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UNIQUE CONTINUATION FOR FRACTIONAL SCHRÖDINGER OPERATORS IN THREE AND HIGHER DIMENSIONS

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ABSTRACT. We prove the unique continuation property for the differential inequality $|(-\Delta)^{\alpha/2}u| \le |V(x)u|$, where $0 < \alpha < n$ and $V \in L^{n/\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, $n \ge 3$.

1. Introduction

In this note we are concerned with the unique continuation property for solutions of the differential inequality

$$(1.1) \qquad |(-\Delta)^{\alpha/2}u| \le |V(x)u|, \quad x \in \mathbb{R}^n, \quad n \ge 2,$$

where $(-\Delta)^{\alpha/2}$, $0 < \alpha < n$, is defined by means of the Fourier transform $\mathcal{F}f\ (= \widehat{f}\,)$:

$$\mathcal{F}[(-\Delta)^{\alpha/2}f](\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

In particular, the equation $((-\Delta)^{\alpha/2} + V(x))u = 0$ has attracted interest from quantum mechanics in the case $1 < \alpha < 2$ as well as the case $\alpha = 2$. Recently, by generalizing the Feynman path integral to the Lévy one, Laskin [5] introduced the fractional quantum mechanics in which it is conjectured that physical realizations may be limited to $1 < \alpha < 2$, where averaged quantities are finite, and the fractional Schrödinger operator $(-\Delta)^{\alpha/2}+V(x)$ plays a central role. Of course, the case $\alpha = 2$ becomes equivalent to an ordinary quantum mechanics.

The unique continuation property means that a solution of (1.1) which vanishes in an open subset of $\mathbb{R}^n$ must vanish identically. In the case of $\alpha = 2$, Jerison and Kenig [1] proved the property for $V \in L^{n/2}_{\mathrm{loc}}$, $n \ge 3$. An extension to $L^{n/2,\infty}_{\mathrm{loc}}$ was obtained by Stein [9] with small norm in the sense that

$$\sup_{a\in\mathbb{R}^n} \lim_{r\to 0} \|V\|_{L^{n/2,\infty}(B(a,r))}$$

is sufficiently small. Here, $B(a,r)$ denotes the ball of radius $r > 0$ centered at $a \in \mathbb{R}^n$. These results later turn out to be optimal in the context of L^p spaces ([2,3]).

On the other hand, the results when $\alpha \ne 2$ are rather scarce. Laba [4] considered the higher orders where $\alpha/2$ are integers, and obtained the property for $V \in L^{n/\alpha}_{\mathrm{loc}}$. Recently, there was an attempt [7] to handle the non-integer orders when $n - 1 \le \alpha < n$, $n \ge 2$, from which it turns out that the condition $V \in L^p$, $p > n/\alpha$, is sufficient to have the property. Hence this particularly gives a unique continuation

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result for the fractional Schrödinger operator in the full range $1 < \alpha < 2$ when $n = 2$. Our aim here is to fill the gap, $0 < \alpha < n - 1$, for $n \geq 3$, which allows us to have the unique continuation for the fractional Schrödinger operator when $n \geq 3$ with the full range of α.

Theorem 1.1. *Let $n \geq 3$ and $0 < \alpha < n$. Assume that $V \in L^{n/\alpha,\infty}_{\text{loc}}$ and u is a non-zero solution of* (1.1) *such that*

$$u \in L^1 \cap L^{p,q} \quad \text{and} \quad (-\Delta)^{\alpha/2} u \in L^q, \tag{1.2}$$

where $p = 2n/(n-\alpha)$ and $q = 2n/(n+\alpha)$. Then it cannot vanish in any non-empty open subset of $\mathbb{R}^n$ if

$$\sup_{a \in \mathbb{R}^n} \lim_{r \to 0} \|V\|_{L^{n/\alpha,\infty}(B(a,r))} \tag{1.3}$$

is sufficiently small. Here, $L^{p,q}$ denotes the usual Lorentz space.

Remarks. (a) The smallness condition (1.3) is trivially satisfied for $V \in L^{n/\alpha}_{\text{loc}}$ because $L^{n/\alpha}_{\text{loc}} \subset L^{n/\alpha,\infty}_{\text{loc}}$. Hence the above theorem can be seen as natural extensions to (1.1) of the results obtained in [1, 9] for the Schrödinger operator ($\alpha = 2$). As an immediate consequence of the theorem, the same result also holds for the stationary equation

$$((-\Delta)^{\alpha/2} + V(x))u = Eu, \quad E \in \mathbb{C},$$

because $(-\Delta)^{\alpha/2} u = (E - V(x))u$ and the condition (1.3) is trivially satisfied for the constant E.

