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A QUADRATIC FORMULA FOR BASIC HYPERGEOMETRIC SERIES RELATED TO ASKEY-WILSON POLYNOMIALS

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Dedicated to Srinivasa Ramanujan on the occasion of his 125th birth anniversary

Abstract. We prove a general quadratic formula for basic hypergeometric series, from which simple proofs of several recent determinant and Pfaffian formulas are obtained. A special case of the quadratic formula is actually related to a Gram determinant formula for Askey-Wilson polynomials. We also show how to derive a recent double-sum formula for the moments of Askey-Wilson polynomials from Newton's interpolation formula.

1. Introduction

Throughout this paper we assume that q is a fixed number in $(0,\ 1)$. A q-shifted factorial is defined by

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),\quad \text{and}\quad (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},\quad \text{for}\quad n\in\mathbb{Z}.$$

Following Gasper and Rhaman [7] we shall use the abbreviated notation

$$(a_1,a_2,\ldots,a_r;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n,\quad \text{for}\quad n\in\mathbb{Z}.$$

A *basic hypergeometric series with r numerators and s denominators* is then defined by

$${}_r\phi_s\left[\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};\,q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}z^n.$$

The Askey-Wilson polynomials $p_n(x;a,b,c,d;q)$ $(n\in\mathbb{N})$ are the ${}_4\phi_3$ polynomials [3, 7]

$$\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left[\begin{matrix}q^{-n},\,abcdq^{n-1},\,ae^{I\theta},\,ae^{-I\theta}\\ ab,\,ac,\,ad\end{matrix};q,q\right],$$

where $x=\cos\theta$ and I is a complex number such that $I^2=-1$.

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Our main result is the following quadratic formula for basic hypergeometric series, which was discovered by applying the Desnanot–Jacobi adjoint matrix theorem to compute some determinants of Mehta-Wang type [8, 14, 16, 18] or deformed Gram determinants of orthogonal polynomials [20].

Theorem 1.1. *Let $r, s \geq 0$, $a, b, c, d, q \in \mathbb{C}$, $\mathbf{E}_r = (e_1, e_2, \dots, e_r) \in \mathbb{C}^r$, $\mathbf{F}_s = (f_1, f_2, \dots, f_s) \in \mathbb{C}^s$. Then we have*

$$\begin{aligned}
(1.1)\quad & (a-b)(a-c)(bc-d)(1-d) \\
& \times {}_{r+4}\phi_{s+3}\left[\begin{matrix} a^{-1}bc, bcq^{-2}, c, dq^{-1}, \mathbf{E}_r \\ aq^{-1}, bq^{-1}, bcd^{-1}, \mathbf{F}_s \end{matrix}; q, z\right] {}_{r+4}\phi_{s+3}\left[\begin{matrix} a^{-1}bc, bc, c, dq, \mathbf{E}_r q \\ aq, bq, bcd^{-1}, \mathbf{F}_s q \end{matrix}; q, q^{s-r}z\right] \\
&= (a-d)(1-b)(1-c)(bc-ad) \\
& \times {}_{r+4}\phi_{s+3}\left[\begin{matrix} a^{-1}bc, bcq^{-2}, cq^{-1}, d, \mathbf{E}_r \\ aq^{-1}, b, bcd^{-1}q^{-1}, \mathbf{F}_s \end{matrix}; q, z\right] {}_{r+4}\phi_{s+3}\left[\begin{matrix} a^{-1}bc, bc, cq, d, \mathbf{E}_r q \\ aq, b, bcd^{-1}q, \mathbf{F}_s q \end{matrix}; q, q^{s-r}z\right] \\
& - (1-a)(b-d)(c-d)(a-bc) \\
& \times {}_{r+4}\phi_{s+3}\left[\begin{matrix} a^{-1}bcq^{-1}, bcq^{-2}, c, d, \mathbf{E}_r \\ a, bq^{-1}, bcd^{-1}q^{-1}, \mathbf{F}_s \end{matrix}; q, z\right] {}_{r+4}\phi_{s+3}\left[\begin{matrix} a^{-1}bcq, bc, c, d, \mathbf{E}_r q \\ a, bq, bcd^{-1}q, \mathbf{F}_s q \end{matrix}; q, q^{s-r}z\right].
\end{aligned}$$

Let $s = r$, $d = a_0$, $c = a_1$, $b = 0$, $a = b_1$, $e_1 = f_1 = 0$, $e_i = a_i$ and $f_j = b_j$ $(2 \leq j \leq r)$. We get the following result by shifting r to $r - 3$.

