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RESCALING PRINCIPLE FOR ISOLATED ESSENTIAL SINGULARITIES OF QUASIREGULAR MAPPINGS

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Abstract. We establish a rescaling theorem for isolated essential singularities of quasiregular mappings. As a consequence we show that the class of closed manifolds receiving a quasiregular mapping from a punctured unit ball with an essential singularity at the origin is exactly the class of closed quasiregularly elliptic manifolds, that is, closed manifolds receiving a non-constant quasiregular mapping from a Euclidean space.

1. Introduction

A continuous mapping $f\colon M \to N$ between oriented Riemannian n-manifolds is *K-quasiregular* if f belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(M,N)$ and satisfies the distortion inequality

$$\|Df\|^n \leq K J_f \quad \text{a.e.},$$

where $\|Df\|$ is the operator norm and J_f is the Jacobian determinant of the differential Df of f.

The main result of this paper is the following rescaling theorem. We denote the open unit ball about the origin in $\mathbb{R}^n$ by $\mathbb{B}^n$. We say that a quasiregular mapping f from $\mathbb{B}^n \setminus \{0\}$ to a closed and oriented Riemannian n-manifold N has *an essential singularity at the origin* if the limit $\lim_{x\to 0} f(x)$ does not exist in N.

Theorem 1. *Let N be a closed and oriented Riemannian n-manifold, $n \geq 2$, and let $f\colon \mathbb{B}^n \setminus \{0\} \to N$ be a K-quasiregular mapping with an essential singularity at the origin, $K \geq 1$. Then there exist a non-constant K-quasiregular mapping $g\colon X \to N$, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, and sequences (x_k) and (ρ_k) in $\mathbb{B}^n$ and $(0,\infty)$, respectively, such that $\lim_{k\to\infty} x_k = 0$, $\lim_{k\to\infty} \rho_k = 0$ and*

$$\lim_{k\to\infty} f(x_k + \rho_k v) = g(v)$$

locally uniformly on X.

Theorem 1 bears a close resemblance to Miniowitz's Zalcman lemma for quasiregular mappings; see Miniowitz [10] and Zalcman [16]. It seems, however, that this version for isolated essential singularities has gone unnoticed in the quasiregular literature although the heuristic idea behind this rescaling principle is well known

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in the classical holomorphic case ($n = 2$ and $K = 1$); see, e.g., Bergweiler [1] and Minda [9].

Theorem 1 readily yields the following characterization of closed and oriented Riemannian manifolds receiving a quasiregular mapping with an isolated essential singularity.

Theorem 2. *Let N be a closed and oriented Riemannian n-manifold, $n \geq 2$. If there exists a K-quasiregular mapping $f\colon \mathbb{B}^n \setminus \{0\} \to N$ having an essential singularity at the origin, $K \geq 1$, then there exists a non-constant K'-quasiregular mapping $g\colon \mathbb{R}^n \to N$ satisfying $g(\mathbb{R}^n) \subset f(\mathbb{B}^n \setminus \{0\})$.*

Conversely, if there exists a non-constant K-quasiregular mapping $g\colon \mathbb{R}^n \to N$, $K \geq 1$, then there exists a K'-quasiregular mapping $f\colon \mathbb{B}^n \setminus \{0\} \to N$ having an essential singularity at the origin such that $f(\mathbb{B}^n \setminus \{0\}) \subset g(\mathbb{R}^n)$.

Here $K' = K'(n, K) \geq 1$ depends only on n and K, and $K'(2, K) = K$.

Having Theorem 2 at our disposal, we readily obtain "big" versions of Varopoulos's theorem [15, pp. 146-147] and the Bonk–Heinonen theorem [2, Theorem 1.1], which respectively give a bound of the fundamental group and the de Rham cohomology ring of a closed quasiregularly elliptic manifold. Recall that a connected and oriented Riemannian n-manifold N, $n \geq 2$, is called *quasiregularly elliptic* if there exists a non-constant quasiregular mapping from $\mathbb{R}^n$ to N.

