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ANSWER TO A QUESTION OF KOLMOGOROV

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ABSTRACT. More than 80 years ago Kolmogorov asked the following question. Let $E \subseteq \mathbb{R}^2$ be a measurable set with $\lambda^2(E) < \infty$, where λ^2 denotes the two-dimensional Lebesgue measure. Does there exist for every $\varepsilon > 0$ a contraction $f\colon E \to \mathbb{R}^2$ such that $\lambda^2(f(E)) \geq \lambda^2(E) - \varepsilon$ and $f(E)$ is a polygon? We answer this question in the negative by constructing a bounded, simply connected open counterexample. Our construction can easily be modified to yield an analogous result in higher dimensions.

1. Introduction

The following question was posed by M. Laczkovich in [4]. Let λ^d stand for the d-dimensional Lebesgue measure.

Question 1.1 (Laczkovich)**.** Let $E \subseteq \mathbb{R}^d$ $(d \geq 2)$ be a measurable set such that $\lambda^d(E) > 0$. Does there exist a Lipschitz onto map $f\colon E \to [0,1]^d$?

For $d = 2$ the positive answer to Question 1.1 follows from a result of N. X. Uy [6], and D. Preiss also solved this partial problem by completely different methods. J. Matoušek [5] proved the following stronger, 'absolute constant' version based on a well-known combinatorial lemma due to Erdős and Szekeres. (For the definition of 1-Lipschitz map see the Preliminaries section.)

Theorem 1.2 (Matoušek)**.** *There exists a constant $c > 0$ such that for any measurable set $E \subseteq \mathbb{R}^2$ with $\lambda^2(E) = 1$ there exists a 1-Lipschitz onto map $f\colon E \to [0,c]^2$.*

Question 1.1 is still open for dimensions $d > 2$. Theorem 1.2 states that we can contract every set of the plane with positive measure onto a square such that it 'does not lose too much from its measure'. Can we do this so that the loss of the measure is arbitrarily small? It is easy to see that this is not possible if we require the range to be a square, but how about polygons? Note that by polygons we mean a wider class of objects than its standard definition does:

Definition 1.3. We say that $P \subseteq \mathbb{R}^2$ is a *polygon* if ∂P can be covered by finitely many line segments.

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The next question is due to A. N. Kolmogorov; it was quoted by P. Alexandroff in a letter written to F. Hausdorff (see [1] and [2]).

Question 1.4 (Kolmogorov)**.** Let $E \subseteq \mathbb{R}^2$ be a measurable set with $\lambda^2(E) < \infty$, and let $\varepsilon > 0$. Does there exist a contraction $f\colon E \to \mathbb{R}^2$ such that $\lambda^2(f(E)) \geq \lambda^2(E) - \varepsilon$ and $f(E)$ is a polygon?

The main goal of the paper is to answer Question 1.4 in the negative.

Theorem 1.5. *There exist a bounded, simply connected open set $U \subseteq \mathbb{R}^2$ and $\varepsilon > 0$ such that if $f\colon U \to \mathbb{R}^2$ is a contraction with $\lambda^2(f(U)) \geq \lambda^2(U) - \varepsilon$, then $f(U)$ is not a polygon.*

In contrast to Question 1.1 the higher-dimensional versions of Question 1.4 are not more difficult than the original one. The analogue of Theorem 1.5 can be proved similarly for every dimension $d > 2$ with straightforward modifications.

The structure of the paper is as follows. In Section 2 we recall some notation and definitions which we use in this paper. In Section 3 we prove Theorem 1.5. Finally, in Section 4 we collect the open problems.

2. Preliminaries

Let $B(x, r)$ stand for the closed ball of radius r centered at x. For a set $A \subseteq \mathbb{R}^2$ we denote by $\operatorname{int} A$, $\operatorname{cl} A$ and ∂A the interior, closure and boundary of A, respectively. A function $f\colon A \to \mathbb{R}^2$ is said to be *Lipschitz* if there exists a constant $c \in \mathbb{R}$ such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in A$. The smallest such constant c is called the Lipschitz constant of f and denoted by $\operatorname{Lip}(f)$. If $\operatorname{Lip}(f) \leq 1$, then f is a *1-Lipschitz map*, and if $\operatorname{Lip}(f) < 1$, then f is a *contraction*. A function $f\colon A \to \mathbb{R}^2$ is called an *isometry* if $|f(x) - f(y)| = |x - y|$ for all $x, y \in A$, and f is a *local isometry* if every point of A has a neighborhood U such that $f|_{U \cap A}$ is an isometry. If $A, B \subseteq \mathbb{R}^2$, then let $\operatorname{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$.