Corollary 1.2. *For $r \geq 1$, there holds*

$$\begin{aligned}
(1.2)\quad & (a_0-1)(a_1-b_1) \\
& \times {}_{r+1}\phi_r\left[\begin{matrix} a_0/q, a_1, a_2, \dots, a_r \\ b_1/q, b_2, \dots, b_r \end{matrix}; q, z\right] {}_{r+1}\phi_r\left[\begin{matrix} a_0q, a_1, a_2q \dots, a_rq \\ b_1q, b_2q, \dots, b_rq \end{matrix}; q, z\right] \\
&= (a_0-a_1)(1-b_1) \\
& \times {}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right] {}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, a_2q \dots, a_rq \\ b_1, b_2q, \dots, b_rq \end{matrix}; q, z\right] \\
& - (1-a_1)(a_0-b_1) \\
& \times {}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1/q, a_2, \dots, a_r \\ b_1/q, b_2, \dots, b_r \end{matrix}; q, z\right] {}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1q, a_2q \dots, a_rq \\ b_1q, b_2q, \dots, b_rq \end{matrix}; q, z\right].
\end{aligned}$$

Taking $r = 3$, $z = q$, $a_0 = q^{-n+1}$, $a_1 = abcdq^{n-1}$, $a_2 = ae^{i\theta}$, $a_3 = ae^{-i\theta}$, $b_1 = abq$, $b_2 = ac$ and $b_3 = ad$ in Corollary 1.2, we obtain the following quadratic relation for Askey-Wilson polynomials.

Corollary 1.3. *Let n be a positive integer. There holds*

$$\begin{aligned}
(1.3)\quad & ab(1-q^{n-1})(1-cdq^{n-2})p_n(x; a, b, c, d; q)p_{n-2}(x; aq, bq, c, d; q) \\
&= (1-abq^{n-1})(1-abcdq^{n-1})p_{n-1}(x; a, b, c, d; q)p_{n-1}(x; aq, bq, c, d; q) \\
&\quad - (1-ab)(1-abcdq^{2n-2})p_{n-1}(x; aq, b, c, d; q)p_{n-1}(x; a, bq, c, d; q).
\end{aligned}$$

We shall prove Theorem 1.1 in the next section. In Section 3, we use the Desnanot-Jacobi adjoint matrix theorem and Corollary 1.3 to derive a determinant formula for Askey-Wilson polynomials (cf. Theorem 3.1), which turns out to be a generalization of several recent determinant and Pffafian evaluations in [6, 10, 11, 14, 16]. In Section 4, we connect the determinant formula in Theorem 3.1 to a Gram determinant formula of Askey-Wilson polynomials and show how to derive a recent double–sum formula for the moments of Askey-Wilson polynomials in [4, 9] from Newton's interpolation formula.

2. Proof of Theorem 1.1

For $0 \le k \le n$, we write

$$\begin{aligned}
A_k &= (a-b)(a-c)(bc-d)(1-d)\alpha_k\alpha_{n-k+1}\\
&\quad\times \frac{(a^{-1}bc, bcq^{-2}, c, dq^{-1}, \mathbf{E}_r; q)_k (a^{-1}bc, bc, c, dq, \mathbf{E}_r q; q)_{n-k}}{(q, aq^{-1}, bq^{-1}, bcd^{-1}, \mathbf{F}_s; q)_k (q, aq, bq, bcd^{-1}, \mathbf{F}_s q; q)_{n-k}},\\
B_k &= (a-d)(1-b)(1-c)(bc-ad)\alpha_k\alpha_{n-k+1}\\
&\quad\times \frac{(a^{-1}bc, bcq^{-2}, cq^{-1}, d, \mathbf{E}_r; q)_k (a^{-1}bc, bc, cq, d, \mathbf{E}_r q; q)_{n-k}}{(q, aq^{-1}, b, bcd^{-1}q^{-1}, \mathbf{F}_s; q)_k (q, aq, b, bcd^{-1}q, \mathbf{F}_s q; q)_{n-k}},\\
C_k &= (1-a)(b-d)(c-d)(a-bc)\alpha_k\alpha_{n-k+1}\\
&\quad\times \frac{(a^{-1}bcq^{-1}, bcq^{-2}, c, d, \mathbf{E}_r; q)_k (a^{-1}bcq, bc, c, d, \mathbf{E}_r q; q)_{n-k}}{(q, a, bq^{-1}, bcd^{-1}q^{-1}, \mathbf{F}_s; q)_k (q, a, bq, bcd^{-1}q, \mathbf{F}_s q; q)_{n-k}},
\end{aligned}$$

where $\alpha_k = \left\{(-1)^k q^{\frac{k(k-1)}{2}}\right\}^{s-r}$ for $k \ge 0$. Equating the coefficients of z^n on both sides of (1.1) yields the equivalent identity

$$\sum_{k=0}^{n}(A_k - B_k + C_k) = 0. \tag{2.1}$$

The key point to prove (2.1) is the observation that

$$A_k - B_k + C_k + \big(A_{n-k+1} - B_{n-k+1} + C_{n-k+1}\big) = 0 \tag{2.2}$$

for $0 \le k \le n+1$, where $A_{n+1} = B_{n+1} = C_{n+1} = 0$. Indeed, summing (2.2) over k from 0 to $n+1$ on both sides immediately yields (2.1).