Corollary 1. *Let N be a closed, connected, and oriented Riemannian n-manifold, $n \geq 2$, with a K-quasiregular mapping $\mathbb{B}^n \setminus \{0\} \to N$ having an essential singularity at the origin, $K \geq 1$. Then the fundamental group $\pi_1(N)$ of N has polynomial growth of order at most n, and the de Rham cohomology ring $H^*(N)$ of N satisfies*

$$\dim H^*(N) := \sum_{k=0}^{n} \dim H^k(N) \leq C, \tag{1.1}$$

where $C = C(n, K) > 0$ depends only on n and K.

Although the former half of Corollary 1, the big Varopoulos theorem, is well known to the experts, we have been unable to find it in the literature. For a direct proof of the big Bonk–Heinonen theorem, i.e., the bound (1.1), see [12].

We would also like to note that together with the Holopainen–Rickman Picard theorem for quasiregularly elliptic manifolds [6], we obtain a big Picard type theorem for quasiregular mappings into closed manifolds; see also [5].

Corollary 2. *Let N be a closed, oriented, and connected Riemannian n-manifold, $n \geq 2$, and $f\colon \mathbb{B}^n \setminus \{0\} \to N$ be a K-quasiregular mapping with an essential singularity at the origin, $K \geq 1$. Then for every $x \in N$, except for at most $q - 1$ points, it holds that $\#f^{-1}(x) = \infty$, where $q = q(n, K) \in \mathbb{N}$ depends only on n and K.*

We conclude this introduction with an application of Theorem 1 to the Ahlfors five islands theorem; see, e.g., Bergweiler [1] or Nevanlinna [11, XII §7, §8] for a detailed discussion.

Let f be a *quasimeromorphic function* on a domain U in the 2-sphere $\mathbb{S}^2$, i.e., a quasiregular mapping from U to $\mathbb{S}^2$. We say that f has a *simple island Ω over a Jordan domain D' in $\mathbb{S}^2$* if Ω is a relatively compact subdomain in U and is mapped univalently onto D' by f. The Ahlfors five islands theorem states that *given five Jordan domains in $\mathbb{S}^2$ with pair-wise disjoint closures, any non-constant*

quasimeromorphic function on $\mathbb{R}^2$ has a simple island over one of these Jordan domains.

Corollary 3. *Let f be a quasimeromorphic function on $\mathbb{B}^2\setminus\{0\}$ having an essential singularity at the origin. Then given five Jordan domains $D_1,\dots,D_5$ in $\mathbb{S}^2$ with pairwise disjoint closures, f has a simple island over one of $D_1,\dots,D_5$.*

Proof. Applying Theorem 1 to f, we obtain sequences (x_k) and (ρ_k), and a non-constant quasimeromorphic function g on $\mathbb{R}^2\setminus\{0\}$, where f_k is the mapping $v\mapsto f(x_k+\rho_k v)$ and g is the locally uniform limit of (f_k), as in Theorem 1. We may fix Jordan domains $D_1',\dots,D_5'$ in $\mathbb{S}^2$ satisfying $D_j\Subset D_j'$ for every $j\in\{1,2,3,4,5\}$ and having pair-wise disjoint closures. By the Ahlfors five islands theorem, the quasimeromorphic function $g\circ\exp$ on $\mathbb{R}^2$ has a simple island $\tilde\Omega'$ over one of these Jordan domains, say D_j'. Hence g has a simple island Ω' over D_j'. By Rouché's theorem, for every $k\in\mathbb{N}$ large enough, f_k has a simple island $\Omega_k\Subset\Omega'$ over $D_j\Subset D_j'$. □

2. Preliminaries

Let $\mathbb{B}^n(x,r)$ be the open ball in $\mathbb{R}^n$ about $x\in\mathbb{R}^n$ of radius $r>0$. Set $\mathbb{B}^n(r):=\mathbb{B}^n(0,r)$ for each $r>0$ and set $\mathbb{B}^n:=\mathbb{B}^n(1)$. The corresponding closed balls are denoted by $\bar{\mathbb{B}}^n(x,r)$, $\bar{\mathbb{B}}^n(r)$, and $\bar{\mathbb{B}}^n$, respectively.