For the sake of simplicity, let $\lambda = \lambda^2$ stand for the two-dimensional Lebesgue measure.

3. The proof

First we need the following lemma.

Lemma 3.1. *Assume that $U \subseteq \mathbb{R}^2$ is a non-empty, bounded, connected open set and $f\colon U \to \mathbb{R}^2$ is a 1-Lipschitz map such that $\lambda(f(U)) = \lambda(U)$. Then f is the restriction of an (affine) isometry $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$.*

Proof. Recall that if the set $A \subseteq \mathbb{R}^2$ is not collinear and $g\colon A \to \mathbb{R}^2$ is an isometry, then there exists a unique affine isometry $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $g = \psi|_A$. Indeed, let $S \subseteq A$ be a non-degenerate triangle; then it is easy to see that there exists a unique affine isometry $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $g|_S = \psi|_S$. Assume to the contrary that there is an $x \in A$ such that $g(x) \neq \psi(x)$. Then $x \neq \psi^{-1}g(x)$, and for all $s \in S$

$$|s - x| = |g(s) - g(x)| = |\psi^{-1}g(s) - \psi^{-1}g(x)| = |s - \psi^{-1}g(x)|.$$

Hence S is covered by the perpendicular bisector of x and $\psi^{-1}g(x)$; therefore S is collinear, which is a contradiction.

Now we prove that f is a local isometry. Let $z \in U$ be arbitrary. Then there exists $r > 0$ such that $B(z, 2r) \subseteq U$. We prove that f is an isometry on $B(z, r)$.

As f is 1-Lipschitz, we have $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in U$. Assume to the contrary that there are $x, y \in B(z, r)$ such that $|f(x) - f(y)| < |x - y|$. Consider

$$C_{x,y} = B(x, |x - y|/2) \cup B(y, |x - y|/2).$$

Clearly, $C_{x,y} \subseteq B(z, 2r) \subseteq U$. Since f is 1-Lipschitz, $f(C_{x,y})$ is contained in the union of two balls $B(f(x), |x-y|/2) \cup B(f(y), |x-y|/2)$. Since $|f(x)-f(y)| < |x-y|$, the area of this union is smaller than the area of $C_{x,y}$; thus $\lambda(f(C_{x,y})) < \lambda(C_{x,y})$. Applying that λ is subadditive and f is 1-Lipschitz we obtain

$$\lambda\left(f(U)\right) \leq \lambda\left(f(C_{x,y})\right) + \lambda\left(f(U \setminus C_{x,y})\right) < \lambda(C_{x,y}) + \lambda\left(U \setminus C_{x,y}\right) = \lambda(U),$$

which is a contradiction.

Since f is a local isometry, it is locally a restriction of unique affine maps. As U is open and connected, f is the restriction of a unique affine map $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$, and clearly ψ is an isometry. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let B be the closed unit ball centered at the origin and let $C \subseteq [0, 1]$ be a nowhere dense compact set with positive one-dimensional Lebesgue measure. Set $U = \operatorname{int}(B) \setminus (C \times [0, 1])$. Clearly, U is open and path-connected. It is easy to see that every simple closed curve can be shrunk to a point continuously in U, so U is simply connected. Clearly, $\operatorname{cl} U = B$ and $\lambda(U) < \lambda(B)$.