To prove (2.2) we start from the identity

$$\begin{aligned}
&(a-b)(a-c)(d-x)(bc-dx)(x-ay)\\
&\quad\times (x-by)(x-cy)(y-dz)(ax-bcy)(dy-bcz)\\
&\quad - (a-d)(b-x)(c-x)(ad-bc)(x-ay)\\
&\quad\times (y-bz)(y-cz)(x-dy)(ax-bcy)(dx-bcy)\\
&\quad + (b-d)(c-d)(a-x)(ax-bc)(y-az)\\
&\quad\times (x-by)(x-cy)(x-dy)(ay-bcz)(dx-bcy)\\
&\quad = xy(a-b)(a-c)(a-d)(b-d)(c-d)\\
&\qquad\times (1-y)(ad-bc)(x-bcz)(x^2-bcy)(y^2-xz),
\end{aligned} \tag{2.3}$$

which can be easily checked either by hands or by Maple. Replacing (x, y, z) by (q, q^k, q^n) in (2.3) and multiplying both sides of the resulting identity by

$$\frac{(c, d, a^{-1}bc; q)_{k-1}(bcq^{-2}, \mathbf{E}_r; q)_k (c, d, a^{-1}bc, bc, \mathbf{E}_r q; q)_{n-k}}{(aq^{-1}, bq^{-1}, bcd^{-1}q^{-1}; q)_{k+1}(q, \mathbf{F}_s; q)_k (aq, bq, bcd^{-1}q, q, \mathbf{F}_s q; q)_{n-k}},$$

we obtain

$$A_k - B_k + C_k = (q^{n-k+1} - q^k) G_k G_{n-k+1} \Xi, \tag{2.4}$$

where

$$G_k = \frac{(1-q^k)(1-bcq^{k-2})(a^{-1}bc, c, d; q)_{k-1}(bc; q)_{k-2}(\mathbf{E}_r; q)_k \alpha_k}{(a, b, bcd^{-1}, q; q)_k (\mathbf{F}_s; q)_k}$$

and

$$\begin{aligned}\Xi = (a-b)(a-c)(a-d)(b-d)(c-d)(ad-bc)(1-bcq^{n-1}) \\ \times \frac{(1-a)(1-b)(1-bcd^{-1})(1-bcq^{-2})(1-bcq^{-1})(\mathbf{F}_s; q)_1}{adq^2(1-aq^{-1})(1-bq^{-1})(1-bcd^{-1}q^{-1})(\mathbf{E}_r; q)_1},\end{aligned}$$

which is independent of k. Clearly (2.4) implies (2.2).

3. Application to determinant and Pfaffian evaluation

Given a matrix M, if $i_1, \dots, i_r$ (resp. $j_1, \dots, j_r$) are row (resp. column) indices, we denote by $M_{i_1,\dots,i_r}^{j_1,\dots,j_r}$ the matrix that remains when the rows $i_1, \dots, i_r$ and columns $j_1, \dots, j_r$ are deleted. Let $n \geq 2$ and M be an $n \times n$ matrix. Then the Desnanot-Jacobi adjoint matrix theorem [1, Lemma 7.7] reads

$$\det M \ \det M_{1\,n}^{1\,n} = \det M_1^1 \ \det M_n^n - \det M_n^1 \ \det M_1^n, \tag{3.1}$$

where we set $\det M_{1\,n}^{1\,n} = 1$ if $n = 2$.

Theorem 3.1. *For $n \geq 1$, $0 \leq i \leq n-1$ and $1 \leq j \leq n$, let*

$$\begin{aligned}B_{i,j} =& \frac{(ab; q)_{i+j-1}(-bq^{-1+j})}{(abcd; q)_{i+j}} \\ &\times \left[c + d - 2x + (1-cd)(aq^i + bq^{j-1}) - ab(c+d-2cdx)q^{i+j-1}\right].\end{aligned} \tag{3.2}$$

Then

$$\det(B_{i,j})_{0 \leq i \leq n-1, 1 \leq j \leq n} = D_n(a, b) \cdot p_n(x; a, b, c, d; q), \tag{3.3}$$

where

$$D_n(a, b) = a^{n(n-1)/2} b^{n(n+1)/2} q^{n(n-1)(2n-1)/6} \prod_{i=0}^{n-1} \frac{(ab, cd, q; q)_i}{(abcd; q)_{n+i}}. \tag{3.4}$$

Proof. By (3.4) we have

$$D_n(a, b)/D_{n-1}(a, b) = a^{n-1} b^n q^{(n-1)^2} \frac{(ab, cd; q)_{n-1}(q; q)_{n-1}}{(abcdq^{n-1}; q)_n (abcd; q)_{2n-2}},$$

$$D_n(aq, b)/D_n(a, b) = q^{n(n-1)/2} \frac{(abq; q)_{n-1}}{(abcdq^n; q)_n} \frac{(1-abcd)^n}{(1-ab)^{n-1}}.$$

Therefore

$$\frac{D_n(a, b)/D_{n-1}(a, b)}{D_{n-1}(aq, bq)/D_{n-2}(aq, bq)} = \frac{ab(ab; q)_2(1-cdq^{n-2})(1-q^{n-1})}{(1-abq^{n-1})(1-abcdq^{n-1})(abcd; q)_2} \tag{3.5}$$

and

$$\frac{D_{n-1}(aq,b)/D_{n-1}(a,b)}{D_{n-1}(aq,bq)/D_{n-1}(a,bq)} = \frac{1-abq}{1-abq^{n-1}}\frac{1-abcdq^{2n-2}}{1-abcdq^{n-1}}\Big(\frac{1-abq}{1-ab}\Big)^{n-2}\Big(\frac{1-abcd}{1-abcdq}\Big)^{n-1}. \tag{3.6}$$