Let M be an oriented Riemannian n-manifold, $n\geq 2$. We denote by $|x-y|$ the distance between x and y in M, and by $B(x,r)$ the Riemannian ball $\{y\in M\colon |x-y|<r\}$ about $x\in M$ of radius $r>0$ in M. Similarly, we denote by $\bar B(x,r)$ the corresponding closed ball about $x\in M$ of radius $r>0$.

By [14, III.1.11], every K-quasiregular mapping from an open set $U\subset\mathbb{R}^n$ to $\mathbb{R}^n$ is locally α-Hölder continuous with $\alpha=(1/K)^{1/(n-1)}$. We refer to [14] and [7] for the Euclidean theory of quasiregular mappings.

For every $x\in M$, there exist $r>0$ and a 2-bilipschitz chart $B(x,r)\to\mathbb{R}^n$. Thus every K-quasiregular mapping from an open set $U\subset\mathbb{R}^n$ to M is locally β-Hölder continuous with $\beta=\beta(n,K)$ depending only on n and K.

The local Hölder continuity plays a key role in the proof of the following manifold version ([7, Theorem 19.9.3]) of Miniowitz's Zalcman lemma [10, §4]. Recall that a family $\mathcal{F}$ of K-quasiregular mappings from a domain Ω in $\mathbb{R}^n$ to M is *normal on an open subset U in Ω* if every infinite sequence (f_k) in $\mathcal{F}$ contains a subsequence which converges locally uniformly on U, and is *normal at a point $a\in\Omega$* if $\mathcal{F}$ is normal on some open neighborhood of a.

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^n$, and N be a closed, connected, and oriented Riemannian n-manifold, $n\geq 2$. Let $\mathcal{F}$ be a family of K-quasiregular mappings from Ω to N, $K\geq 1$. If $\mathcal{F}$ is not normal at $a\in\Omega$, then there exist sequences (x_j), (ρ_j), and (f_j) in Ω, $(0,\infty)$, and $\mathcal{F}$, respectively, and a non-constant K-quasiregular mapping $g\colon\mathbb{R}^n\to N$ such that $\lim_{j\to\infty}x_j=a$, $\lim_{j\to\infty}\rho_j=0$ and*

$$\lim_{j\to\infty} f_j(x_j+\rho_j v)=g(v)$$

locally uniformly on $\mathbb{R}^n$.

3. Proof of Theorem 1

We begin the proof of Theorem 1 by showing a manifold version of a classical lemma on isolated essential singularities due to Lehto and Virtanen [8]; see also Heinonen and Rossi [4, Theorem 2.3] and Gauld and Martin [3].

Lemma 3.1. *Let M be a closed and oriented Riemannian n-manifold, $n \geq 2$, and $f\colon \mathbb{B}^n \setminus \{0\} \to M$ be a quasiregular mapping with an essential singularity at the origin. Then*

$$\limsup_{r\to 0} \operatorname{diam} f(\partial \mathbb{B}^n(r)) > 0. \tag{3.1}$$

Proof. The proof follows the argument of Heinonen and Rossi in [4]. Set $A(r', r) := \mathbb{B}^n(r) \setminus \bar{\mathbb{B}}^n(r')$ for each $r, r' > 0$, $r > r'$.

Since the origin is an isolated essential singularity of f and M is compact, there exist sequences (z_k) and (w_j) in $\mathbb{B}^n \setminus \{0\}$ such that $\lim_{k\to\infty} z_k = \lim_{j\to\infty} w_j = 0$ and that both limits

$$a := \lim_{k\to\infty} f(z_k) \quad \text{and} \quad b := \lim_{j\to\infty} f(w_j)$$

exist in M and are distinct.

