COHEN-MACAULAY BIPARTITE GRAPHS IN ARBITRARY CODIMENSION

HASSAN HAGHIGHI, SIAMAK YASSEMI, AND RAHIM ZAARE NAHANDI

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ABSTRACT. Let G be an unmixed bipartite graph of dimension $d - 1$. Assume that $K_{n,n}$, with $n \geq 2$, is a maximal complete bipartite subgraph of G of minimum dimension. Then G is Cohen-Macaulay in codimension t if and only if $t \geq d - n + 1$. This is derived from a characterization of Cohen-Macaulay bipartite graphs by Herzog and Hibi and generalizes a recent result of Cook and Nagel on unmixed Buchsbaum graphs. Furthermore, we show that any unmixed bipartite graph G which is Cohen-Macaulay in codimension t , is obtained from a Cohen-Macaulay graph by replacing certain edges of G with complete bipartite graphs. Thus, in light of combinatorial characterization of Cohen-Macaulay bipartite graphs, our result may be considered purely combinatorial.

1. INTRODUCTION

Cohen-Macaulay simplicial complexes are among central research topics in combinatorial commutative algebra. While characterization of such complexes is a far reaching problem, one appeals to study specific families of Cohen-Macaulay simplicial complexes. Flag complexes are among important families of complexes recommended to study [10, page 100]. However, it is known that a simplicial complex is Cohen-Macaulay if and only if its barycentric subdivision is a Cohen-Macaulay flag complex. Therefore, a characterization of Cohen-Macaulay flag complexes is equivalent to a characterization of Cohen-Macaulay simplicial complexes. Nevertheless, after all, the ideal of a flag complex is generated by quadratic square-free monomials, which are simpler compared to arbitrary square-free monomial ideals. Furthermore, it seems that, expressing many combinatorial properties in terms of graphs is more convenient. As some evidences, the characterization of unmixed bipartite graphs by Villarreal [11] and Cohen-Macaulay bipartite graphs by Herzog and Hibi [5] are well expressed in terms of graphs.

On the other hand, in the hierarchy of families of graphs with respect to Cohen-Macaulay property, Buchsbaum complexes appear right after Cohen-Macaulay ones. Unmixed bipartite Buchsbaum graphs were characterized by Cook and Nagel [1] (also by the authors [3]). Natural families of graphs in this hierarchy are bipartite CM_t graphs, namely, graphs for which the independence complex is pure and Cohen-Macaulay in codimension t . The concept of CM_t simplicial complexes was introduced in [4] which is the pure version of simplicial complexes Cohen-Macaulay in codimension t studied by Miller, Novik and Swartz [6]. In this note, we give

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characterizations of unmixed bipartite CM_t graphs in terms of its dimension and the minimum dimension of its maximal nontrivial complete bipartite subgraphs. Cook and Nagel showed that the only non-Cohen-Macaulay unmixed Buchsbaum bipartite graphs are complete bipartite graphs [1, Theorem 4.10] and [3, Theorem 1.3]. We give a generalization of this fact to unmixed bipartite graphs which are Cohen-Macaulay in arbitrary codimension. In the next section we gather necessary definitions and known results to be used in the rest of the paper. In Section 3 we improve some results on joins of simplicial complexes and disjoint unions of graphs with respect to the CM_t property. Section 4 is devoted to two characterizations of bipartite CM_t graphs and some examples.

2. PRELIMINARIES

For basic definitions and general facts on simplicial complexes we refer to the book of Stanley [10]. By a complex we will always mean a simplicial complex. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E . The *inclusive neighborhood* of $v \in V$ is the set $N[v]$ consisting of v and vertices adjacent to v in G . The *independence complex* of $G = (V, E)$ is the complex $\text{Ind}(G)$ with vertex set V and with faces consisting of independent sets of vertices of G , i.e., sets of vertices of G where no two of them are adjacent. These complexes are called *flag complexes*, and their Stanley-Reisner ideal is generated by quadratic square-free monomials. By *dimension* of a graph G we mean the dimension of the complex $\text{Ind}(G)$ (= independence number of $G - 1$). A graph G is said to be *unmixed* if $\text{Ind}(G)$ is pure.

For an integer $t \geq 0$, a complex Δ is called CM_t if it is pure and for every face $F \in \Delta$ with $\#(F) \geq t$, $\text{link}_\Delta(F)$ is Cohen-Macaulay. This is the same as pure complexes which are Cohen-Macaulay in codimension t . Accordingly, CM_0 and CM_1 complexes are precisely Cohen-Macaulay and Buchsbaum complexes, respectively. Clearly, a CM_t complex is CM_r for all $r \geq t$ and a complex of dimension $d - 1$ is always CM_{d-1} . One uses the convention that for $t < 0$, CM_t would mean CM_0 . A graph G is called CM_t if $\text{Ind}(G)$ is CM_t . A basic tool for checking CM_t property of complexes is the following lemma.