It is enough to prove that there is an $\varepsilon > 0$ such that if $f\colon U \to \mathbb{R}^2$ is a contraction with $\lambda(f(U)) \geq \lambda(U) - \varepsilon$, then $\lambda\left(\partial f(U)\right) > 0$. Assume to the contrary that for all $n \in \mathbb{N}^+$ there are contractions $g_n\colon U \to \mathbb{R}^2$ such that $\lambda(g_n(U)) \geq \lambda(U) - 1/n$ and $\lambda\left(\partial g_n(U)\right) = 0$. Clearly, we may assume that $\bigcup_{n=1}^{\infty} g_n(U)$ is bounded. Let $\{z_i : i \in \mathbb{N}\}$ be a dense set in U. By Cantor's diagonal argument we can choose a strictly increasing subsequence of the positive integers $\langle n_k \rangle$ such that for every $i \in \mathbb{N}$ the limit $\lim_{k\to\infty} g_{n_k}(z_i)$ exists. Since the maps g_{n_k} are contractions, the sequence of functions $\langle g_{n_k} \rangle$ is uniformly convergent on U. Hence by passing to this subsequence and renumbering it we may assume that g_n converges uniformly to a map $g\colon U \to \mathbb{R}^2$. The uniform convergence implies that g is 1-Lipschitz.

First we prove that $\lambda(g(U)) = \lambda(U)$. Since g is 1-Lipschitz, $\lambda(g(U)) \leq \lambda(U)$, so it is enough to prove the opposite direction. As a continuous image of an open set, $g(U)$ is σ-compact and so measurable. Let $\delta > 0$ be arbitrary. The regularity of the Lebesgue measure implies that there is an open set V such that $g(U) \subseteq V$ and $\lambda(V) < \lambda(g(U)) + \delta$. Similarly, there exists a compact set K such that $K \subseteq U$ and $\lambda(U \setminus K) < \delta$. Since the maps g_n are contractions, we obtain for all $n \in \mathbb{N}^+$

$$\lambda(g_n(U)) - \lambda(g_n(K)) \leq \lambda(g_n(U \setminus K)) \leq \lambda(U \setminus K) < \delta. \tag{1}$$

The uniform convergence $g_n \to g$ yields that there is an integer L such that for all $n > L$ we have $g_n(K) \subseteq V$. Therefore (1) and the definition of the maps g_n imply that for all $n > L$ we obtain

$$\lambda(g(U)) + \delta > \lambda(V) \geq \lambda(g_n(K)) > \lambda(g_n(U)) - \delta \geq \lambda(U) - 1/n - \delta.$$

As $\delta > 0$ is arbitrary, we obtain $\lambda(g(U)) \geq \lambda(U)$, so $\lambda(g(U)) = \lambda(U)$. Then Lemma 3.1 implies that g is the restriction of an (affine) isometry $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$. By replacing g_n with $\psi^{-1} \circ g_n$ we may assume that g is the identity, that is, $g = \operatorname{id}_U$.

Since $\operatorname{cl} U = B$, one can extend the maps g_n to contractions $\widehat{g}_n : B \to \mathbb{R}^2$. Clearly, $\widehat{g}_n \to \mathrm{id}_B$ uniformly on B. Let $D \subseteq B$ be a closed ball centered at the origin such that $\lambda(U) < \lambda(D) < \lambda(B)$. There exists $M \in \mathbb{N}$ such that for all $n > M$ we have

$$\max_{x \in B} |\widehat{g}_n(x) - x| < \operatorname{dist}(D, \partial B). \tag{2}$$

We prove for all $n > M$ that

$$D \subseteq \operatorname{cl}(g_n(U)). \tag{3}$$

Let us fix $n > M$. As $g_n(U)$ is dense in $\widehat{g}_n(B)$, we obtain $\operatorname{cl}(g_n(U)) = \widehat{g}_n(B)$. Thus we need to prove $D \subseteq \widehat{g}_n(B)$ for (3). Assume to the contrary that there is an $x_0 \in D$ such that $x_0 \notin \widehat{g}_n(B)$. Set $r = \operatorname{dist}(D, \partial B)$. Then $B(x_0, r) \subseteq B$, so we can define the map $\phi : B(x_0, r) \to \mathbb{R}^2$ by $\phi(x) = -\widehat{g}_n(x) + x + x_0$. Inequality (2) implies $|\phi(x) - x_0| < r$, so $\phi(B(x_0, r)) \subseteq B(x_0, r)$. Since $x_0 \notin \widehat{g}_n(B)$, we obtain that $\phi(x) \neq x$ for all $x \in B(x_0, r)$. Hence ϕ is a continuous self-map of the ball $B(x_0, r)$ without any fixed points, which contradicts the Brouwer Fixed Point Theorem [3, Proposition 4.4.]. Thus (3) holds.