Let $M_n(a,b) := \det(B_{i,j})_{0\le i\le n-1, 1\le j\le n}$. For $n=1$, formula (3.3) is obvious. Assume that $n \ge 2$. Applying (3.1) to the determinant in (3.3) we obtain

$$M_n(a,b)\,M_{n-2}(aq,bq) = \frac{(ab;q)_2}{(abcd;q)_2}M_{n-1}(a,b)\,M_{n-1}(aq,bq) - \Big(\frac{1-ab}{1-abcd}\Big)^n\Big(\frac{1-abcdq}{1-abq}\Big)^{n-2}M_{n-1}(aq,b)\,M_{n-1}(a,bq).$$

It suffices to show that if we substitute $M_n(a,b)$ by $D_n(a,b)p_n(x;a,b,c,d;q)$ the above identity still holds, i.e., for $n\ge 2$,

$$\begin{aligned}&D_n(a,b)D_{n-2}(aq,bq)p_n(x;a,b,c,d;q)p_{n-2}(x;aq,bq,c,d;q)\\ &= \frac{(ab;q)_2}{(abcd;q)_2}D_{n-1}(a,b)D_{n-1}(aq,bq)p_{n-1}(x;a,b,c,d;q)p_{n-1}(x;aq,bq,c,d;q)\\ &\quad - \Big(\frac{1-ab}{1-abcd}\Big)^n\Big(\frac{1-abcdq}{1-abq}\Big)^{n-2}D_{n-1}(aq,b)D_{n-1}(a,bq)\\ &\qquad\times p_{n-1}(x;a,bq,c,d;q)p_{n-1}(x;aq,b,c,d;q).\end{aligned} \tag{3.7}$$

Dividing the two sides of (3.7) by $D_{n-1}(a,b)D_{n-1}(aq,bq)$ and applying the two identities (3.5) and (3.6), we see that (3.7) is exactly the quadratic formula (1.3). The result then follows by induction on n. □

The following formula is an extension of Nishizawa's q-analogue of Mehta-Wang's formula [8, 16, 18].

Corollary 3.2. *For $n \ge 1$, there holds*

$$\begin{aligned}&\det\Big((q^{i-1}-cq^{j-1})\frac{(aq;q)_{i+j-2}}{(abq^2;q)_{i+j-2}}\Big)_{1\le i,j\le n}\\ &= (-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(2n-5)}{6}}(abcq;q^2)_n \prod_{k=1}^{n}\frac{(q;q)_{k-1}(aq;q)_k(bq;q)_{k-2}}{(abq^2;q)_{k+n-2}}\\ &\qquad\times {}_4\phi_3\left[\begin{matrix}q^{-n},\ abq^n,\ (acq)^{\frac12},\ -(acq)^{\frac12}\\ aq,\ (abcq)^{\frac12},\ -(abcq)^{\frac12}\end{matrix};q,q\right].\end{aligned} \tag{3.8}$$

Proof. Since the Askey-Wilson polynomials are symmetric on a and b, in (3.3) replacing $p_n(x;a,b,c,d;q)$ by $p_n(x;b,a,c,d;q)$ and making the following substitution:

$$x\leftarrow 0,\quad a\leftarrow (aq/c)^{1/2}I,\quad b\leftarrow -(acq)^{1/2}I,\qquad c\leftarrow b^{1/2}I,\quad d\leftarrow -b^{1/2}I$$

where $I^2=-1$, gives (3.8) as $(bq;q)_{-1} = 1/(1-b)$. □

If $c = 1$, we can sum the ${}_4\phi_3$ in (3.8) by Andrews' terminating q-analogue of Watson's formula [7, II.17]

$$
{}_4\phi_3\left[\begin{matrix} q^{-n},\, a^2q^{n+1},\, b,\, -b \\ aq,\, -aq,\, b^2 \end{matrix}; q, q\right] = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \dfrac{b^n(q, a^2q^2/b^2; q^2)_{n/2}}{(a^2q^2,\, b^2q; q^2)_{n/2}}, & \text{if } n \text{ is even}, \end{cases}
$$

and deduce the following result, which was first proved in [11], and is also a q-analogue of [14, Theorem 6].

