ON THE CONCEPT OF ANALYTIC HARDNESS

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(Communicated by Mirna Dzamonja)

Abstract. Let $H \subseteq Z \subseteq 2^\omega$. Using only classical descriptive set theory we prove that if Borel functions from $2^\omega$ to $Z$ give as preimages of $H$ all analytic subsets of $2^\omega$, then so do continuous injections. This strengthens a theorem Kechris proved by means of effective descriptive set theory.

Let $H \subseteq Z$ be subsets of the Cantor space $C = 2^\omega$. The pair $(H, Z)$ is called $\Sigma^1_1$-hard, resp. $\Pi^1_1$-hard, if for any $\Sigma^1_1$ set $Q \subseteq C$ there is a continuous, resp. Borel, function $f: C \to Z$ with $Q = f^{-1}[H]$. Using effective descriptive set theory Kechris [1] showed that $(H, Z)$ is $\Sigma^1_1$-hard if it is Borel $\Sigma^1_1$-hard. Since the statement of Kechris’s theorem is purely classical, one would like to have a classical proof, and, in fact, Kechris asked about a possibility of such a proof.

Using only classical methods we prove the following:

Theorem. Let $n \geq 1$ and $H \subseteq Z \subseteq C$. If Borel functions from $2^\omega$ to $Z$ give as preimages of $H$ all $\Sigma^1_n(C)$ sets, then so do continuous injections.

Note that for any separable metrizable space $S$ there exists a Borel injection $e: S \to C$ whose inverse is continuous (e.g., $e(s)(i) = 1 \iff s \in O_i$, where $\{O_i\}_{i \in \omega}$ is a basis of $S$). Moreover, $e$ can be chosen to be continuous if $S$ is zero-dimensional. It follows that we can change in the Theorem the range space $Z$ to any separable metrizable space, and the domain space $C$ to any zero-dimensional uncountable Polish space.

Let $X$ be an arbitrary separable metrizable space. The projective classes $\Sigma^1_n(X)$, $\Pi^1_n(X)$, and $\Delta^1_n(X)$, $n \geq 1$, are defined in the same way they are defined for a Polish space (see [25, A]). In particular, $Q \in \Delta^1_1(X)$ iff $Q \in \Sigma^1_1(X) \cap \Pi^1_1(X)$, and if $X$ is a subspace of a Polish space $\bar{X}$, then $Q \in \Sigma^1_n(X)$ iff there is $\bar{Q} \in \Sigma^1_n(\bar{X})$ with $Q = X \cap \bar{Q}$.

The $\Sigma^1_1(X)$, $\Pi^1_1(X)$, and $\Delta^1_1(X)$ sets are also called, respectively, analytic, coanalytic, and bianalytic in $X$. Recall that Borel subsets of $X$ are always bianalytic in $X$, and if $X$ is analytic in a Polish space, then the converse is true; there are, however, $X \in \Pi^1_1(C)$ for which the converse fails.

A function from one separable metrizable space to another is called bianalytic if preimages of open sets are bianalytic (then preimages of bianalytic sets are also bianalytic).
Let \( \mathcal{P} \) be the set of all nonempty perfect subsets of \( \mathcal{C} \) endowed with the Vietoris topology; this is a Polish space, a homeomorph of the Baire space \( \omega^\omega \). For \( G \subseteq \mathcal{C} \), let \( \mathcal{P}(G) = \mathcal{P} \cap \text{Pow} \ G \). Recall that if \( G \) is \( G_\delta \) in \( \mathcal{C} \), then \( \mathcal{P}(G) \) is \( G_\delta \) in \( \mathcal{P} \), and if \( G \) is comeager in \( \mathcal{C} \), then \( \mathcal{P}(G) \) is comeager in \( \mathcal{P} \).

