

## STRONG COLORINGS YIELD $\kappa$ -BOUNDED SPACES WITH DISCRETELY UNTOUCHABLE POINTS

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ABSTRACT. It is well known that every non-isolated point in a compact Hausdorff space is the accumulation point of a discrete subset. Answering a question raised by Z. Szentmiklóssy and the first author, we show that this statement fails for countably compact regular spaces, and even for  $\omega$ -bounded regular spaces. In fact, there are  $\kappa$ -bounded counterexamples for every infinite cardinal  $\kappa$ . The proof makes essential use of the so-called *strong colorings* that were invented by the second author.

### 1. INTRODUCTION

It is part of topology folklore that a topological space is compact iff any discrete subset in it has compact closure. Since compact subsets of Hausdorff spaces are closed, this implies that every compact Hausdorff space  $X$  is *weakly discretely generated*, i.e. for any non-closed set  $A \subset X$  there is a discrete set  $D \subset A$  such that  $\overline{D} \setminus A \neq \emptyset$ . It immediately follows from this that every non-isolated point in a compact Hausdorff space is the accumulation point of a discrete subset, or in other words, each such point is “discretely touchable”.

Motivated by this fact, Z. Szentmiklóssy and the first author raised the natural question whether this property of non-isolated points in compact Hausdorff spaces remains valid after relaxing compactness to a weaker property, like countable compactness or Lindelöfness. The aim of this note is to give a negative answer to this question and in fact show that no essential relaxation of compactness suffices to preserve the above statement.

Before turning to the proof of this, we present a few preliminary results concerning discretely touchable points. First of all we note that any accumulation point of a right separated (or equivalently, scattered) subspace is discretely touchable. Indeed, this follows from the fact that the set of isolated points of any right separated space, which is clearly discrete, is dense in that space. This simple observation yields the following proposition.

**Proposition 1.1.** *If  $X$  is a Hausdorff space,  $\kappa$  is an infinite cardinal, and the point  $x$  is the limit of a one-to-one  $\kappa$ -sequence in  $X$ , then  $x$  is discretely touchable in  $X$ .*

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*Proof.* Clearly, we may assume that  $\kappa$  is regular; hence if  $s = \langle x_\alpha : \alpha < \kappa \rangle$  is the one-to-one  $\kappa$ -sequence converging to  $x$ , then any intersection of fewer than  $\kappa$  many neighbourhoods of  $x$  contains a tail of the sequence  $s$ . As  $X$  is Hausdorff, the singleton  $\{x\}$  is the intersection of the closed neighbourhoods of the point  $x$ ; hence by a straight-forward transfinite induction we may select a cofinal subsequence  $\langle x_{\alpha_\nu} : \nu < \kappa \rangle$  of  $s$  such that  $\{x_\alpha : \alpha \geq \alpha_\nu\}$  is closed in  $\{x_\alpha : \alpha < \kappa\}$  for each  $\nu < \kappa$ . But then  $\{x_{\alpha_\nu} : \nu < \kappa\}$  is clearly a right separated subset of  $X$  that accumulates (even converges) to the point  $x$ .  $\square$

**Corollary 1.2.** *If we have  $\chi(p, X) = \psi(p, X) \geq \omega$  for the point  $p$  in the Hausdorff space  $X$ , then  $p$  is discretely touchable in  $X$ .*

*Proof.* It is straight-forward to show that if  $\chi(p, X) = \psi(p, X) = \kappa \geq \omega$ , then there is one-to-one  $\kappa$ -sequence in  $X$  converging to  $p$ .  $\square$

Since  $\chi(p, X) = \psi(p, X)$  for each point  $p$  of a compact Hausdorff space  $X$ , Corollary 1.2 yields an alternative way of showing our starting point which was the fact that every non-isolated point in a compact Hausdorff space is discretely touchable.

This also leads us to the following result that perhaps explains why it seems to be non-trivial to find a discretely untouchable non-isolated point in a regular countably compact space.