As the maps g_n are contractions, $\lambda(g_n(U)) \leq \lambda(U)$. Therefore $\lambda(U) < \lambda(D)$ and (3) imply, for all $n > M$, that

$$\begin{aligned} \lambda(\partial g_n(U)) &\geq \lambda(\operatorname{cl}(g_n(U)) \setminus g_n(U)) \\ &\geq \lambda(\operatorname{cl}(g_n(U))) - \lambda(g_n(U)) \\ &\geq \lambda(D) - \lambda(U) > 0. \end{aligned}$$

Thus $\lambda(\partial g_n(U)) > 0$, which contradicts the definition of g_n. This concludes the proof. □

Remark 3.2. It is easy to see that for all Lebesgue null sets $N \subseteq \mathbb{R}^2$ the sets $U \Delta N$ are also counterexamples to Question 1.4. On the other hand, one can show that for all $\varepsilon > 0$ there exist a contraction $f : U \to \mathbb{R}^2$ and a Lebesgue null set $N \subseteq \mathbb{R}^2$ such that $\lambda^2(f(U)) \geq \lambda^2(U) - \varepsilon$ and $f(U) \Delta N$ is a polygon. Thus U will not be a counterexample to Question 4.4 below.

4. Open questions

The most important question is the following.

Question 4.1. Let $K \subseteq \mathbb{R}^2$ be a compact set, and let $\varepsilon > 0$. Does there exist a contraction $f : K \to \mathbb{R}^2$ such that $\lambda^2(f(K)) \geq \lambda^2(K) - \varepsilon$ and $f(K)$ is a polygon?

In order to answer Question 4.1 we consider the next question.

Question 4.2. Let $C \subseteq \mathbb{R}^2$ be a compact set with $\lambda^2(C) = 0$, and let $\varepsilon > 0$. Does there exist a contraction $f : C \to \mathbb{R}^2$ such that $|f(x) - x| \leq \varepsilon$ for all $x \in C$ and $f(C)$ can be covered by finitely many line segments?

If the compact set C is a counterexample to Question 4.2 with $\varepsilon > 0$, then consider $K = C \cup R$, where R is a closed ring such that the bounded component of its complement contains C. Then K is a counterexample to Question 4.1; the sketch of the proof is the following. Assume to the contrary that there are contractions $f_n : K \to \mathbb{R}^2$ ($n \in \mathbb{N}^+$) such that $\lambda(f_n(K)) \geq \lambda(K) - 1/n$ and $f_n(K)$ is a polygon; that is, $\partial f_n(K)$ can be covered by finitely many line segments. Similarly as in the proof of Theorem 1.5, one can show that f_n converges uniformly to an isometry, f. We may assume that $f = \mathrm{id}_K$. Let us fix $n \in \mathbb{N}^+$ such that $|f_n(x) - x| \leq \varepsilon$

for all $x \in C$ and $f_n(C) \cap f_n(R) = \emptyset$. As f_n is a contraction and C has zero Lebesgue measure, $f_n(C)$ also has zero measure, so $\operatorname{dist}(f_n(C), f_n(R)) > 0$ implies $f_n(C) \subseteq \partial f_n(K)$. Therefore $f_n(C)$ can be covered by finitely many line segments, which contradicts the choice of C and ε.

Remark 4.3. We do not know whether the Sierpiński carpet is a counterexample to Question 4.2.

Finally, our last question is the following.

Question 4.4. Let $E \subseteq \mathbb{R}^2$ be a measurable set with $\lambda^2(E) < \infty$, and let $\varepsilon > 0$. Do there exist a contraction $f\colon E \to \mathbb{R}^2$ and a Lebesgue null set $N \subseteq \mathbb{R}^2$ such that $\lambda^2(f(E)) \geq \lambda^2(E) - \varepsilon$ and $f(E)\Delta N$ is a polygon? Is this true at least for compact sets?

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