Using the exponential map $\exp_a\colon T_aM \to M$, we find $R \in (0, |b-a|/4)$ and a 2-bilipschitz map $\varphi\colon B(a, 4R) \to \mathbb{B}^n(4R)$ satisfying $\varphi(a) = 0$. Note that $b \notin \bar{B}(a, 4R)$ by our choice of R.

Suppose that (3.1) does not hold. Then there is $r_0 > 0$ such that, for every $r \in (0, r_0)$, $\operatorname{diam}(f(\partial \mathbb{B}^n(r))) < R/8$.

We fix $k_1 \in \mathbb{N}$ so large that $|z_{k_1}| < r_0$ and that $|f(z_{k_1}) - a| \leq R/8$. Let $r_1 := |z_{k_1}|$. Since $\operatorname{diam}(f(\partial \mathbb{B}^n(r_1))) < R/8$, we have

$$f(\partial \mathbb{B}^n(r_1)) \subset \varphi^{-1}(\mathbb{B}^n(R/4)) \subset B(a, R). \tag{3.2}$$

Since $f(w_j) \to b \notin \bar{B}(a, 4R)$ as $j \to \infty$, the continuity of f implies the existence of the maximal element, say $r_2 \in (0, r_1)$, of the subset

$$\{r \in (0, r_1]\colon f(\partial \mathbb{B}^n(r)) \not\subset B(a, 2R)\}.$$

By maximality of r_2, $f(\partial \mathbb{B}^n(r_2)) \cap \partial B(a, 2R) \neq \emptyset$. Note that $f(A(r_2, r_1)) \subset B(a, 2R)$ and $b \notin f(A(r_2, r_1))$.

Let $c \in f(\partial \mathbb{B}^n(r_2)) \cap \partial B(a, 2R)$. Then $\operatorname{diam}(f(\partial \mathbb{B}^n(r_2))) < R/8$ and

$$f(\partial \mathbb{B}^n(r_2)) \subset \varphi^{-1}(\mathbb{B}^n(\varphi(c), R/4)) \subset B(c, R/2). \tag{3.3}$$

We join $\partial \mathbb{B}^n(r_1)$ and $\partial \mathbb{B}^n(r_2)$ by a line segment ℓ, which is contained, except for the end points, in the ring domain $A(r_2, r_1)$. Then the path $f(\ell)$ in M joins $f(\partial \mathbb{B}^n(r_1))$ and $f(\partial \mathbb{B}^n(r_2))$, and by (3.2), (3.3) and the choice of r_1 and r_2, we may fix $y_0 \in \ell$ such that

$$f(y_0) \in B(a, 2R) \setminus (\bar{B}(a, R) \cup \bar{B}(c, R/2)).$$

Since both $\mathbb{B}^n(4R) \setminus (\mathbb{B}^n(R/4) \cup \mathbb{B}^n(\varphi(c), R/4))$ and $\varphi^{-1}(\partial \mathbb{B}^n(3R))$ are connected, also $M \setminus (\varphi^{-1}(\mathbb{B}^n(R/4)) \cup \varphi^{-1}(\mathbb{B}^n(\varphi(c), R/4)))$ is connected.

Thus $f(y_0)$ can be joined with $b \notin \bar{B}(a, 4R)$ by a path $\beta\colon [0, 1] \to M$ such that $\beta([0, 1]) \cap f(\partial(A(r_2, r_1))) = \emptyset$.