**Corollary 1.3.** *If  $x$  is a discretely untouchable non-isolated point in a regular countably compact space  $X$ , then we have*

$$\omega < \psi(x, X) < \chi(x, X).$$

*In particular, then  $\chi(x, X) \geq \omega_2$ .*

*Proof.* Indeed, it is well known that if  $p \in X$  and  $\psi(p, X) = \omega$  in a regular countably compact space  $X$ , then we also have  $\chi(p, X) = \omega$ .  $\square$

A completely similar argument as above, using the fact that any point  $x$  in an initially  $\kappa$ -compact regular space  $X$  with pseudo-character  $\psi(x, X) \leq \kappa$  satisfies  $\chi(x, X) = \psi(x, X)$ , yields the following more general result.

**Proposition 1.4.** *If  $x$  is a discretely untouchable non-isolated point in a regular initially  $\kappa$ -compact space  $X$ , then we have*

$$\kappa < \psi(x, X) < \chi(x, X).$$

*In particular, then  $\chi(x, X) \geq \kappa^{++}$ .*

Finally, we recall that a space  $X$  is called  $\kappa$ -bounded if every subset of  $X$  of cardinality  $\leq \kappa$  has compact closure in  $X$ . It is obvious that every  $\kappa$ -bounded space  $X$  is initially  $\kappa$ -compact, i.e. every open cover of  $X$  of size at most  $\kappa$  has a finite subcover, or equivalently that every infinite subset of  $X$  of cardinality at most  $\kappa$  has a complete accumulation point.

## 2. MAIN RESULTS

Our main results make essential use of certain *strong colorings* that were introduced and established by the second author. Therefore we start with defining these colorings.

**Definition 2.1.** Let  $\lambda$  and  $\kappa$  be infinite cardinals. We shall denote by  $\text{Col}(\lambda, \kappa)$  the following statement: There is a coloring  $c : [\lambda]^2 \rightarrow 2$  such that, given any ordinal  $\xi < \kappa^+$  and a map  $h : \xi \times \xi \rightarrow 2$ , for every family  $\{A_\alpha : \alpha < \lambda\}$  of  $\lambda$  many pairwise disjoint subsets of  $\lambda$  of order type  $\xi$  there are  $\alpha < \beta < \lambda$  for which

$$c(a_{\alpha,i}, a_{\beta,j}) = h(i, j)$$

holds for all pairs  $\langle i, j \rangle \in \xi \times \xi$ . Here, of course,  $a_{\alpha,i}$  denotes the  $i$ th member of  $A_\alpha$  in its increasing ordering (of type  $\xi$ ).

Thus our relation  $\text{Col}(\lambda, \kappa)$  is identical with the relation  $\text{Pr}_0(\lambda, \lambda, 2, \kappa^+)$  that was defined by the second author e.g. in [6], Appendix 1, def. 1.1.

We mention that, simply putting together the results given in 4.6C(5) and 4.5(3) from chapter III of [6] (the first result can be found on page 172 and the second on page 170), one obtains the following fact.

**Proposition 2.2.** *For every infinite cardinal  $\kappa$  the relation  $\text{Pr}(\lambda, \lambda, \lambda, \kappa^+)$  (that is stronger than  $\text{Pr}_0(\lambda, \lambda, 2, \kappa^+) \equiv \text{Col}(\lambda, \kappa)$ ) holds for the cardinal*

$$\lambda = (2^\kappa)^{++} + \omega_4.$$

We note that  $(2^\kappa)^{++} < \omega_4$  can only occur if  $\kappa = \omega$  and the continuum hypothesis holds; hence in every other case we have  $\text{Col}((2^\kappa)^{++}, \kappa)$ .

Motivated by the work on this paper, the second author has achieved some further improvements on this proposition that will appear in [7].