For any \( Q \subseteq X \times Y \), \( f: X \times Y \to Z \), and \( x \in X \), define the sections \( Q_x \subseteq Y \) and \( f_x: Y \to Z \) by \( y \in Q_x \iff (x,y) \in Q \) and \( f_x(y) = f(x,y) \).

Fix also a continuous function \( \pi: \mathcal{P} \times \mathcal{C} \to \mathcal{C} \) such that each section \( \pi_p \), \( p \in \mathcal{P} \), is a homeomorphism from \( \mathcal{C} \) onto \( p \) (e.g., let \( \pi_p \) be induced by the unique bijection from \( 2^{<\omega} \) onto the split nodes of the tree \( \{s|l : s \in p, l \in \omega \} \) which preserves the lexicographic ordering).

**Proposition.** Let \( X \subseteq \mathcal{C} \). Given a bianalytic function \( b: X \times \mathcal{C} \to \mathcal{C} \), there exists a bianalytic function \( p: X \to \mathcal{P} \) such that for each \( x \in X \), \( b_x|p(x) \) is continuous injective or constant.

**Proof.** Let \( B \) consist of all pairs \( (x,p) \in X \times \mathcal{P} \) such that \( b_x|p \) is continuous injective or constant.

We claim that (1) \( B \in \Pi_1^1(X \times \mathcal{P}) \), and (2) \( \forall x \in X \ B_x \) is nonmeager in \( \mathcal{P} \); so we can use the “large sections” uniformization for coanalytic sets ([2 36.F]) to get a bianalytic \( p: X \to \mathcal{P} \) uniformizing \( B \).

(1) First, letting \( \{I_n\}_{n \in \omega} \) be an enumeration of all clopen subsets of \( \mathcal{C} \), note that \( b_x|p \) is continuous iff
\[
\forall n \exists m \forall y \in p \ y \in I_m \iff b(x,y) \in I_n.
\]
This defines a \( \Pi_1^1(X \times \mathcal{P}) \) set since \( b(x,y) \in I_n \) defines a \( \Delta_1^1(X \times \mathcal{C} \times \omega) \) set.

Next, note that \( b_x|p \) is injective iff
\[
\forall y,y' \in p \ b(x,y) = b(x,y') \Rightarrow y = y',
\]
and constant iff
\[
\forall y,y' \in p \ b(x,y) = b(x,y').
\]
Clearly, both these formulas define \( \Pi_1^1(X \times \mathcal{P}) \) sets.

(2) Fix \( x \in X \). Since the section \( b_x \) is Borel, it is continuous on a dense \( G_\delta \) set \( G \subseteq \mathcal{C} \). In \( G^2 \) consider the open set
\[
\nabla = \{(y,y') \in G^2 : b_x(y) \neq b_x(y')\}.
\]
If the section \( b_x \) is constant on a nonempty open in \( G \) set \( U \), then the set \( \mathcal{P}_{\text{const}} = \mathcal{P}(U) \) is nonempty and open in \( \mathcal{P}(G) \), hence nonmeager in \( \mathcal{P} \); clearly \( b_x \) is constant on each \( p \in \mathcal{P}_{\text{const}} \).

Otherwise the set \( G^2 \cap \nabla \) is dense open in \( G^2 \), and then, by the Kuratowski-Mycielski theorem ([2 19.1]), the set
\[
\mathcal{P}_{\text{inj}} = \{p \in \mathcal{P} : p^2 \subseteq \nabla \cup \{(y,y) : y \in G\}\}
\]
is comeager in \( \mathcal{P}(G) \), hence also in \( \mathcal{P} \); clearly \( b_x \) is injective on each \( p \in \mathcal{P}_{\text{inj}} \).

**Corollary.** Let \( X \subseteq \mathcal{C} \). Given bianalytic functions \( b: X \times \mathcal{C} \to \mathcal{C} \) and \( b: X \to \mathcal{P} \), there exists a bianalytic function \( p: X \to \mathcal{P} \) such that for each \( x \in X \), \( p(x) \subseteq b(x) \) and \( b_x|p(x) \) is continuous injective or constant.