Lemma 2.1 ([4, Lemma 2.3]). *Let $t \geq 1$ and let Δ be a nonempty complex. Then the following are equivalent:*

- (i) Δ is a CM_t complex.
- (ii) Δ is pure and $\text{link}_\Delta(v)$ is CM_{t-1} for every vertex $v \in \Delta$.

By the straightforward identity $\text{link}_{\text{Ind}(G)}(v) = \text{Ind}(G \setminus N[v])$, the counterpart of this lemma for graphs is the following:

Lemma 2.2. *Let $t \geq 1$ and let G be a graph. Then the following are equivalent:*

- (i) G is a CM_t graph.
- (ii) G is unmixed and $G \setminus N[v]$ is a CM_{t-1} graph for every vertex $v \in G$.

We recall some basic relevant facts on bipartite graphs. A graph $G = (V, E)$ is called *bipartite* if V is a disjoint union of a partition V_1 and V_2 such that $E \subset V_1 \times V_2$. If $\#(V_1) = m$ and $\#(V_2) = n$ and $E = V_1 \times V_2$, then G is the *complete* bipartite graph $K_{m,n}$. We will be interested in unmixed complete bipartite graphs $K_{n,n}$.

Unmixed bipartite graphs are characterized by Villarreal in the following result.

Theorem 2.3 ([11, Theorem 1.1]). *Let G be a bipartite graph without an isolated vertex. Then G is unmixed if and only if there is a partition $V_1 = \{x_1, \dots, x_n\}$ and $V_2 = \{y_1, \dots, y_n\}$ of vertices of G such that*

- (1) $x_i y_i$ is an edge in G for $1 \leq i \leq n$ and
- (2) if $x_i y_j$ and $x_j y_k$ are edges in G , for some distinct i, j and k , then $x_i y_k$ is an edge in G .

In this case, such a partition and ordering is called a *pure order* of G . The edges $x_i y_i$, $i = 1, \dots, n$ are called *perfect matching* edges of G . A pure order is said to have a *cross* if, for some $i \neq j$, $x_i y_j$ and $x_j y_i$ are both edges in G . Otherwise, the order is called *cross-free* (see [1, § 4]). For unmixed bipartite graphs, being cross-free is independent from an ordering of vertices of G . More precisely, if G has a cross in some pure ordering, it has a cross in every pure ordering [1, Lemma 4.5].

An immediate consequence of Theorem 2.3 is the following useful lemma.

Lemma 2.4. *Let G be an unmixed bipartite graph with pure order of vertices $(\{x_1, \dots, x_d\}, \{y_1, \dots, y_d\})$ and let $K_{n,n}$ be a complete bipartite subgraph of G on $(\{x_{i_1}, \dots, x_{i_n}\}, \{y_{i_1}, \dots, y_{i_n}\})$.*

- (i) *If $x_j y_{i_k}$ is an edge in G for some j and k , then $x_j y_{i_l}$ is an edge in G for all $l = 1, \dots, n$.*
- (ii) *If $x_{i_k} y_j$ is an edge in G for some k and j , then $x_{i_l} y_j$ is an edge in G for all $l = 1, \dots, n$.*

Proof. The assertion (i) is immediate by Theorem 2.3 because $x_{i_k} y_{i_l}$ is an edge in $K_{n,n} \subset G$ for all $l = 1, \dots, n$. Also (ii) follows because $x_{i_l} y_{i_k}$ is an edge in $K_{n,n} \subset G$ for all $l = 1, \dots, n$. $\square$

There are also at least two nice characterizations of Cohen-Macaulay bipartite graphs.

Theorem 2.5 ([5, Theorem 3.4]). *Let G be a bipartite graph without isolated vertices. Then G is Cohen-Macaulay if and only if there is a pure ordering $V_1 = \{x_1, \dots, x_n\}$ and $V_2 = \{y_1, \dots, y_n\}$ of vertices of G such that $x_i y_j$ being in G implies $i \leq j$.*

The ordering in Theorem 2.5 is called a *Macaulay order* of vertices of G .

Proposition 2.6 ([1, Proposition 4.8]). *Let G be a bipartite graph. Then G is Cohen-Macaulay if and only if G has a cross-free pure order.*

Remark 2.7. Both characterizations of Cohen-Macaulay bipartite graphs are purely combinatorial (Theorem 2.3 together with Theorem 2.5 or Proposition 2.6). These characterizations may be taken as combinatorial definitions of Cohen-Macaulayness for bipartite graphs. Therefore, for bipartite graphs, the CM_t property can also be stated in combinatorial terms.