Corollary 3.3. *For $m \geq 1$, there holds*

$$
\begin{aligned}
&\det\left((q^{i-1} - q^{j-1})\frac{(aq;q)_{i+j-2}}{(abq^2;q)_{i+j-2}}\right)_{1\leq i,j\leq 2m} \\
&= a^{2m(m-1)} q^{\frac{2m(m-1)(4m+1)}{3}} \prod_{k=1}^{m}\left(\frac{(q, aq; q)_{2k-1}(bq;q)_{2k-2}}{(abq^2;q)_{2(k+m)-3}}\right)^2.
\end{aligned}
$$

Recall [1] that the Pfaffian of a skew-symmetric matrix $A = (A_{i,j})_{1\leq i,j\leq 2m}$ is defined by

$$
\operatorname{Pf} A = \sum_{\pi\in\mathcal{M}[1,\ldots,2m]} \operatorname{sgn}\pi \prod_{\substack{i<j \\ i,j \text{ matched in } \pi}} A_{i,j}. \tag{3.9}
$$

Here $\mathcal{M}[a,\ldots,b]$ is the set of perfect matchings of the complete graph on $\{a,\ldots,b\}$ for any nonnegative integers a and b such that $a < b$, and $\operatorname{sgn}\pi = (-1)^{\operatorname{cr}\pi}$, where $\operatorname{cr}\pi$ is the number of matched pairs $(i,\, j)$ and $(i',\, j')$ in π such that $i < i' < j < j'$. The following result was first proved by Ishikawa et al. [11].

Corollary 3.4. *For $m \geq 1$, there holds*

$$
\begin{aligned}
&\operatorname{Pf}\left((q^{i-1} - q^{j-1})\frac{(aq;q)_{i+j-2}}{(abq^2;q)_{i+j-2}}\right)_{1\leq i,j\leq 2m} \\
&\qquad = a^{m(m-1)} q^{\frac{m(m-1)(4m+1)}{3}} \prod_{k=1}^{m}\frac{(q, aq; q)_{2k-1}(bq;q)_{2k-2}}{(abq^2;q)_{2(k+m)-3}}.
\end{aligned}
$$

Proof. As the square of the Pfaffian of any skew-symmetric matrix is its determinant, we derive from Corollary 3.3 that

$$
\begin{aligned}
&\operatorname{Pf}\left((q^{i-1} - q^{j-1})\frac{(aq;q)_{i+j-2}}{(abq^2;q)_{i+j-2}}\right)_{1\leq i,j\leq 2m} \\
&\qquad = \varepsilon_m a^{m(m-1)} q^{\frac{m(m-1)(4m+1)}{3}} \prod_{k=1}^{m}\frac{(q, aq; q)_{2k-1}(bq;q)_{2k-2}}{(abq^2;q)_{2(k+m)-3}},
\end{aligned}
\tag{3.10}
$$

where $\varepsilon_m^2 = 1$. By (3.10), the factor ε_m is a rational function of a, b and q, and only takes values 1 or -1. Hence, for fixed m, we must have $\varepsilon_m = 1$ or $\varepsilon_m = -1$

regardless of the values of a, b and q. It remains to show that $\varepsilon_m = 1$ for all $m \geq 1$. Obviously we have $\varepsilon_1 = 1$. Suppose that $m \geq 2$. Taking $b = 0$ and replacing a by q^{a-1} the identity (3.10) reduces to

$$\begin{aligned} &\operatorname{Pf}\big((q^i - q^j)(q^a;q)_{i+j}\big)_{0\leq i,j\leq 2m-1} \\ &\quad= \varepsilon_m q^{m(m-1)(a-1)+\frac{m(m-1)(4m+1)}{3}} \prod_{k=1}^{m} (q, q^a; q)_{2k-1}. \end{aligned} \tag{3.11}$$

Using the q-gamma function [7, p. 20]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad 0 < q < 1,$$

we can rewrite (3.11) as

$$\begin{aligned} &\operatorname{Pf}\left(q^i[j-i]_q\Gamma_q(a+i+j)\right)_{0\leq i,j\leq 2m-1} \\ &\quad= \varepsilon_m q^{m(m-1)(a-1)+\frac{m(m-1)(4m+1)}{3}} \prod_{k=1}^{m} [2k-1]_q!\,\Gamma_q(a+2k-1), \end{aligned} \tag{3.12}$$

where $[n]_q! = [1]_q[2]_q\cdots[n]_q$ and $[k]_q = (1-q^k)/(1-q)$. If we multiply both sides of (3.12) by $[a+1]_q$ and let a tend to -1, then, the right-hand side becomes

$$\varepsilon_m q^{\frac{m(m-1)(4m-5)}{3}} \prod_{k=1}^{m} [2k-1]_q! \prod_{k=1}^{m-1} \Gamma_q(2k). \tag{3.13}$$

On the other hand, by (3.9), the left-hand side can be written as

$$\sum_{\pi\in\mathcal{M}[0,\ldots,2m-1]} \operatorname{sgn}\pi \lim_{a\to-1}[a+1]_q \prod_{\substack{i<j\\ i,j\text{matched in }\pi}} q^i[j-i]_q\Gamma_q(a+i+j). \tag{3.14}$$

In this sum, matchings π for which all matched pairs i, j satisfy $i + j > 1$ will not contribute, because the corresponding summands vanish. Therefore, the survival matchings must match 0 and 1 and the sum in (3.14) reduces to

$$\begin{aligned} &\sum_{\pi'\in\mathcal{M}[2,\ldots,2m-1]} \operatorname{sgn}\pi' \prod_{\substack{i<j\\ i,j\text{matched in }\pi'}} q^i[j-i]_q\Gamma_q(i+j-1) \\ &\quad= \operatorname{Pf}\left(q^i[j-i]_q\Gamma_q(i+j-1)\right)_{2\leq i,j\leq 2m-1} \\ &\quad= \operatorname{Pf}\left(q^{i+2}[j-i]_q\Gamma_q(i+j+3)\right)_{0\leq i,j\leq 2m-3} \\ &\quad= \varepsilon_{m-1} q^{\frac{m(m-1)(4m-5)}{3}} \prod_{k=1}^{m} [2k-1]_q! \prod_{k=1}^{m-1} \Gamma_q(2k). \end{aligned}$$

