mathbf{\Pi}^1_1(X)$, and $\mathbf{\Delta}^1_1(X)$ sets are also called, respectively, analytic, co-analytic, and bianalytic in X. Recall that Borel subsets of X are always bianalytic in X, and if X is analytic in a Polish space, then the converse is true; there are, however, $X \in \mathbf{\Pi}^1_1(\mathcal{C})$ for which the converse fails.

A function from one separable metrizable space to another is called bianalytic if preimages of open sets are bianalytic (then preimages of bianalytic sets are also bianalytic).

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Let $\mathcal{P}$ be the set of all nonempty perfect subsets of $\mathcal{C}$ endowed with the Vietoris topology; this is a Polish space, a homeomorph of the Baire space ω^ω. For $G \subseteq \mathcal{C}$, let $\mathcal{P}(G) = \mathcal{P} \cap \mathsf{Pow}\, G$. Recall that if G is G_δ in $\mathcal{C}$, then $\mathcal{P}(G)$ is G_δ in $\mathcal{P}$, and if G is comeager in $\mathcal{C}$, then $\mathcal{P}(G)$ is comeager in $\mathcal{P}$.

For any $Q \subseteq X \times Y$, $f\colon X \times Y \to Z$, and $x \in X$, define the sections $Q_x \subseteq Y$ and $f_x\colon Y \to Z$ by $y \in Q_x \Leftrightarrow (x,y) \in Q$ and $f_x(y) = f(x,y)$.

Fix also a continuous function $\pi\colon \mathcal{P} \times \mathcal{C} \to \mathcal{C}$ such that each section π_p, $p \in \mathcal{P}$, is a homeomorphism from $\mathcal{C}$ onto p (e.g., let π_p be induced by the unique bijection from $2^{<\omega}$ onto the split nodes of the tree $\{s|l\colon s \in p,\ l \in \omega\}$ which preserves the lexicographic ordering).

Proposition. *Let $X \subseteq \mathcal{C}$. Given a bianalytic function $b\colon X \times \mathcal{C} \to \mathcal{C}$, there exists a bianalytic function $\mathfrak{p}\colon X \to \mathcal{P}$ such that for each $x \in X$, $b_x|\mathfrak{p}(x)$ is continuous injective or constant.*

Proof. Let B consist of all pairs $(x,p) \in X \times \mathcal{P}$ such that $b_x|p$ is continuous injective or constant.

We claim that (1) $B \in \mathbf{\Pi}^1_1(X \times \mathcal{P})$, and (2) $\forall\, x \in X\ B_x$ is nonmeager in $\mathcal{P}$; so we can use the "large sections" uniformization for coanalytic sets ([2, 36.F]) to get a bianalytic $\mathfrak{p}\colon X \to \mathcal{P}$ uniformizing B.

(1) First, letting $\{I_n\}_{n\in\omega}$ be an enumeration of all clopen subsets of $\mathcal{C}$, note that $b_x|p$ is continuous iff

$$\forall n\, \exists m\, \forall\, y \in p \quad y \in I_m \quad \Leftrightarrow \quad b(x,y) \in I_n\,.$$

This defines a $\mathbf{\Pi}^1_1(X \times \mathcal{P})$ set since "$b(x,y) \in I_n$" defines a $\mathbf{\Delta}^1_1(X \times \mathcal{C} \times \omega)$ set.

Next, note that $b_x|p$ is injective iff

$$\forall\, y, y' \in p \quad b(x,y) = b(x,y') \ \Rightarrow\ y = y',$$

and constant iff

$$\forall\, y, y' \in p \quad b(x,y) = b(x,y').$$

Clearly, both these formulas define $\mathbf{\Pi}^1_1(X \times \mathcal{P})$ sets.

(2) Fix $x \in X$. Since the section b_x is Borel, it is continuous on a dense G_δ set $G \subseteq \mathcal{C}$. In G^2 consider the open set

$$\nabla = \{(y,y') \in G^2\colon b_x(y) \neq b_x(y')\}.$$

If the section b_x is constant on a nonempty open in G set U, then the set $\mathcal{P}_{const} = \mathcal{P}(U)$ is nonempty and open in $\mathcal{P}(G)$, hence nonmeager in $\mathcal{P}$; clearly b_x is constant on each $p \in \mathcal{P}_{const}$.

Otherwise the set $G^2 \cap \nabla$ is dense open in G^2, and then, by the Kuratowski-Mycielski theorem ([2, 19.1]), the set

$$\mathcal{P}_{injct} \;=\; \{p \in \mathcal{P}\colon\ p^2 \subseteq \nabla \cup \{(y,y)\colon y \in G\}\}$$

is comeager in $\mathcal{P}(G)$, hence also in $\mathcal{P}$; clearly b_x is injective on each $p \in \mathcal{P}_{injct}$. $\square$

Corollary. *Let $X \subseteq \mathcal{C}$. Given bianalytic functions $b\colon X \times \mathcal{C} \to \mathcal{C}$ and $\mathfrak{b}\colon X \to \mathcal{P}$, there exists a bianalytic function $\mathfrak{p}\colon X \to \mathcal{P}$ such that for each $x \in X$, $\mathfrak{p}(x) \subseteq \mathfrak{b}(x)$ and $b_x|\mathfrak{p}(x)$ is continuous injective or constant.*

