

## POLYHARMONIC MAPS OF ORDER $k$ WITH FINITE $L^p$ K-ENERGY INTO EUCLIDEAN SPACES

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ABSTRACT. We consider polyharmonic maps  $\phi : (M, g) \rightarrow \mathbb{E}^n$  of order  $k$  from a complete Riemannian manifold into the Euclidean space and let  $p$  be a real constant satisfying  $2 \leq p < \infty$ . (i) If  $\int_M |W^{k-1}|^p dv_g < \infty$  and  $\int_M |\bar{\nabla} W^{k-2}|^2 dv_g < \infty$ , then  $\phi$  is a polyharmonic map of order  $k-1$ . (ii) If  $\int_M |W^{k-1}|^p dv_g < \infty$  and  $\text{Vol}(M, g) = \infty$ , then  $\phi$  is a polyharmonic map of order  $k-1$ . Here,  $W^s = \bar{\Delta}^{s-1} \tau(\phi)$  ( $s = 1, 2, \dots$ ) and  $W^0 = \phi$ . As a corollary, we give an affirmative partial answer to the generalized Chen conjecture.

### 1. INTRODUCTION

The theory of harmonic maps has been applied to various fields in differential geometry. Harmonic maps between two Riemannian manifolds are critical points of the energy functional  $E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g$ , for smooth maps  $\phi : (M^m, g) \rightarrow (N^n, h)$  from an  $m$ -dimensional Riemannian manifold into an  $n$ -dimensional Riemannian manifold, where  $dv_g$  denotes the volume element of  $g$ . The Euler-Lagrange equation of  $E$  is  $\tau(\phi) = \text{Trace} \nabla d\phi = 0$ , where  $\tau(\phi)$  is called the tension field of  $\phi$ . A map  $\phi : (M, g) \rightarrow (N, h)$  is called a harmonic map if  $\tau(\phi) = 0$ .

In 1983, J. Eells and L. Lemaire [6] proposed the problem to consider polyharmonic maps of order  $k$  ( $k$ -harmonic maps), which are critical points of the  $k$ -energy functional  $E_k(\phi) = \frac{1}{2} \int_M |(d + d^*)^{k-2} \tau(\phi)|^2 dv_g$ , on the space of smooth maps between two Riemannian manifolds. The  $k$ -energy functional  $E_k(\phi)$  for  $\phi : (M, g) \rightarrow \mathbb{E}^n$  is given as follows:

$$E_k(\phi) = \begin{cases} \frac{1}{2} \int_M |W^\ell|^2 dv_g & (k = 2\ell), \\ \frac{1}{2} \int_M |\bar{\nabla} W^\ell|^2 dv_g & (k = 2\ell + 1), \end{cases}$$

where

$$W^\ell = \underbrace{\bar{\Delta} \cdots \bar{\Delta}}_{\ell-1} \tau(\phi) = \bar{\Delta}^{\ell-1} \tau(\phi)$$

and

$$W^0 = \phi.$$

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Polyharmonic maps are, by definition, a generalization of harmonic maps. If  $\phi : (M, g) \rightarrow \mathbb{E}^n$  is a smooth map, then the Euler-Lagrange equation of  $E_k$  is

$$\overline{\Delta}^{k-1} \tau(\phi) = 0,$$

where  $\overline{\Delta} := \sum_{i=1}^m (\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} - \overline{\nabla}_{\nabla_{e_i} e_i})$ , and  $\overline{\nabla}$  is the induced connection.

Polyharmonic maps of order 2 are called biharmonic maps. There are many studies for biharmonic theory. One of the most interesting problems in biharmonic theory is Chen’s conjecture. In 1988, B. Y. Chen raised the following problem:

**Conjecture 1** ([2]). *Any biharmonic submanifold in  $\mathbb{E}^n$  is minimal.*

Here, we say a submanifold  $M$  in  $\mathbb{E}^n$  is biharmonic if an isometric immersion  $\phi : (M, g) \rightarrow \mathbb{E}^n$  is biharmonic. There are many affirmative partial answers to Chen’s conjecture (cf. [1], [4], [5], [8], [11], [14], [17], etc.).