Comparing with (3.13) we see that $\varepsilon_m = \varepsilon_{m-1} = \cdots = \varepsilon_1 = 1$. □

Remark 3.5. Except the trivial but crucial point that ε_m is independent of a, b and q, the above proof is a q-adaptation of Ciucu and Krattenthaler's proof [6] for the $q \to 1$ case of (3.12):

$$\operatorname{Pf}((j-i)\Gamma(a+i+j))_{0\leq i,j\leq 2m-1} = \prod_{k=1}^{m}(2k-1)!\Gamma(a+2k-1). \tag{3.15}$$

As we have shown that ε_m is actually independent of q, we could also reduce the proof directly to (3.15) by taking the limit $q \to 1$ in (3.12).

4. Link to Gram determinants of Askey-Wilson polynomials

In this section we show that the determinant formula in Theorem 3.1 is actually related to a Gram determinant for the Askey-Wilson orthogonal polynomials (see [20]). This permits us to enlighten the origin of the peculiar matrix coefficients (3.2).

Let $\{p_n(x)\}$ be a sequence of orthogonal polynomials with respect to a measure $d\mu$, and $\{\phi_k\}$ and $\{\psi_k\}$ be two sequences of polynomials such that ϕ_k and ψ_k are of exact degree k. Then

$$\begin{vmatrix} \mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n} \\ \mu_{1,0} & \mu_{1,1} & \cdots & \mu_{1,n} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1,0} & \mu_{n-1,1} & \cdots & \mu_{n-1,n} \\ \phi_0(x) & \phi_1(x) & \cdots & \phi_n(x) \end{vmatrix} = C \cdot p_n(x), \tag{4.1}$$

where $\mu_{i,j} = \int \psi_i(x)\phi_j(x)d\mu$ and C represents a factor of normalization.

Note that the three variables x, θ, z are related to each other as follows:

$$z = e^{I\theta}, \quad x = \cos\theta, \quad x = \frac{z+z^{-1}}{2}.$$

For $|a|, |b|, |c|, |d| < 1$ and $n \ge 0$, let $h_n := h_n(a,b,c,d,q)$ with

$$h_0 = \frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty},$$
$$h_n = h_0 \frac{(1-q^{n-1}abcd)(q,ab,ac,ad,bc,bd,cd;q)_n}{(1-q^{2n-1}abcd)(abcd;q)_n}.$$

Then, the orthogonality of Askey-Wilson polynomials reads

$$\oint_{\mathcal{C}} p_m(\cos\theta;a,b,c,d;q)p_n(\cos\theta;a,b,c,d;q)w(\cos\theta)\frac{dz}{4\pi i z} = \frac{h_n}{h_0}\delta_{mn}, \tag{4.2}$$

where the contour $\mathcal{C}$ is the unit circle with suitable deformations (see [3, 4]) and the weight function $w(x) := w(x,a,b,c,d;q)$ is defined by

$$w(\cos\theta) = \frac{(e^{2I\theta}, e^{-2I\theta};q)_\infty}{h_0 \cdot (ae^{I\theta}, ae^{-I\theta}, be^{I\theta}, be^{-I\theta}, ce^{I\theta}, ce^{-I\theta}, de^{I\theta}, de^{-I\theta};q)_\infty}.$$

Except for the constant factor C, the following representation for the Askey-Wilson polynomials in terms of moments was already given by Atakishiyev and Suslov [2]. Their proof was based on a generalization of Hahn's approach and Wilson [20] suggested a method to evaluate this determinant directly.

Theorem 4.1. *For $n \ge 1$, $0 \le i \le n-1$ and $0 \le j \le n$ let*

$$A_{i,j} = (ac,ad;q)_i(bc,bd;q)_j \frac{(ab;q)_{i+j}}{(abcd;q)_{i+j}}, \tag{4.3}$$

and $A_{n,j} = (bz, b/z;q)_j$. Then

$$\det(A_{i,j})_{0\le i,j\le n} = C \cdot p_n(x;a,b,c,d;q), \tag{4.4}$$

where $x = (z+z^{-1})/2$ *and*

$$C = (-1)^n a^{n(n-1)/2} b^{n(n+1)/2} q^{n(n-1)(2n-1)/6} \prod_{i=0}^{n-1} \frac{(ab, ac, ad, bc, bd, cd, q; q)_i}{(abcd; q)_{n+i}}.$$