For every coloring  $c : [\lambda]^2 \rightarrow 2$  one can naturally define a subspace  $F[c]$  of the Cantor cube  $2^\lambda$  as follows:  $F[c] = \{c_\alpha : \alpha < \lambda\}$ , where, for any  $\alpha < \lambda$ , the point  $c_\alpha \in 2^\lambda$  is defined by the stipulation

$$c_\alpha(\beta) = \begin{cases} c(\alpha, \beta) & \text{if } \beta < \alpha, \\ 1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta > \alpha. \end{cases}$$

Here, and in what follows, we committed the innocent abuse of notation by writing  $c(\alpha, \beta)$  instead of  $c(\{\alpha, \beta\})$ . The requirement  $c_\alpha(\alpha) = 1$  is purely technical, just to ensure that  $\alpha \neq \beta$  implies  $c_\alpha \neq c_\beta$ .

We shall need the following lemma in the proof of our main result.

**Lemma 2.3.** *If  $\lambda$  is an uncountable regular cardinal and  $\text{Col}(\lambda, \kappa)$  holds, then there is a coloring  $d : [\lambda]^2 \rightarrow 2$  establishing  $\text{Col}(\lambda, \kappa)$  with the extra property that the set  $F[d]$  is dense in the Cantor cube  $2^\lambda$ .*

*Proof.* Assume that the coloring  $c : [\lambda]^2 \rightarrow 2$  witnesses  $\text{Col}(\lambda, \kappa)$ . It is obvious that then for each  $\alpha < \lambda$  the coloring  $c \upharpoonright [\lambda \setminus \alpha]^2$ , i.e.  $c$  restricted to the pairs from a tail of  $\lambda$ , when “translated” back to  $\lambda$  is also a witness for  $\text{Col}(\lambda, \kappa)$ . This translated coloring  $c^{(\alpha)} : [\lambda]^2 \rightarrow 2$  is naturally defined by the formula

$$c^{(\alpha)}(\xi, \zeta) = c(\alpha \dot{+} \xi, \alpha \dot{+} \zeta).$$

Here we use  $\dot{+}$  to denote ordinal addition.

Next we show that there is an  $\alpha < \lambda$  for which  $F[c]$  is  $\lambda$ -dense in the tail product  $2^{\lambda \setminus \alpha}$ . This means that for every finite function  $\varepsilon \in Fn(\lambda \setminus \alpha, 2)$  we have  $[[\varepsilon] \cap F[c]] = \lambda$ , where  $[\varepsilon] = \{f \in 2^\lambda : \varepsilon \subset f\}$  is the elementary open set in the Cantor cube  $2^\lambda$  coded by  $\varepsilon$ .

Assume, arguing indirectly, that there is no such  $\alpha < \lambda$ . We may then define a  $\lambda$ -sequence  $\langle \varepsilon_\alpha : \alpha < \lambda \rangle$  of members of  $Fn(\lambda, 2)$  as follows. Assume that  $\alpha < \lambda$

and  $\langle \varepsilon_\beta : \beta < \alpha \rangle$  have already been defined in such a way that for each  $\beta < \alpha$  we have  $|\varepsilon_\beta \cap F[c]| < \lambda$ . For  $\beta < \alpha$  we shall write  $E_\beta = \{i : c_i \in \varepsilon_\beta\}$ ; then  $|E_\beta| = |\varepsilon_\beta \cap F[c]| < \lambda$ .

Since  $\lambda$  is regular, we may then find an ordinal  $\nu_\alpha < \lambda$  such that

$$\bigcup \{ \text{dom}(\varepsilon_\beta) \cup E_\beta : \beta < \alpha \} \subset \nu_\alpha .$$

We then choose  $\varepsilon_\alpha \in Fn(\lambda \setminus \nu_\alpha)$  so that  $|\varepsilon_\alpha \cap F[c]| < \lambda$ . This is possible by our indirect assumption.