Let $\alpha\colon I_0 \to \mathbb{B}^n \setminus \{0\}$, where $I_0 = [0, 1]$ or $[0, t_0)$ for some $t_0 \in (0, 1]$, be a maximal lift of β under f starting at $y_0 = \alpha(0) \in A(r_2, r_1)$. If $I_0 = [0, 1]$, then $f(\alpha(1)) = b \notin f(A(r_2, r_1))$, so $\alpha(1) \notin A(r_2, r_1)$. If $I_0 \neq [0, 1]$, then $\operatorname{dist}(\alpha(t), \partial \mathbb{B}^n \cup \{0\}) \to 0$

as $t \to t_0$. In both cases, $\beta(I_0) \cap f(\partial A(r_2, r_1)) = f(\alpha(I_0) \cap \partial A(r_2, r_1)) \neq \emptyset$. This is a contradiction and (3.1) holds. □

Proof of Theorem 1. Define a function $Q_f \colon \mathbb{B}^n(2/3) \setminus \{0\} \to [0, \infty)$ by

$$Q_f(x) := \sup_{y, y' \in \mathbb{B}^n(x, |x|/2), y \neq y'} \frac{|f(y) - f(y')|}{|y - y'|^\beta},$$

where $\beta = \beta(n, K)$ is as in Section 2. Put

$$M_f := \limsup_{x \to 0} Q_f(x)|x|^\beta.$$

Suppose first that

$$M_f = \infty, \tag{3.4}$$

or, equivalently, that there exists a sequence (y_k) in $\mathbb{B}^n \setminus \{0\}$ satisfying $y_k \to 0$ and $Q_f(y_k)|y_k|^\beta \to \infty$ as $k \to \infty$.

Fix $\delta \in (0, 1)$ small enough, and, for each $k \in \mathbb{N}$, define a mapping $g_k \colon \mathbb{B}^n(1 + \delta) \to N$ by

$$g_k(z) := f(y_k + \frac{|y_k|}{2} z).$$

By (3.4), there exist sequences (z_k) and (w_k) in $\mathbb{B}^n$ satisfying

$$\limsup_{k \to \infty} \frac{|g_k(z_k) - g_k(w_k)|}{|z_k - w_k|^\beta} \geq \limsup_{k \to \infty} \frac{1}{2} Q_f(y_k) \left(\frac{|y_k|}{2} \right)^\beta = \infty.$$

Hence the family $\{g_k \colon k \in \mathbb{N}\}$ is not normal on $\mathbb{B}^n(1 + \delta)$. Indeed, otherwise, there exists a locally uniform limit point of $\{g_k \colon k \in \mathbb{N}\}$, which is K-quasiregular on $\mathbb{B}^n(1 + \delta)$ but not β-Hölder continuous on $\bar{\mathbb{B}}^n$. This is impossible.

The non-normality of $\{g_k \colon k \in \mathbb{N}\}$ on $\mathbb{B}^n(1+\delta)$ is equivalent to the non-normality of $\{g_k \colon k \in \mathbb{N}\}$ at some $a \in \mathbb{B}^n(1 + \delta)$. By Theorem 2.1, there exist sequences (z_j), (ρ_j) and (k_j) in $\mathbb{B}^n(1 + \delta)$, $(0, \infty)$, and $\mathbb{N}$, respectively, and a non-constant K-quasiregular mapping $g \colon \mathbb{R}^n \to N$ such that $\lim_{j \to \infty} z_j = a$, $\lim_{j \to \infty} \rho_j = 0$, $\lim_{j \to \infty} k_j = \infty$, and

$$\lim_{j \to \infty} g_{k_j}(z_j + \rho_j v) = g(v)$$

locally uniformly on $\mathbb{R}^n$. Observing that

$$f\left((y_{k_j} + (|y_{k_j}|/2) z_j) + ((|y_{k_j}|/2)\rho_j) v\right) = g_{k_j}(z_j + \rho_j v),$$

$\lim_{j \to \infty}(y_{k_j} + (|y_{k_j}|/2) z_j) = 0$, and $\lim_{j \to \infty}(|y_{k_j}|/2)\rho_j = 0$ completes the proof in this case.