On the other hand, Chen’s conjecture was generalized as follows (cf. [12]; see also [3]):

**Conjecture 2** ([12]). *Any polyharmonic submanifold of order  $k$  in  $\mathbb{E}^n$  is minimal.*

Obviously, polyharmonic maps of order  $k$  ( $k = 1, 2, \dots$ ) from a compact Riemannian manifold into the Euclidean space are harmonic. In fact, if  $\overline{\Delta}^\ell \tau(\phi) = 0$  or  $\overline{\nabla}_{e_i} \overline{\Delta}^\ell \tau(\phi) = 0$ , then  $\tau(\phi) = 0$  (cf. [12]). The author showed that polyharmonic curves parametrized by arc length of order  $k$  is a straight line (cf. [12]). Recently, N. Nakauchi and H. Urakawa gave an affirmative partial answer to Conjecture 2 as follows.

**Theorem 1.1** ([16]). *Let  $\phi : (M, g) \rightarrow \mathbb{E}^n$  be a polyharmonic map of order  $k$ , that is,  $\overline{\Delta}^{k-1} \tau(\phi) = 0$ , from a complete Riemannian manifold  $(M, g)$  into  $\mathbb{E}^n$ .*

(i) *If*

$$E_{2q-2}(\phi) = \frac{1}{2} \int_M |W^{q-1}|^2 dv_g < \infty, \quad \text{for all } q = 2, 3, \dots, k,$$

and

$$E_{2q-1}(\phi) = \frac{1}{2} \int_M |\overline{\nabla} W^{q-1}|^2 dv_g < \infty, \quad \text{for all } q = 1, 2, \dots, k-1$$

then  $\phi$  is harmonic.

(ii) *If*

$$E_{2q-2}(\phi) = \frac{1}{2} \int_M |W^{q-1}|^2 dv_g < \infty, \quad \text{for all } q = 2, 3, \dots, k,$$

and

$$\text{Vol}(M, g) = \infty,$$

then  $\phi$  is harmonic.

Here,  $W^s = \overline{\Delta}^{s-1} \tau(\phi)$  ( $s = 1, 2, \dots$ ) and  $W^0 = \phi$ .

Our main result of this paper is the following.

**Theorem 1.2.** *Let  $\phi : (M, g) \rightarrow \mathbb{E}^n$  be a polyharmonic map of order  $k$ , that is,  $\overline{\Delta}^{k-1} \tau(\phi) = 0$ , from a complete Riemannian manifold  $(M, g)$  into  $\mathbb{E}^n$  and let  $p$  be a real constant satisfying  $2 \leq p < \infty$ .*

(i) If

$$\int_M |W^{k-1}|^p dv_g < \infty$$

and

$$E_{2k-3}(\phi) = \frac{1}{2} \int_M |\bar{\nabla} W^{k-2}|^2 dv_g < \infty,$$

then  $\phi$  is a polyharmonic map of order  $k-1$ .

(ii) If

$$\int_M |W^{k-1}|^p dv_g < \infty$$

and

$$\text{Vol}(M, g) = \infty,$$

then  $\phi$  is a polyharmonic map of order  $k-1$ .

Here,  $W^s = \bar{\Delta}^{s-1} \tau(\phi)$  ( $s = 1, 2, \dots$ ) and  $W^0 = \phi$ .

By Theorem 1.2, we obtain the following corollary.

**Corollary 1.3.** Let  $\phi : (M, g) \rightarrow \mathbb{E}^n$  be a polyharmonic map of order  $k$ , that is,  $\bar{\Delta}^{k-1} \tau(\phi) = 0$ , from a complete Riemannian manifold  $(M, g)$  into  $\mathbb{E}^n$  and let  $p$  be a real constant satisfying  $2 \leq p < \infty$ .