By  $\lambda = \text{cf}(\lambda) > \omega$ , after an appropriate thinning out the sequence  $\langle \varepsilon_\alpha : \alpha < \lambda \rangle$  may be assumed to be such that for some positive natural number  $n$  and function  $\varepsilon : n \rightarrow 2$  each  $\varepsilon_\alpha$  is a “translate” of  $\varepsilon$ , that is,  $|\text{dom}(\varepsilon_\alpha)| = n$  and  $\varepsilon_\alpha(\xi_{\alpha,k}) = \varepsilon(k)$  for each  $k < n$ , where, of course,  $\xi_{\alpha,k}$  denotes the  $k$ th element of  $\text{dom}(\varepsilon_\alpha)$  in its increasing order.

Now, let  $h : n \times n \rightarrow 2$  be any map with the property that  $h(0, k) = \varepsilon(k)$  for all  $k < n$ . Since  $c$  witnesses  $\text{Col}(\lambda, \kappa)$ , we may then find  $\beta < \alpha < \lambda$  such that for each  $k < n$  we have

$$c_{\xi_{\alpha,0}}(\xi_{\beta,k}) = c(\xi_{\alpha,0}, \xi_{\beta,k}) = h(0, k) = \varepsilon(k) = \varepsilon_\beta(\xi_{\beta,k}) .$$

In other words, this means that  $c_{\xi_{\alpha,0}} \in \varepsilon_\beta$ , i.e.  $\xi_{\alpha,0} \in E_\beta$ , which is a contradiction as  $E_\beta \subset \nu_\alpha$ , while  $\xi_{\alpha,0} \in \text{dom}(\varepsilon_\alpha) \subset \lambda \setminus \nu_\alpha$ .

So fix  $\alpha < \lambda$  for which  $F[c]$  is  $\lambda$ -dense in  $2^{\lambda \setminus \alpha}$ . We claim that then  $F[c^{(\alpha)}]$  is dense, even  $\lambda$ -dense, in  $2^\lambda$ . To see this, consider any  $\varepsilon \in Fn(\lambda, 2)$  and define  $\widehat{\varepsilon} \in Fn(\lambda \setminus \alpha)$  as the natural translate of  $\varepsilon$  by  $\alpha$ . In other words,  $\text{dom}(\widehat{\varepsilon}) = \{\alpha \dot{+} \xi : \xi \in \text{dom}(\varepsilon)\}$  and  $\widehat{\varepsilon}(\alpha \dot{+} \xi) = \varepsilon(\xi)$  for each  $\xi \in \text{dom}(\varepsilon)$ .

Then  $|\widehat{\varepsilon} \cap F[c]| = \lambda$ , hence the set

$$E = \{ \zeta : c_{\alpha \dot{+} \zeta} \in \widehat{\varepsilon} \text{ and } \text{dom}(\widehat{\varepsilon}) \subset \alpha \dot{+} \zeta \}$$

is also of cardinality  $\lambda$ . But this means that for every  $\zeta \in E$  and  $\xi \in \text{dom}(\varepsilon)$  we have

$$(c^{(\alpha)})_\zeta(\xi) = c_{\alpha \dot{+} \zeta}(\alpha \dot{+} \xi) = \widehat{\varepsilon}(\alpha \dot{+} \xi) = \varepsilon(\xi) ,$$

hence  $(c^{(\alpha)})_\zeta \in \varepsilon$  holds for each  $\zeta \in E$ . But this clearly implies  $|\varepsilon \cap F[c^{(\alpha)}]| = \lambda$ , showing that  $d = c^{(\alpha)}$  satisfies the requirements of the lemma.  $\square$