Suppose next that

$$M_f < \infty. \tag{3.5}$$

Let $\{g_k : \mathbb{B}^n(e^k) \setminus \{0\} \to N; k \in \mathbb{N}\}$ be the family of K-quasiregular mappings defined as

$$g_k(v) := f(e^{-k} v).$$

By (3.5), we have for every $v \in \mathbb{R}^n \setminus \{0\}$,

$$\limsup_{k \to \infty} \sup_{w \in \mathbb{B}^n(v, |v|/2), w \neq v} \frac{|g_k(w) - g_k(v)|}{|w - v|^\beta} \leq M_f |v|^{-\beta} < \infty,$$

so the family $\{g_k\colon k \in \mathbb{N}\}$ is locally equicontinuous on $\mathbb{R}^n \setminus \{0\}$.

By Lemma 3.1, there exists a sequence (r_j) in $(0,\infty)$ tending to 0 as $j \to \infty$ such that $\lim_{j\to\infty} \operatorname{diam} f(\partial\mathbb{B}(r_j)) > 0$. Fix a sequence (k_j) in $\mathbb{N}$ such that for every $j \in \mathbb{N}$, $e^{-k_j-1} < r_j \le e^{-k_j}$. Then for every $j \in \mathbb{N}$, $g_{k_j}(\bar{\mathbb{B}}^n \setminus \mathbb{B}^n(e^{-1})) = f(\bar{\mathbb{B}}^n(e^{-k_j}) \setminus \mathbb{B}^n(e^{-k_j-1})) \supset f(\partial\mathbb{B}(r_j))$. In particular,

$$\liminf_{j\to\infty} \operatorname{diam}(g_{k_j}(\bar{\mathbb{B}}^n \setminus \mathbb{B}^n(e^{-1}))) > 0. \tag{3.6}$$

By passing to a further subsequence of (g_{k_j}) if necessary, we may assume, by the Arzelà–Ascoli theorem, that (g_{k_j}) converges locally uniformly on $\mathbb{R}^n \setminus \{0\}$ to a mapping $g\colon \mathbb{R}^n \setminus \{0\} \to N$. Since (g_{k_j}) is a sequence of K-quasiregular mappings, g is K-quasiregular and, by (3.6), non-constant. This completes the proof. □

Example 3.2. To see that both cases in the proof of Theorem 1 actually occur, we give two examples which are similar to Examples 23 and 24 in [13].

For $M_f < \infty$, we may take the conformal mapping $f\colon \mathbb{B}^n \setminus \{0\} \to \mathbb{S}^{n-1} \times \mathbb{S}^1$, $x \mapsto (x/|x|, e^{-i\log|x|})$. Then f is the composition $\psi \circ h$ of a conformal homeomorphism $h\colon \mathbb{B}^n \setminus \{0\} \to \mathbb{S}^{n-1} \times \mathbb{R}$, $x \mapsto (x/|x|, -\log|x|)$, and a locally isometric covering map $\psi\colon \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{S}^{n-1} \times \mathbb{S}^1$. Since $|h(sx) - h(ty)| \le |\log s - \log t| + |x - y|$ for all $s,t \in (0,1)$ and all $x,y \in \mathbb{S}^{n-1}$, we easily observe that $M_f < \infty$.

For $M_f = \infty$, we construct $f\colon \mathbb{B}^n \setminus \{0\} \to \mathbb{S}^n$ using the winding map $h\colon \mathbb{S}^n \to \mathbb{S}^n$,

$$(x_1, \dots, x_{n-2}, re^{i\theta}) \mapsto (x_1, \dots, x_{n-2}, re^{i3\theta}),$$

which is a quasiregular endomorphism on $\mathbb{S}^n$; we identify $\mathbb{R}^{n+1}$ with $\mathbb{R}^{n-1} \times \mathbb{C}$.

Let $\sigma\colon \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{R}^n$ be the stereographic projection, and S the lower hemisphere $\{(x_1, \dots, x_{n+1}) \in \mathbb{S}^n\colon x_{n+1} \le 0\}$ of $\mathbb{S}^n$. We note that $h|\partial S$ is the identity.