(b) The index n/α is quite natural, in view of the standard rescaling: $u_\varepsilon(x) = u(\varepsilon x)$ takes the equation $(-\Delta)^{\alpha/2} u = Vu$ into $(-\Delta)^{\alpha/2} u_\varepsilon = V_\varepsilon u_\varepsilon$, where $V_\varepsilon(x) = \varepsilon^\alpha V(\varepsilon x)$. So, $\|V_\varepsilon\|_{L^{p,\infty}} = \varepsilon^{\alpha - n/p}\|V\|_{L^{p,\infty}}$. Hence the $L^{p,\infty}$ norm of V_ε is independent of ε precisely when $p = n/\alpha$.

(c) When $\alpha = n$ in (1.1), there are some unique continuation results with $V \in L^p_{\text{loc}}$, $p > 1$. (See [1] and [6] for $\alpha = 2$ and $\alpha = 2m$ ($m \in \mathbb{N}$), respectively.)

2. Proof of the theorem

From now on, we will use the letter C to denote a constant that may be different at each occurrence.

Without loss of generality, we need to prove that u must vanish identically if it vanishes in a sufficiently small neighborhood of zero.

Our proof is based on the following Carleman estimate which will be shown below: If $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $(-\Delta)^{\alpha/2} f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, then there is a constant C depending only on $\delta_t := \min_{k \in \mathbb{Z}} |t - k|$ and n such that for $t \notin \mathbb{Z}$ with $\delta_t < n - \alpha$

$$\big\||x|^{-t-n/p} f\big\|_{L^{p,q}} \leq C \big\||x|^{-t+\alpha-n/q} (-\Delta)^{\alpha/2} f\big\|_{L^q}, \tag{2.1}$$

where p, q are given as in the theorem (i.e., $1/p + 1/q = 1$ and $1/q - 1/p = \alpha/n$).

Indeed, since we are assuming that $u \in L^1 \cap L^{p,q}$ and $(-\Delta)^{\alpha/2} u \in L^q$ vanish near zero (see (1.2), (1.1)), from (2.1) (with a standard limiting argument involving a C_0^∞ approximate identity), we see that

$$\big\||x|^{-t-n/p} u\big\|_{L^{p,q}} \leq C \big\||x|^{-t+\alpha-n/q} (-\Delta)^{\alpha/2} u\big\|_{L^q}.$$

Hence,

$$\big\||x|^{-t-n/p}u\big\|_{L^{p,q}(B(0,r))} \le C\big\||x|^{-t+\alpha-n/q}Vu\big\|_{L^q(B(0,r))} + C\big\||x|^{-t+\alpha-n/q}(-\Delta)^{\alpha/2}u\big\|_{L^q(\mathbb{R}^n\setminus B(0,r))}.$$

The first term on the right-hand side can be absorbed into the left-hand side as follows:

$$C\big\||x|^{-t+\alpha-n/q}Vu\big\|_{L^q(B(0,r))} \le C\|V\|_{L^{n/\alpha,\infty}(B(0,r))}\big\||x|^{-t+\alpha-n/q}u\big\|_{L^{p,q}(B(0,r))} \le \frac{1}{2}\big\||x|^{-t-n/p}u\big\|_{L^{p,q}(B(0,r))}$$

if we choose r small enough (see (1.3)). Here, recall that $\alpha - n/q = -n/p$, and $\||x|^{-t-n/p}u\|_{L^{p,q}(B(0,r))}$ is finite since $u \in L^{p,q}$ vanishes near zero. So, we get

$$\|(r/|x|)^{t+n/p}u\|_{L^{p,q}(B(0,r))} \le 2C\|(-\Delta)^{\alpha/2}u\|_{L^q(\mathbb{R}^n\setminus B(0,r))} < \infty.$$

Now, we choose a sequence $\{t_i\}$ of values of t tending to infinity such that δ_{t_i} is independent of $i \in \mathbb{N}$. Then, by letting $i \to \infty$, we see that $u = 0$ on $B(0,r)$, which implies $u \equiv 0$ by a standard connectedness argument.