**Proof.** Get \( p': X \to \mathcal{P} \) by the Proposition applied to the function
\[
b'(x,y) = b(x,\pi(b(x),y)).
\]
Then the function $x \mapsto \pi_{b(\varepsilon)}[p'(x)]$ is our required $p$. Just note that the function $(p, p') \mapsto \pi_p[p']$ is continuous. \hfill \Box

Fix now a bianalytic function that is universal for Borel functions. For this, choose $E \in \Pi^1_1(C)$ and $U \in \Delta^1_1(E \times (\omega \times C))$ such that $\{U_\varepsilon\}_{\varepsilon \in E}$ is the family of all Borel subsets of $\omega \times C$ (see [2], 35.B), and define the function $u: E \times C \to C$ by
\[
u(\varepsilon, y)(n) = 1 \iff (\varepsilon, n, y) \in U.
\]
Then $u$ is bianalytic and $\{u_\varepsilon\}_{\varepsilon \in E}$ is the family of all Borel functions from $C$ to $C$.

Proof of the Theorem. Let $(H, Z)$ be as postulated. For $z \in C$, define $z^0 \in C$ by $z^0(i) = z(2i)$. Let $p: X \to P$ be obtained by the Corollary applied to $X = E$, $b = u$, and $b$ given by $\varepsilon \to \{z \in C : z^0 = \varepsilon\}$.

Consider the following bianalytic injection of $E \times C$ into $C$:
\[
h(\varepsilon, y) = \pi(p(\varepsilon), y).
\]

If $Q \in \Sigma^1_n(C)$, then
\[
h[E \times Q] = \{z \in C : \exists y \in Q \ h(z^0, y) = z\} \in \Sigma^1_n(h[E \times C]).
\]
Indeed, we have here the projection along $Q \in \Sigma^1_n(C)$ of the $\Delta^1_1(h(E \times C) \times C)$ set given by the preimage of $\{(z, z) : z \in C\}$ by the bianalytic function
\[
h[E \times C] \times C \ni (z, y) \mapsto (h(z^0, y), z).
\]
It follows that $h[E \times Q] = \tilde{Q} \cap h[E \times C]$ for some $\tilde{Q} \in \Sigma^1_1(C)$. So, by our assumptions about $(H, Z)$, there exists a Borel function $f: C \to Z$ such that
\[
h[E \times Q] \subseteq f^{-1}[H] \cap h[E \times C],
\]
hence, since $h$ is injective,
\[
E \times Q = h^{-1}[f^{-1}[H]].
\]

Find $\varepsilon$ with $f = u_\varepsilon$. Then
\[
Q = h^{-1}_\varepsilon[u^{-1}_\varepsilon[H]] = (u_\varepsilon h_\varepsilon)^{-1}[H].
\]
The function $u_\varepsilon h_\varepsilon$ is continuous injective or constant, as $h_\varepsilon$ is continuous bijective onto $p(\varepsilon)$, and $p(\varepsilon)$ is continuous injective or constant.

If the function $u_\varepsilon h_\varepsilon$ is injective, we are done. Otherwise it is constant, and it follows that $Q \in \{C, \varnothing\}$. But then there is a continuous injective $e: C \to Z$ with $Q = e^{-1}[H]$, since both sets $H$ and $Z \setminus H$ contain copies of $C$. \hfill \Box

REFERENCES


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\[\text{\footnotesize 1} \text{ Fix } G \in G_\delta(C) \setminus F_\sigma(C). \text{ Let } g: C \to Z \text{ be continuous with } G = g^{-1}[H]. \text{ Then } g(G) \text{ is uncountable, as otherwise } G = g^{-1}[g(G)] \text{ would be } F_\sigma. \text{ Being an uncountable } \Sigma^1_1 \text{ set, } g(G) \text{ contains a copy of } C. \text{ The same argument works for } Z \setminus H.\]