We are now ready to state and prove our main result. Before formulating it, however, we recall that the  $\kappa$ -closure  $cl_\kappa(A)$  of a subset  $A$  of a topological space  $X$  is defined by

$$cl_\kappa(A) = \bigcup \{ \overline{B} : B \in [A]^{\leq \kappa} \},$$

where  $\overline{B}$  denotes the closure of  $B$  in  $X$ . Moreover, for every point  $x \in 2^\lambda$  its support  $\text{supp}(x)$  is defined by

$$\text{supp}(x) = \{ \alpha < \lambda : x(\alpha) = 1 \} .$$

**Theorem 2.4.** *Assume that  $\kappa$  is an infinite cardinal and  $\lambda > \kappa^+$  is a regular cardinal; moreover, the coloring  $c : [\lambda]^2 \rightarrow 2$  witnesses the relation  $\text{Col}(\lambda, \kappa)$ . Let us denote by  $H_\kappa[c]$  the  $\kappa$ -closure of the set  $F[c]$  in the Cantor cube  $2^\lambda$ . Then for every right separated subset  $S$  of  $H_\kappa[c]$  there is an  $\alpha < \lambda$  for which*

$$\bigcup \{ \text{supp}(x) : x \in S \} \subset \alpha .$$

*Proof.* Assume that the statement of the theorem fails. Then we clearly may find a subset  $S = \{x_\alpha : \alpha < \lambda\} \subset H_\kappa[c]$  that is right separated by the well-ordering given by the indices  $\alpha$ ; moreover, for each  $\alpha < \lambda$  we have  $\text{supp}(x_\alpha) \setminus \alpha \neq \emptyset$ . The first part of this means that for every  $\alpha < \lambda$  there is a finite set  $a_\alpha \in [\lambda]^{<\omega}$  such that  $\varepsilon_\alpha = x_\alpha \upharpoonright a_\alpha$  codes an elementary open right separating neighbourhood of the point  $x_\alpha$  in  $S$ , i.e.  $x_\beta \notin [\varepsilon_\alpha]$  for all  $\beta > \alpha$ . The second part implies that for every  $\alpha < \lambda$  there is a set  $A_\alpha \in [\lambda \setminus \alpha]^\kappa$  such that  $x_\alpha \in \overline{\{c_i : i \in A_\alpha\}}$ .

A standard delta-system and counting argument allows us to thin out the sequence  $\{x_\alpha : \alpha < \lambda\}$  in such a way that the sets  $a_\alpha$  are pairwise disjoint and of the same size  $n$ . Moreover, similarly as above in the proof of Lemma 2.3, we may in addition assume that for some  $\varepsilon \in 2^n$  we have  $\varepsilon_\alpha = \varepsilon * a_\alpha$  for each  $\alpha < \lambda$ . Let us now set  $B_\alpha = a_\alpha \cup A_\alpha$  for  $\alpha < \lambda$ . After some further thinning out, using  $\lambda = \text{cf}(\lambda) > \kappa^+$ , we may also assume that all the sets  $B_\alpha$  have the same order type  $\xi < \kappa^+$ ; moreover,  $\sup B_\beta < \min B_\alpha$  whenever  $\beta < \alpha < \lambda$ . Finally, we may also assume that there is some fixed set  $a \in [\xi]^n$  so that for each  $\alpha < \lambda$  we have  $a_\alpha = \{\zeta_{\alpha,\nu} : \nu \in a\}$ , where  $B_\alpha = \{\zeta_{\alpha,\nu} : \nu \in \xi\}$  is the increasing enumeration of  $B_\alpha$ .

Now let  $h : \xi \times \xi \rightarrow 2$  be any map satisfying  $h(\eta, \nu_k) = \varepsilon(k)$  for all  $\eta < \xi$  and  $k < n$ , where  $\nu_k$  is the  $k$ th member of the set  $a \in [\xi]^n$  in its increasing order. Since the coloring  $c$  witnesses the relation  $\text{Col}(\lambda, \kappa)$ , we may then find  $\beta < \alpha < \lambda$  such that  $h(\nu, \mu) = c(\zeta_{\alpha,\nu}, \zeta_{\beta,\mu})$  holds for each pair  $\langle \nu, \mu \rangle \in \xi \times \xi$ . But according to our above arrangements this implies  $c_i \in [\varepsilon_\beta]$  for each  $i \in B_\alpha \supset A_\alpha$ ; consequently,  $x_\alpha \in [\varepsilon_\beta]$  as well because  $[\varepsilon_\beta]$  is a closed (in fact clopen) set and  $x_\alpha \in \overline{\{c_i : i \in A_\alpha\}}$ . This, however, is a contradiction because  $[\varepsilon_\beta]$  was assumed to be a right separating neighbourhood of  $x_\beta$  which thus cannot contain the point  $x_\alpha$ . This contradiction then completes the proof of Theorem 2.4. □