Proof. Get $\mathfrak{p}'\colon X \to \mathcal{P}$ by the Proposition applied to the function

$$b'(x,y) = b(x, \pi(\mathfrak{b}(x), y)).$$

Then the function $x \mapsto \pi_{\mathfrak{b}(x)}[\mathfrak{p}'(x)]$ is our required $\mathfrak{p}$. Just note that the function $(p, p') \mapsto \pi_p[p']$ is continuous. □

Fix now a bianalytic function that is universal for Borel functions. For this, choose $\mathcal{E} \in \mathbf{\Pi}^1_1(\mathcal{C})$ and $U \in \mathbf{\Delta}^1_1(\mathcal{E} \times (\omega \times \mathcal{C}))$ such that $\{U_\varepsilon\}_{\varepsilon \in \mathcal{E}}$ is the family of all Borel subsets of $\omega \times \mathcal{C}$ (see [2, 35.B]), and define the function $u \colon \mathcal{E} \times \mathcal{C} \to \mathcal{C}$ by

$$u(\varepsilon, y)(n) = 1 \;\Leftrightarrow\; (\varepsilon, n, y) \in U.$$

Then u is bianalytic and $\{u_\varepsilon\}_{\varepsilon \in \mathcal{E}}$ is the family of all Borel functions from $\mathcal{C}$ to $\mathcal{C}$.

Proof of the Theorem. Let (H, Z) be as postulated. For $z \in \mathcal{C}$, define $z^0 \in \mathcal{C}$ by $z^0(i) = z(2i)$. Let $\mathfrak{p} \colon X \to \mathcal{P}$ be obtained by the Corollary applied to $X = \mathcal{E}$, $b = u$, and $\mathfrak{b}$ given by $\varepsilon \to \{z \in \mathcal{C} : z^0 = \varepsilon\}$.

Consider the following bianalytic injection of $\mathcal{E} \times \mathcal{C}$ into $\mathcal{C}$:

$$h(\varepsilon, y) = \pi(\mathfrak{p}(\varepsilon), y).$$

If $Q \in \mathbf{\Sigma}^1_n(\mathcal{C})$, then

$$h[\mathcal{E} \times Q] = \{z \in \mathcal{C} \colon \exists\, y \in Q\; h(z^0, y) = z\} \in \mathbf{\Sigma}^1_n(h[\mathcal{E} \times \mathcal{C}]).$$

Indeed, we have here the projection along $Q \in \mathbf{\Sigma}^1_n(\mathcal{C})$ of the $\mathbf{\Delta}^1_1(h(\mathcal{E} \times \mathcal{C}) \times \mathcal{C})$ set given by the preimage of $\{(z, z) \colon z \in \mathcal{C}\}$ by the bianalytic function

$$h[\mathcal{E} \times \mathcal{C}] \times \mathcal{C} \ni (z, y) \mapsto (h(z^0, y), z).$$

It follows that $h[\mathcal{E} \times Q] = \tilde{Q} \cap h[\mathcal{E} \times \mathcal{C}]$ for some $\tilde{Q} \in \mathbf{\Sigma}^1_1(\mathcal{C})$. So, by our assumptions about (H, Z), there exists a Borel function $f \colon \mathcal{C} \to Z$ such that

$$h[\mathcal{E} \times Q] = f^{-1}[H] \cap h[\mathcal{E} \times \mathcal{C}],$$

hence, since h is injective,

$$\mathcal{E} \times Q = h^{-1}[f^{-1}[H]].$$

Find ε with $f = u_\varepsilon$. Then

$$Q = h_\varepsilon^{-1}[u_\varepsilon^{-1}[H]] = (u_\varepsilon h_\varepsilon)^{-1}[H].$$

The function $u_\varepsilon h_\varepsilon$ is continuous injective or constant, as h_ε is continuous bijective onto $\mathfrak{p}(\varepsilon)$, and $u_\varepsilon | \mathfrak{p}(\varepsilon)$ is continuous injective or constant.

If the function $u_\varepsilon h_\varepsilon$ is injective, we are done. Otherwise it is constant, and it follows that $Q \in \{\mathcal{C}, \varnothing\}$. But then there is a continuous injective $e \colon \mathcal{C} \to Z$ with $Q = e^{-1}[H]$, since both sets H and $Z \setminus H$ contain copies of $\mathcal{C}$.[1] □

References

[1] Alexander S. Kechris, *On the concept of* $\mathbf{\Pi}^1_1$*-completeness*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1811–1814, DOI 10.1090/S0002-9939-97-03770-2. MR1372034 (97g:03052)

[2] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597 (96e:03057)

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[1] Fix $G \in G_\delta(\mathcal{C}) \setminus F_\sigma(\mathcal{C})$. Let $g \colon \mathcal{C} \to Z$ be continuous with $G = g^{-1}[H]$. Then $g[G]$ is uncountable, as otherwise $G = g^{-1}[g[G]]$ would be F_σ. Being an uncountable $\mathbf{\Sigma}^1_1$ set, $g[G]$ contains a copy of $\mathcal{C}$. The same argument works for $Z \setminus H$.