Proof. By (4.2) we have $\oint_{\mathcal{C}} w(\cos\theta) \frac{dz}{4\pi i z} = 1$. It follows that

$$\begin{aligned}
&\oint_{\mathcal{C}} w(x, a, b, c, d; q)(az, a/z; q)_i (bz, b/z; q)_j \frac{dz}{4\pi i z} \\
&\qquad = \frac{h_0(aq^i, bq^j, c, d, q)}{h_0(a, b, c, d, q)} \oint_{\mathcal{C}} w(x, aq^i, bq^j, c, d; q) \frac{dz}{4\pi i z} \qquad (4.5)\\
&\qquad = (ac, ad; q)_i (bc, bd; q)_j \frac{(ab; q)_{i+j}}{(abcd; q)_{i+j}},
\end{aligned}$$

which coincides with $A_{i,j}$ in (4.3) for $i, j = 0, \ldots, n$. Thus, by choosing $\psi_i = (az, a/z; q)_i$ and $\phi_j = (bz, b/z; q)_j$ in (4.1) we obtain the determinant in (4.4). It remains to compute the factor C. As

$$p_n(x; a, b, c, d; q) = 2^n (abcdq^{n-1}; q)_n x^n + \text{ lower terms},$$

$$(bz, b/z; q)_n = \prod_{k=0}^{n-1} (1 - 2bxq^k + b^2 q^{2k}),$$

comparing the coefficients of x^n in (4.1) we get

$$C = \frac{(-1)^n b^n q^{n(n-1)/2}}{(abcdq^{n-1}; q)_n} \det(A_{i,j})_{0 \le i,j \le n-1}. \tag{4.6}$$

The determinant in (4.6) is essentially the Hankel determinant associated to the moments of little q-Jacobi polynomials (see [10])

$$\det\left(\frac{(ab; q)_{i+j}}{(abcd; q)_{i+j}}\right)_{0 \le i,j \le n-1} = (ab)^{\frac{n(n-1)}{2}} q^{\frac{n(n-1)(n-2)}{3}} \prod_{k=0}^{n-1} \frac{(q, ab, cd; q)_k}{(abcd; q)_{k+n-1}}, \tag{4.7}$$

which is also a special case of (3.8). It follows from (4.7) that

$$\begin{aligned}
&\det\left((ac, ad; q)_i (bc, bd; q)_j \frac{(ab; q)_{i+j}}{(abcd; q)_{i+j}}\right)_{0 \le i,j \le n-1} \qquad (4.8)\\
&\qquad = (ab)^{\frac{n(n-1)}{2}} q^{\frac{n(n-1)(n-2)}{3}} \prod_{j=1}^{n-1} (ac, ad, bc, bd; q)_j \prod_{k=0}^{n-1} \frac{(q, ab, cd; q)_k}{(abcd; q)_{k+n-1}}.
\end{aligned}$$

Substituting this in (4.6) we recover the factor C in formula (4.4). $\square$

Remark 4.2. Using (4.8), formula (4.4) can also be proven from the Desnanot-Jacobi adjoint matrix theorem and a special case of the known contiguous relation (see [13, (3.3)])

$$\begin{aligned}
&{}_r\phi_s \left[\begin{matrix} aq, \mathbf{A} \\ bq, \mathbf{B} \end{matrix}; q, z\right] - {}_r\phi_s \left[\begin{matrix} a, \mathbf{A} \\ b, \mathbf{B} \end{matrix}; q, z\right] \qquad (4.9)\\
&\qquad = \frac{(-1)^{1+s-r} z(a-b)}{(1-b)(1-bq)} \frac{\prod_{i=1}^{r-1}(1-A_i)}{\prod_{i=1}^{s-1}(1-B_i)} {}_r\phi_s \left[\begin{matrix} aq, \mathbf{A}q \\ bq^2, \mathbf{B}q \end{matrix}; q, q^{1+s-r} z\right],
\end{aligned}$$

where $\mathbf{A} = (A_1, \ldots, A_{r-1})$ and $\mathbf{B} = (B_1, \ldots, B_{s-1})$ for $r, s \ge 2$.

Actually it is not hard to see that Theorems 3.1 and 4.1 are equivalent.

Proposition 4.3. *The two determinant formulae* (4.4) *and* (3.3) *are equivalent.*

Proof. Let $z = e^{I\theta}$ and $x = \cos\theta$. Consider the matrix $A := (A_{i,j})_{0\le i,j\le n}$, where $A_{i,j}$ are given in (4.3). Upon multiplying the $(j-1)$st column of A by $1-2bxq^{j-1}+b^2q^{2j-2}$ and subtracting from the jth column, for $j = n, n-1, \dots, 1$, the last row of the matrix becomes $(1, 0, \dots, 0)$, and

$$A_{i,j} - (1 - 2bxq^{j-1} + b^2q^{2j-2})A_{i,j-1} = (ac, ad; q)_i(bc, bd; q)_{j-1}B_{i,j},$$

where $B_{i,j}$ is given in (3.2) for $0 \le i \le n-1$ and $1 \le j \le n$. Hence, the two formulae (4.4) and (3.3) are equivalent. ☐

By (4.2) the linear functional $\mathcal{L} : \mathbb{C}[x] \mapsto \mathbb{C}$ associated to the orthogonal measure of the Askey-Wilson polynomials has the explicit integral expression:

$$\mathcal{L}\big(x^n\big) = \oint_{\mathcal{C}} \left(\frac{z+z^{-1}}{2}\right)^n w(\cos\theta)\frac{dz}{4\pi iz}. \tag{4.10}$$