Proof of (2.1). We will show (2.1) using Stein's complex interpolation, as in [9], on an analytic family of operators T_z defined by

$$T_z g(x) = \int_{\mathbb{R}^n} K_z(x,y)g(y)|y|^{-n}dy,$$

where $K_z(x,y) = H_z(x,y)/\Gamma(n/2 - z/2)$ with

$$H_z(x,y) = |x|^{-t}|y|^{n+t-z}c_z\bigg(|x-y|^{-n+z} - \sum_{j=0}^{m-1}\frac{1}{j!}\Big(\frac{\partial}{\partial s}\Big)^j|sx-y|^{-n+z}|_{s=0}\bigg).$$

Note that for $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$

$$T_\alpha(|y|^{-t+\alpha}(-\Delta)^{\alpha/2}f(y))(x) = |x|^{-t}f(x)/\Gamma(n/2-\alpha/2) \tag{2.2}$$

(see Lemma 2.1 in [7]).

Let m be a fixed positive integer such that $m-1 < t < m$, and recall the following two estimates for the cases of $\operatorname{Re} z = 0$ (Lemma 2.3 in [1]) and $n-1 < \operatorname{Re} z < n-\delta_t$ (Lemma 4 in [9]): There is a constant C depending only on δ_t and n such that

$$\|T_{i\gamma}g\|_{L^2(dx/|x|^n)} \le Ce^{c|\gamma|}\|g\|_{L^2(dx/|x|^n)}, \quad \gamma \in \mathbb{R}, \tag{2.3}$$

and

$$\|T_z g\|_{L^r(dx/|x|^n)} \le Ce^{c|\gamma|}\|g\|_{L^s(dx/|x|^n)}, \quad \gamma = \operatorname{Im} z \in \mathbb{R}, \tag{2.4}$$

where $n-1 < \beta = \operatorname{Re} z < n - \delta_t$, $1/s - 1/r = \beta/n$ and $1 < s < n/\beta$.

We first consider the case where $n-1 < \alpha < n$. Note that we can choose β so that $\alpha < \beta < n - \delta_t$, since we are assuming $\delta_t < n - \alpha$. Hence, by Stein's complex interpolation ([8]) between (2.3) and (2.4), we see that

$$\|T_\alpha g\|_{L^r(dx/|x|^n)} \le C\|g\|_{L^s(dx/|x|^n)}, \tag{2.5}$$

where $1/s - 1/r = \alpha/n$ and $1 < s < n/\alpha$. From this and (2.2), we get

$$\big\||x|^{-t-n/r}f\big\|_{L^r} \le C\big\||x|^{-t+\alpha-n/s}(-\Delta)^{\alpha/2}f\big\|_{L^s} \tag{2.6}$$

with the same r, s in (2.5), since we are assuming $(-\Delta)^{\alpha/2} f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Note that $1 < q < n/\alpha$. So, we can choose r_j, s_j, $j = 1, 2$, such that

$$1 < s_1 < q < s_2 < n/\alpha, \quad 1/s_j - 1/r_j = \alpha/n,$$

and for $t_j = t + n(1/p - 1/r_j)$

$$m - 1 < t_j < m, \quad \delta_t/2 \le \delta_{t_j} \le 3\delta_t/2.$$

Hence we can apply (2.6) with $t = t_j$ to obtain

$$\big\| |x|^{-t-n/p} f \big\|_{L^{r_j}} \le C \big\| |x|^{-t+\alpha-n/q} (-\Delta)^{\alpha/2} f \big\|_{L^{s_j}} \tag{2.7}$$

for $j = 1, 2$. Since $r_1 < p < r_2$ and $s_1 < q < s_2$, by real interpolation ([8]) between the estimates in (2.7), we see that for $1 \le w \le \infty$

$$\big\| |x|^{-t-n/p} f \big\|_{L^{p,w}} \le C \big\| |x|^{-t+\alpha-n/q} (-\Delta)^{\alpha/2} f \big\|_{L^{q,w}}.$$

By choosing $w = q$, we get (2.1).

Now we turn to the remaining case where $0 < \alpha \le n - 1$. In this case, (2.5) is valid for $1/s - 1/r = \alpha/n$ and

$$\frac{1}{2}(1 - \frac{\alpha}{n-1}) + \frac{\alpha}{n} < \frac{1}{s} < \frac{1}{2} + \frac{\alpha}{2(n-1)}, \tag{2.8}$$

because we can choose β so that $n - 1 < \beta < n - \delta_t$. Since (2.8) holds for s replaced by q, repeating the previous argument, one can show (2.1). We omit the details. $\square$

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