(i) If

$$\int_M |W^{q-1}|^p dv_g < \infty, \quad \text{for all } q = 2, 3, \dots, k,$$

and

$$E_{2q-1}(\phi) = \frac{1}{2} \int_M |\bar{\nabla} W^{q-1}|^2 dv_g < \infty, \quad \text{for all } q = 1, 2, \dots, k-1,$$

then  $\phi$  is harmonic.

(ii) If

$$\int_M |W^{q-1}|^p dv_g < \infty, \quad \text{for all } q = 2, 3, \dots, k,$$

and

$$\text{Vol}(M, g) = \infty,$$

then  $\phi$  is harmonic.

Here,  $W^s = \bar{\Delta}^{s-1} \tau(\phi)$  ( $s = 1, 2, \dots$ ) and  $W^0 = \phi$ .

*Remark 1.4.* In Corollary 1.3, if a polyharmonic submanifold, that is, an isometric immersion  $\phi : (M, g) \rightarrow \mathbb{E}^n$ , is polyharmonic of order  $k$ , we do not need  $E(\phi) = \frac{1}{2} \int_M |\bar{\nabla} W^0|^2 dv_g = \frac{1}{2} \int_M |d\phi|^2 dv_g < \infty$  (cf. [15]).

This corollary is a generalization of Theorem 1.1 and gives an affirmative partial answer to the generalized Chen conjecture.

The remaining sections are organized as follows. In section 2, we recall polyharmonic maps of order  $k$ . In section 3, we show our main theorem.

2. PRELIMINARIES

In this section, we recall polyharmonic maps.

Let  $(M, g)$  be an  $m$ -dimensional Riemannian manifold. Assume that  $\phi : (M, g) \rightarrow \mathbb{E}^n$  is a smooth map. We denote by  $\nabla$  the Levi-Civita connection on  $(M, g)$  and by  $\bar{\nabla}$  the induced connection.

Let us recall the definition of a harmonic map  $\phi : (M, g) \rightarrow \mathbb{E}^n$ . For a smooth map  $\phi : (M, g) \rightarrow \mathbb{E}^n$ , the energy of  $\phi$  is defined by

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g.$$

The Euler-Lagrange equation of  $E$  is

$$\tau(\phi) = \sum_{i=1}^m \{ \bar{\nabla}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i) \} = 0,$$

where  $\tau(\phi)$  is called the tension field of  $\phi$  and  $\{e_i\}_{i=1}^m$  is an orthonormal frame field on  $M$ . A map  $\phi$  is called a harmonic map if  $\tau(\phi) = 0$ .

In 1983, J. Eells and L. Lemaire [6] proposed the problem to consider polyharmonic maps of order  $k$  ( $k$ -harmonic maps), which are critical points of the  $k$ -energy functional  $E_k(\phi) = \frac{1}{2} \int_M |(d + d^*)^{k-2} \tau(\phi)|^2 dv_g$ , on the space of smooth maps  $\phi : (M, g) \rightarrow \mathbb{E}^n$ . The  $k$ -energy functional  $E_k(\phi)$  for  $\phi : (M, g) \rightarrow \mathbb{E}^n$  is given as follows:

$$E_k(\phi) = \begin{cases} \frac{1}{2} \int_M |W^\ell|^2 dv_g & (k = 2\ell), \\ \frac{1}{2} \int_M |\bar{\nabla} W^\ell|^2 dv_g & (k = 2\ell + 1), \end{cases}$$

where

$$W^\ell = \underbrace{\bar{\Delta} \cdots \bar{\Delta}}_{\ell-1} \tau(\phi) = \bar{\Delta}^{\ell-1} \tau(\phi)$$

and

$$W^0 = \phi.$$

The Euler-Lagrange equation of  $E_k$  is

$$\bar{\Delta}^{k-1} \tau(\phi) = 0.$$

3. PROOF OF THEOREM 1.2

In this section, we shall give a proof of our main theorem (Theorem 1.2). We first show the following lemma.