It follows from (4.5) with $j = 0$ and $i = n$ that

$$\mathcal{L}\big((az, a/z; q)_n\big) = \frac{(ab, ac, ad; q)_n}{(abcd; q)_n}. \tag{4.11}$$

Clearly, if we take $\psi_i(x) = \phi_i(x) = x^i$ for $i \ge 0$ in (4.1), then $\mu_{i,j} = \mathcal{L}\big(x^{i+j}\big)$ are the moments of Askey-Wilson polynomials and we obtain another determinant expression for the Askey-Wilson polynomials. Such a formula would be interesting if we have a simple formula for the moments (4.10). Recently Corteel et al. [4] and Ismail-Rhaman [9] have published a double-sum formula for the moments of Askey-Wilson polynomials. We would like to point out that their formula does follow straightforwardly from (4.11) and the Newton interpolation formula (see [17, Chapter 1]), that we recall below.

Theorem 4.4 (Newton's interpolation formula)**.** *Let $b_0, b_1, \dots, b_{n-1}$ be distinct complex numbers. Then, for any polynomial f of degree less than or equal to n we have*

$$f(x) = \sum_{k=0}^{n}\left(\sum_{j=0}^{k}\frac{f(b_j)}{\prod_{r=0,r\ne j}^{k}(b_j - b_r)}\right)(x-b_0)\cdots(x-b_{k-1}). \tag{4.12}$$

Indeed, if $b_j = (q^{-j}/a + aq^j)/2$ for $j = 0, \dots, n-1$, then

$$\prod_{r=0}^{j-1}(b_j - b_r) = (-1)^j 2^{-j} a^j q^{\binom{j}{2}}(q, q^{-2j+1}/a^2; q)_j,$$

$$\prod_{r=j+1}^{k}(b_j - b_l) = (-1)^{k-j}2^{j-k}a^{j-k}q^{-(j+1)(k-j)-\binom{k-j}{2}}(q, a^2q^{2j+1}; q)_{k-j}.$$

As $x = (z + 1/z)/2$ we have

$$(x-b_0)\cdots(x-b_{k-1}) = (-1)^k 2^{-k} a^{-k} q^{-\binom{k}{2}}(az, a/z; q)_k.$$

Substituting the above into (4.12) we obtain Proposition 3.1 of [4], i.e.,

$$f(x) = \sum_{k=0}^{n}(az, a/z; q)_k \sum_{j=0}^{k}\frac{q^{k-j^2}a^{-2j}f\big(\frac{q^ja+q^{-j}/a}{2}\big)}{(q, q^{-2j+1}/a^2; q)_j(q, q^{2j+1}a^2; q)_{k-j}}. \tag{4.13}$$

In particular, applying the linear functional $\mathcal{L}$ to the two sides of (4.13) with $f(x) = (t+x)^n$ we obtain Theorem 1.13 of [4] with a typo corrected, which also appears in [9].

Theorem 4.5 (Corteel-Stanley-Stanton-Williams, Ismail-Rahman)**.** *For any fixed $t \in \mathbb{C}$, we have*

$$\mathcal{L}((t+x)^n) = \sum_{k=0}^{n} \frac{(ac, ab, ad; q)_k}{(abcd; q)_k} \sum_{j=0}^{k} \frac{q^{k-j^2} a^{-2j} \left(t + \frac{q^j a + q^{-j}/a}{2}\right)^n}{(q, q^{-2j+1}/a^2; q)_j (q, q^{2j+1}a^2; q)_{k-j}}. \tag{4.14}$$

Remark 4.6. The proof of the above formula in [4, 9] was based on a result of Ismail and Stanton [5, Theorem 3.3], which is equivalent to (4.13). Since Ismail and Stanton's formula was originally proved using the Askey-Wilson operator in [5, Theorem 20], our proof seems more accessible for people who are not familiar with an Askey-Wilson operator. The $t = 0$ of (4.14) is the starting point of [12].

Finally, it is interesting to note that the following special case of Newton's formula (4.12) has been rediscovered recently by Mansour et al. [15].

Corollary 4.7. *We have*

$$(x + a_0) \cdots (x + a_{n-1}) = \sum_{k=0}^{n} u(n,k)(x - b_0) \cdots (x - b_{k-1}), \tag{4.15}$$

where

$$u(n,k) = \sum_{r=0}^{k} \frac{\prod_{j=0}^{n-1} (b_r + a_j)}{\prod_{j=0, j\neq r}^{k} (b_r - b_j)} \quad \textit{for} \quad k = 0, \dots, n.$$

Remark 4.8. Clearly the connection coefficients $u(n,k)$ in (4.15) are characterized by the recurrence

$$u(n,k) = u(n-1,k-1) + (a_{n-1} + b_k) u(n-1,k)$$

with boundary conditions $u(n,0) = \prod_{i=0}^{n-1} (a_i + b_0)$ and $u(0,k) = \delta_{0,k}$ (the Kronecker delta function). Hence we recover the main result of Mansour et al. [15] (see also [19]).

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