In what follows, let us denote by  $\Sigma_\lambda$  the subset of the the Cantor cube  $2^\lambda$  that consists of all points  $x \in 2^\lambda$  whose support is bounded in  $\lambda$ . (Of course, if  $\lambda$  is regular, this is equivalent with  $|\text{supp}(x)| < \lambda$ .) Using this notation, for every coloring  $c : [\lambda]^2 \rightarrow 2$  we have, by definition,  $F[c] \subset \Sigma_\lambda$ . Moreover, if  $c$  witnesses the relation  $\text{Col}(\lambda, \kappa)$  and  $\lambda > \kappa^+$  is a regular cardinal, then, by Theorem 2.4, we even have  $\overline{S} \subset \Sigma_\lambda$  whenever  $S \subset H_\kappa[c]$  is right separated. Thus we have arrived at the following result that makes the statement made in the title of our paper precise.

**Corollary 2.5.** *If  $\lambda > \kappa^+$  is a regular cardinal and  $\text{Col}(\lambda, \kappa)$  holds, then there is a dense  $\kappa$ -bounded subspace of the the Cantor cube  $2^\lambda$  that has a discretely untouchable (non-isolated) point.*

*Proof.* By Lemma 2.3 there is a coloring  $c : [\lambda]^2 \rightarrow 2$  witnessing  $\text{Col}(\lambda, \kappa)$  for which  $F[c]$  is dense in  $2^\lambda$ . Now pick any point  $x \in 2^\lambda \setminus \Sigma_\lambda$ , i.e. with  $|\text{supp}(x)| = \lambda$ , and set  $X = H_\kappa[c] \cup \{x\}$ . Then  $H_\kappa[c]$  is  $\kappa$ -bounded, being the  $\kappa$ -closure of a subset of the compact space  $2^\lambda$ ; hence so is  $X$ . Moreover,  $x$  is an accumulation point of  $H_\kappa[c]$ , as already  $F[c] \subset H_\kappa[c]$  is dense in  $2^\lambda$ . But by Theorem 2.4 no discrete (or equivalently, right separated) subset of  $H_\kappa[c]$  has  $x$  in its closure. □

For each infinite cardinal  $\kappa$ , according to Proposition 2.2 from the beginning of this section,  $\lambda = (2^\kappa)^{++} + \omega_4$  satisfies the assumption of Corollary 2.5. In particular, for  $\kappa = \omega$  the smallest value we get for such a  $\lambda$  is  $\omega_4$ , provided that the continuum is  $\leq \omega_2$ .

Note that in Corollary 2.5 the character  $\chi(x, X)$  of the discretely untouchable point  $x \in X$  is  $\lambda$ . On the other hand, Proposition 1.4 yields the lower bound  $\kappa^{++}$  for the character of a discretely untouchable non-isolated point in an initially  $\kappa$ -compact space. So it is natural to raise the question if the value for the character of such a point could be lower than  $(2^\kappa)^{++} + \omega_4$ . The following problem seems to be the most intriguing.

**Problem 1.** Is it consistent with (or even provable from) ZFC that there is a discretely untouchable non-isolated point of character  $\omega_2$  (or  $\omega_3$ ) in some countably compact (or  $\omega$ -bounded) regular space?