**Lemma 3.1.** *Let  $\phi : (M, g) \rightarrow \mathbb{E}^n$  be a polyharmonic map of order  $k$  from a complete Riemannian manifold  $(M, g)$  into  $\mathbb{E}^n$ .*

*Assume that  $p$  satisfies  $2 \leq p < \infty$ . If for such a  $p$ ,*

$$\int_M |W^{k-1}|^p dv_g < \infty,$$

*then  $\bar{\nabla}_X W^{k-1} = 0$  for any vector field  $X$  on  $M$ . In particular,  $|W^{k-1}|$  is constant. Here,  $W^s = \bar{\Delta}^{s-1} \tau(\phi)$  ( $s = 1, 2, \dots$ ) and  $W^0 = \phi$ .*

*Proof.* For a fixed point  $x_0 \in M$ , and for every  $0 < r < \infty$ , we first take a cutoff function  $\lambda$  on  $M$  satisfying that

$$(3.1) \quad \begin{cases} 0 \leq \lambda(x) \leq 1 & (x \in M), \\ \lambda(x) = 1 & (x \in B_r(x_0)), \\ \lambda(x) = 0 & (x \notin B_{2r}(x_0)), \\ |\nabla \lambda| \leq \frac{C}{r} & (x \in M), \end{cases} \quad \text{for some constant } C \text{ independent of } r,$$

where  $B_r(x_0)$  and  $B_{2r}(x_0)$  are the balls centered at a fixed point  $x_0 \in M$  with radius  $r$  and  $2r$  respectively (cf. [10]). Since  $\bar{\Delta}W^{k-1} = W^k = \bar{\Delta}^{k-1}\tau(\phi) = 0$ , we have

$$(3.2) \quad 0 = \int_M \langle -\bar{\Delta}W^{k-1}, \lambda^2 |W^{k-1}|^{p-2} W^{k-1} \rangle dv_g.$$

By (3.2), we have

$$(3.3) \quad \begin{aligned} 0 &= \int_M \langle -\bar{\Delta}W^{k-1}, \lambda^2 |W^{k-1}|^{p-2} W^{k-1} \rangle dv_g \\ &= \int_M \langle \bar{\nabla}W^{k-1}, \bar{\nabla}(\lambda^2 |W^{k-1}|^{p-2} W^{k-1}) \rangle dv_g \\ &= \int_M \sum_{i=1}^m \langle \bar{\nabla}_{e_i} W^{k-1}, (e_i \lambda^2) |W^{k-1}|^{p-2} W^{k-1} + \lambda^2 e_i \{ (|W^{k-1}|^2)^{\frac{p-2}{2}} \} W^{k-1} \\ &\quad + \lambda^2 |W^{k-1}|^{p-2} \bar{\nabla}_{e_i} W^{k-1} \rangle dv_g \\ &= \int_M \sum_{i=1}^m \langle \bar{\nabla}_{e_i} W^{k-1}, 2\lambda(e_i \lambda) |W^{k-1}|^{p-2} W^{k-1} \rangle dv_g \\ &\quad + \int_M \sum_{i=1}^m \langle \bar{\nabla}_{e_i} W^{k-1}, \lambda^2 (p-2) |W^{k-1}|^{p-4} \langle \bar{\nabla}_{e_i} W^{k-1}, W^{k-1} \rangle W^{k-1} \rangle dv_g \\ &\quad + \int_M \sum_{i=1}^m \langle \bar{\nabla}_{e_i} W^{k-1}, \lambda^2 |W^{k-1}|^{p-2} \bar{\nabla}_{e_i} W^{k-1} \rangle dv_g. \end{aligned}$$