Let (r_k) be a sequence tending to 0 in $(0, 1/2)$ and put $x_k := r_k e_1$ and $B_k := \mathbb{B}^n(x_k, r_k/2)$ for each $k \in \mathbb{N}$. We may assume that balls B_k, $k \in \mathbb{N}$, are mutually disjoint. For each $k \in \mathbb{N}$, put $x_k' := \sigma^{-1}(x_k)$ and $B_k' := \sigma^{-1}(B_k)$ and let ρ_k be the the Möbius transformation on $\mathbb{S}^n$ defined as

$$\rho_k(y) = \begin{cases} \sigma^{-1} \circ \alpha_k \circ \sigma(y), & y \ne e_{n+1}, \\ e_{n+1}, & y = e_{n+1}, \end{cases}$$

where α_k is the affine transformation $x \mapsto r_k^{-1}(x - x_k)$ on $\mathbb{R}^n$. Then $\rho_k(B_k') = S$ and $\rho_k(x_k') = -e_{n+1}$, so the mapping $f\colon \mathbb{B}^n \setminus \{0\} \to \mathbb{S}^n$ defined by

$$f(x) = \begin{cases} \sigma^{-1}(x), & x \in (\mathbb{B}^n \setminus \{0\}) \setminus \bigcup_{k=1}^\infty B_k, \\ (\rho_k^{-1} \circ h \circ \rho_k) \circ \sigma^{-1}(x), & x \in B_k, \end{cases}$$

is quasiregular with the same distortion constant K as h. Recall the definition of $\beta = \beta(n,K)$ in Section 2. We observe that, for every $k \in \mathbb{N}$, there exists a unique $y_k \in B_k$ satisfying $f(y_k) = e_1$ and $|x_k - y_k| \le C r_k^2$, where $C > 0$ is independent of k. Since $f(x_k) = e_{n+1}$ for every $k \in \mathbb{N}$, by a direct computation, there is $C' > 0$ such that for every $k \in \mathbb{N}$, $Q_f(x_k)|x_k|^\beta \ge C' r_k^{-\beta}$, so f satisfies $M_f = \infty$.

4. Proof of Theorem 2

Let $Z_n : \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$, $n > 2$, be the Zorich mapping (see [17] or [14, I.3.3] for the construction of Z_n) which is K_n-quasiregular for some $K_n \ge 1$ and an analog

of the exponential function $Z_2 : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. The mapping Z_2 is K_2-quasiregular for $K_2 = 1$. Set $K' = K \cdot K_n \geq 1$ for each $n \geq 2$ and each $K \geq 1$.

Let N be a closed, connected, and oriented Riemannian n-manifold.

Suppose there exists a K-quasiregular mapping $f\colon \mathbb{B}^n \setminus \{0\} \to N$ with an essential singularity at the origin. By Theorem 1 and a manifold version of Hurwitz's theorem (cf. the proof of [10, Lemma 2]), there exists a non-constant K-quasiregular mapping $g\colon X \to N$, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, satisfying $g(X) \subset f(\mathbb{B}^n \setminus \{0\})$. If $X = \mathbb{R}^n$, then the mapping g has the desired properties. If $X = \mathbb{R}^n \setminus \{0\}$, then the mapping $g \circ Z_n\colon \mathbb{R}^n \to N$ has the desired properties.

Suppose now that $g\colon \mathbb{R}^n \to N$ is a non-constant K-quasiregular mapping. Let ι be an orientation preserving conformal involution of $\mathbb{R}^n \setminus \{0\}$ satisfying

$$\iota(\mathbb{B}^n \setminus \{0\}) = \mathbb{R}^n \setminus \overline{\mathbb{B}^n}.$$

If g has an essential singularity at the infinity, $f\colon \mathbb{B}^n \setminus \{0\} \to N$, $x \mapsto g \circ \iota(x)$, has an essential singularity at the origin. If g has a removable singularity at the infinity, g extends to a quasiregular mapping $\mathbb{S}^n \to N$. Then the mapping $f\colon \mathbb{B}^n \setminus \{0\} \to N$, $x \mapsto g \circ Z_n(\iota(x))$, has the desired properties.

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