It is standard to show that the cardinality of the  $\kappa$ -bounded space  $X$  given in Corollary 2.5 is  $|H_\kappa[c]| = |cl_\kappa(F([c]))| = \lambda^\kappa \cdot 2^{2^\kappa}$ . However, if instead of  $\kappa$ -bounded we only want an initially  $\kappa$ -compact example, then this value may be chosen to be just  $\lambda^\kappa$ . Indeed, this can be achieved by constructing a subspace  $Y \subset X$  that includes  $F[c] \cup \{x\}$  and has the property that every infinite set  $A \in [Y]^{\leq \kappa}$  has a complete accumulation point in  $Y$ .

In particular, for  $\kappa = \omega$  this yields us a countably compact regular space with a discretely untouchable non-isolated point of cardinality  $\omega_4$ , provided that the continuum is  $\leq \omega_2$ . Again, it is an intriguing problem if the cardinality of such an example can be, consistently, lowered. Let us note that, since every non-isolated point in a scattered space is discretely touchable, such an example cannot be scattered and hence must be of cardinality at least continuum.

**Problem 2.** Is it consistent with ZFC that there is a countably compact (or  $\omega$ -bounded) regular space of cardinality  $\omega_1$  (or  $\omega_2$ , or  $\omega_3$ ) with a discretely untouchable non-isolated point?

### 3. COUNTABLE EXAMPLES

We have shown in the previous section that for every cardinal  $\kappa$  there is a  $\kappa$ -bounded, and hence initially  $\kappa$ -compact, regular space with a discretely untouchable non-isolated point. In the introduction we also promised to exhibit such points in Lindelöf regular spaces to conclude that basically no weakening of compactness suffices to preserve the property of compact Hausdorff spaces that was our starting point.

In fact, we would like to point out that there are even countable, hence hereditarily Lindelöf regular spaces that are crowded, i.e. have no isolated points, in which all discrete subsets are closed; hence all points are discretely untouchable. Perhaps the first such example, a countable maximal space that is regular, was constructed by E. van Douwen; his example was published in the posthumous paper [1]. A very different such example is the countable submaximal dense subspace of the Cantor cube  $2^{2^\omega}$  that was constructed in theorem 4.1 of [3].

Both of these examples are rather non-trivial, so we decided to include in this paper the following result which shows that actually every crowded irresolvable countable regular space contains such an open subspace.

We recall that a space is called irresolvable if it has no two disjoint dense subsets and that there is a crowded, countable, and regular irresolvable space. The existence of such a space was first established by E. Hewitt in 1943, in his classical paper [2] on resolvability.

**Proposition 3.1.** *Every crowded irresolvable regular space has an open subspace in which all countable discrete subsets are closed.*

*Proof.* Let  $X$  be a crowded irresolvable regular space. It is well known that every irresolvable space has an open subspace that is hereditarily irresolvable, so let  $Y$  be such an open subspace of  $X$ . It suffices to show that the set  $Z$  of all accumulation points of countable discrete sets cannot be dense in  $Y$  because then the interior of  $Y \setminus Z$  is the required open set.

Assume, on the contrary that  $Z$  is dense in  $Y$ . Then  $Y \setminus Z$  cannot be dense in  $Y$  because  $Y$  is irresolvable. This means that  $U = \text{Int}(Z) \neq \emptyset$ . But then, by definition, every point of  $U$  is the accumulation point of a countable discrete subset of  $U$ . Now, every countable discrete set in a regular space is strongly discrete, i.e. its points can be separated by pairwise disjoint open sets. But by theorem 2.1 of [5] (see also theorem 1.3 of [4]), then  $U$  is even  $\omega$ -resolvable, i.e. it has infinitely many pairwise disjoint dense subsets, which contradicts the choice of  $Y$ .  $\square$

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#### ADDED IN PROOF

We only noticed after submitting this paper that its main result yields an affirmative answer to Problem 5.5 raised in: A. Dow, M.G. Tkachenko, V.V. Tkachuk and R.G. Wilson, TOPOLOGIES GENERATED BY DISCRETE SUBSPACES, Glasnik Mat., Vol. 37 (57) (2002), 189-212.

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