From (3.3), we have

$$(3.4) \quad \begin{aligned} 0 &\geq \int_M \sum_{i=1}^m \langle \bar{\nabla}_{e_i} W^{k-1}, 2\lambda(e_i \lambda) |W^{k-1}|^{p-2} W^{k-1} \rangle dv_g \\ &\quad + \int_M \sum_{i=1}^m \langle \bar{\nabla}_{e_i} W^{k-1}, \lambda^2 |W^{k-1}|^{p-2} \bar{\nabla}_{e_i} W^{k-1} \rangle dv_g. \end{aligned}$$

We consider the first term of the right hand side of (3.4):

$$\begin{aligned}
 & - 2 \int_M \sum_{i=1}^m \langle \bar{\nabla}_{e_i} W^{k-1}, \lambda(e_i \lambda) |W^{k-1}|^{p-2} W^{k-1} \rangle dv_g \\
 (3.5) \quad & = - 2 \int_M \sum_{i=1}^m \langle (e_i \lambda) |W^{k-1}|^{\frac{p}{2}-1} W^{k-1}, \lambda |W^{k-1}|^{\frac{p}{2}-1} \bar{\nabla}_{e_i} W^{k-1} \rangle dv_g \\
 & \leq 2 \int_M |\nabla \lambda|^2 |W^{k-1}|^p dv_g \\
 & \quad + \frac{1}{2} \int_M \lambda^2 |W^{k-1}|^{p-2} |\bar{\nabla} W^{k-1}|^2 dv_g,
 \end{aligned}$$

where the inequality of (3.5) follows from the following inequality:

$$(3.6) \quad \pm 2 \langle V, U \rangle \leq \varepsilon |V|^2 + \frac{1}{\varepsilon} |U|^2, \quad \text{for all positive } \varepsilon > 0,$$

because of the inequality  $0 \leq |\sqrt{\varepsilon}V \pm \frac{1}{\sqrt{\varepsilon}}U|^2$ . The inequality (3.6) is called *Young's inequality*. Substituting (3.5) into (3.4), we have

$$\begin{aligned}
 (3.7) \quad & \int_M \lambda^2 |W^{k-1}|^{p-2} |\bar{\nabla} W^{k-1}|^2 dv_g \leq 4 \int_M |\nabla \lambda|^2 |W^{k-1}|^p dv_g \\
 & \leq \int_M \frac{4C^2}{r^2} |W^{k-1}|^p dv_g.
 \end{aligned}$$

By the assumption  $\int_M |W^{k-1}|^p dv_g < \infty$ , the right hand side of (3.7) goes to zero and the left hand side of (3.7) goes to

$$\int_M |W^{k-1}|^{p-2} |\bar{\nabla} W^{k-1}|^2 dv_g,$$

since  $\lambda = 1$  on  $B_r(x_0)$ . Thus, we have

$$\int_M |W^{k-1}|^{p-2} |\bar{\nabla} W^{k-1}|^2 dv_g = 0.$$

Therefore we obtain that  $|W^{k-1}|$  is constant and  $\bar{\nabla} W^{k-1} = 0$ . □

Before proving our main result, we recall Gaffney's theorem (cf. [7]).

**Theorem 3.2** ([7]). *Let  $(M, g)$  be a complete Riemannian manifold. If a  $C^1$  1-form  $\omega$  satisfies that  $\int_M |\omega| dv_g < \infty$  and  $\int_M (\delta\omega) dv_g < \infty$ , or equivalently, a  $C^1$  vector field  $X$  defined by  $\omega(Y) = \langle X, Y \rangle$ , ( $\forall Y \in \mathfrak{X}(M)$ ) satisfies that  $\int_M |X| dv_g < \infty$  and  $\int_M \operatorname{div}(X) dv_g < \infty$ , then*

$$\int_M (\delta\omega) dv_g = \int_M \operatorname{div}(X) dv_g = 0.$$

By using Lemma 3.1 and Theorem 3.2, we shall show Theorem 1.2.

*Proof of Theorem 1.2.* By Lemma 3.1, we have  $\bar{\nabla}_X W^{k-1} = 0$  for any vector field  $X$  on  $M$ , and  $|W^{k-1}|$  is constant.

We shall show case (ii). If  $\operatorname{Vol}(M, g) = \infty$  and  $|W^{k-1}| \neq 0$ , then

$$\int_M |W^{k-1}|^p dv_g = |W^{k-1}|^p \operatorname{Vol}(M, g) = \infty,$$

which yields the contradiction.

We shall show case (i). If  $W^{k-1} = 0$ , we obtain the theorem. We consider the case that  $W^{k-1} \neq 0$ . Define a 1-form  $\omega$  on  $M$  by

$$\omega(X) := |W^{k-1}|^{\frac{p}{2}-1} \langle \bar{\nabla}_X W^{k-2}, W^{k-1} \rangle, \quad (X \in \mathfrak{X}(M)).$$

By the assumption  $\int_M |\bar{\nabla} W^{k-2}|^2 dv_g < \infty$  and  $\int_M |W^{k-1}|^p dv_g < \infty$ , we have

$$\begin{aligned} \int_M |\omega| dv_g &= \int_M \left( \sum_{i=1}^m |\omega(e_i)|^2 \right)^{\frac{1}{2}} dv_g \\ (3.8) \quad &\leq \int_M |W^{k-1}|^{\frac{p}{2}} |\bar{\nabla} W^{k-2}| dv_g \\ &\leq \left( \int_M |W^{k-1}|^p dv_g \right)^{\frac{1}{2}} \left( \int_M |\bar{\nabla} W^{k-2}|^2 dv_g \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

We consider  $-\delta\omega = \sum_{i=1}^m (\nabla_{e_i} \omega)(e_i)$ :

$$\begin{aligned} -\delta\omega &= \sum_{i=1}^m \nabla_{e_i} (\omega(e_i)) - \omega(\nabla_{e_i} e_i) \\ &= \sum_{i=1}^m \left\{ \nabla_{e_i} \left( |W^{k-1}|^{\frac{p}{2}-1} \langle \bar{\nabla}_{e_i} W^{k-2}, W^{k-1} \rangle \right) \right. \\ &\quad \left. - |W^{k-1}|^{\frac{p}{2}-1} \langle \bar{\nabla}_{\nabla_{e_i} e_i} W^{k-2}, W^{k-1} \rangle \right\} \\ (3.9) \quad &= \sum_{i=1}^m \left\{ |W^{k-1}|^{\frac{p}{2}-1} \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_i} W^{k-2}, W^{k-1} \rangle \right. \\ &\quad \left. - |W^{k-1}|^{\frac{p}{2}-1} \langle \bar{\nabla}_{\nabla_{e_i} e_i} W^{k-2}, W^{k-1} \rangle \right\} \\ &= \sum_{i=1}^m \left\{ |W^{k-1}|^{\frac{p}{2}-1} \langle \bar{\Delta} W^{k-2}, W^{k-1} \rangle \right\} \\ &= |W^{k-1}|^{\frac{p}{2}+1}, \end{aligned}$$

where the third equality follows from the fact that  $|W^{k-1}|$  is constant and  $\bar{\nabla}_X W^{k-1} = 0$  ( $X \in \mathfrak{X}(M)$ ). Since  $|W^{k-1}|$  is constant and  $\int_M |W^{k-1}|^p dv_g < \infty$ , the function  $-\delta\omega$  is also integrable over  $M$ . From this and (3.8), we can apply Gaffney's theorem for the 1-form  $\omega$ . Therefore we have

$$0 = \int_M (-\delta\omega) dv_g = \int_M |W^{k-1}|^{\frac{p}{2}+1} dv_g.$$

Then we have  $W^{k-1} = 0$ , which contradicts the fact that  $W^{k-1} \neq 0$ .  $\square$

*Proof of Corollary 1.3.* By using Theorem 1.2, we have  $W^{k-1} = 0$ . By repeating this procedure we have  $W^{k-2} = 0$ ,  $W^{k-3} = 0$ ,  $\dots$ . Finally, we obtain  $\tau(\phi) = W^1 = 0$ .  $\square$

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## REFERENCES

- [1] Kazuo Akutagawa and Shun Maeta, *Biharmonic properly immersed submanifolds in Euclidean spaces*, *Geom. Dedicata* **164** (2013), 351–355, DOI 10.1007/s10711-012-9778-1. MR3054632
- [2] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Michigan State University (1988 version).
- [3] B. Y. Chen, *Recent developments of biharmonic conjecture and modified biharmonic conjectures*, arXiv:1307.0245 [math.DG], to appear in Proceedings of PADGE-2012.
- [4] Filip Defever, *Hypersurfaces of  $E^4$  with harmonic mean curvature vector*, *Math. Nachr.* **196** (1998), 61–69, DOI 10.1002/mana.19981960104. MR1657990 (99j:53004)
- [5] Ivko Dimitrić, *Submanifolds of  $E^m$  with harmonic mean curvature vector*, *Bull. Inst. Math. Acad. Sinica* **20** (1992), no. 1, 53–65. MR1166218 (93g:53087)
- [6] James Eells and Luc Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, vol. 50, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1983. MR703510 (85g:58030)
- [7] Matthew P. Gaffney, *A special Stokes's theorem for complete Riemannian manifolds*, *Ann. of Math. (2)* **60** (1954), 140–145. MR0062490 (15,986d)
- [8] Th. Hasanis and Th. Vlachos, *Hypersurfaces in  $E^4$  with harmonic mean curvature vector field*, *Math. Nachr.* **172** (1995), 145–169. MR1330627 (96c:53085)
- [9] Jiang Guoying, *2-harmonic maps and their first and second variational formulas*, *Note Mat.* **28** (2009), no. [2008 on verso], suppl. 1, 209–232. Translated from the Chinese by Hajime Urakawa. MR2640582 (2011c:53150)
- [10] A. Kasue, *Riemannian Geometry*, in Japanese, Baihu-kan, Tokyo, 2001.
- [11] Y. Luo, *On biharmonic submanifolds in non-positively curved manifolds and  $\varepsilon$ -superbiharmonic submanifolds*, arXiv:1306.6069 [math.DG].
- [12] Shun Maeta, *k-harmonic maps into a Riemannian manifold with constant sectional curvature*, *Proc. Amer. Math. Soc.* **140** (2012), no. 5, 1835–1847, DOI 10.1090/S0002-9939-2011-11049-9. MR2869168
- [13] Shun Maeta, *Biminimal properly immersed submanifolds in the Euclidean spaces*, *J. Geom. Phys.* **62** (2012), no. 11, 2288–2293, DOI 10.1016/j.geomphys.2012.07.006. MR2964661
- [14] S. Maeta, *Biharmonic maps from a complete Riemannian manifold into a non-positively curved manifold*, *Ann. Global Anal. Geom.* **46** (2014), no. 1, 75–85. MR3205803
- [15] Nobumitsu Nakauchi and Hajime Urakawa, *Biharmonic submanifolds in a Riemannian manifold with non-positive curvature*, *Results Math.* **63** (2013), no. 1-2, 467–474, DOI 10.1007/s00025-011-0209-7. MR3009698
- [16] N. Nakauchi and H. Urakawa, *Polyharmonic maps into the Euclidean space*, arXiv:1307.5089 [mathDG].
- [17] N. Nakauchi, H. Urakawa and S. Gudmundsson, *Biharmonic maps into a Riemannian manifold of non-positive curvature*, *Geom. Dedicata* **169** (2014), 263–272. MR3175248
- [18] Ye-Lin Ou and Liang Tang, *On the generalized Chen's conjecture on biharmonic submanifolds*, *Michigan Math. J.* **61** (2012), no. 3, 531–542, DOI 10.1307/mmj/1347040257. MR2975260

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