Recall that a complex is Buchsbaum if and only if it is pure and the link of each vertex is Cohen-Macaulay [9]. Thus, a graph is Buchsbaum if and only if G is unmixed and for every vertex $v \in G$, $G \setminus N[v]$ is Cohen-Macaulay. For bipartite graphs there is a sharper result. Complete bipartite graphs are well known to be Buchsbaum (e.g., see [12, Proposition 2.3]). But indeed, the converse is also true.

Theorem 2.8 (see [1, Theorem 4.10] or [3, Theorem 1.3]). *Let G be a bipartite graph. Then G is Buchsbaum if and only if G is a complete bipartite graph $K_{n,n}$ for some $n \geq 2$, or G is Cohen-Macaulay.*

3. JOINS OF CM_t COMPLEXES AND DISJOINT UNIONS OF CM_t GRAPHS

It is known that the join of two complexes is Cohen-Macaulay if and only if they are both Cohen-Macaulay (see [8] and [2]). In fact, if the join of two complexes is Buchsbaum, it should indeed be Cohen-Macaulay [7, Theorem 2.6]. If Δ is a CM_r complex of dimension $d - 1$ and Δ' is a $\text{CM}_{r'}$ complex of dimension $d' - 1$, then their join $\Delta * \Delta'$ is a CM_t complex where $t = \max\{d + r', d' + r\}$ [4, Proposition 2.10]. However, if one of the complexes is Cohen-Macaulay, this result could be strengthened. Below we combine this with relevant known results.

Theorem 3.1. *Let Δ and Δ' be two complexes of dimensions $d - 1$ and $d' - 1$, respectively. Then*

- (i) *The join complex $\Delta * \Delta'$ is Cohen-Macaulay if and only if both Δ and Δ' are so.*
- (ii) *If Δ is Cohen-Macaulay and Δ' is $\text{CM}_{r'}$ for some $r' \geq 1$, then $\Delta * \Delta'$ is $\text{CM}_{d+r'}$ (independent of d'). This is sharp, i.e., if Δ' is not $\text{CM}_{r'-1}$, then $\Delta * \Delta'$ is not $\text{CM}_{d+r'-1}$. In particular, a cone on Δ' is $\text{CM}_{r'+1}$.*
- (iii) *If Δ is CM_r and Δ' is $\text{CM}_{r'}$ for some $r, r' \geq 1$, then $\Delta * \Delta'$ is CM_t where $t = \max\{d + r', d' + r\}$. Conversely, if $\Delta * \Delta'$ is CM_t , then Δ is $\text{CM}_{t-d'}$ and Δ' is CM_{t-d} .*

Proof. The statement in (i) is proved by Sava [8] and Fröberg [2]. The assertion (iii) is proved in [4, Theorem 2.10]. We prove (ii) using induction on $d + r' \geq 2$. Let $d + r' = 2$, i.e., $d = 1$ and $r' = 1$. Then $\Delta = \{v\}$ is a singleton. Thus $\text{link}_{\Delta * \Delta'}(v) = \Delta'$, which is CM_1 . For $v \in \Delta'$, $\text{link}_{\Delta * \Delta'}(v) = \Delta * \text{link}_{\Delta'}(v)$, which is Cohen-Macaulay by (i). Thus by Lemma 2.1, $\Delta * \Delta'$ is CM_2 . Now let $d + r' \geq 2$. Let $v \in \Delta$. Then, $\text{link}_{\Delta * \Delta'}(v) = \text{link}_{\Delta}(v) * \Delta'$. But $\text{link}_{\Delta}(v)$ is Cohen-Macaulay of dimension less than $d - 1$, and Δ' is $\text{CM}_{r'}$. Thus by induction hypothesis, $\text{link}_{\Delta * \Delta'}(v)$ is $\text{CM}_{d-1+r'}$. If $v \in \Delta'$, then $\text{link}_{\Delta * \Delta'}(v) = \Delta' * \text{link}_{\Delta'}(v)$. But $\text{link}_{\Delta'}(v)$ is $\text{CM}_{r'-1}$ and hence $\text{link}_{\Delta * \Delta'}(v)$ is again $\text{CM}_{d+r'-1}$. Therefore, $\Delta * \Delta'$ is $\text{CM}_{d+r'}$. To prove that this result is sharp, proceed by induction on $d \geq 1$. Indeed, in this case, for any $v \in \Delta$, $\text{link}_{\Delta}(v)$ has dimension less than $d - 1$ and hence by induction hypothesis, $\text{link}_{\Delta * \Delta'}(v) = \text{link}_{\Delta}(v) * \Delta$ is not $\text{CM}_{d+r'-2}$. Therefore, $\Delta * \Delta'$ is not $\text{CM}_{d+r'-1}$. $\square$

Let $G \sqcup G'$ denote the disjoint union of graphs G and G' . By the fact that $\text{Ind}(G \sqcup G') = \text{Ind}(G) * \text{Ind}(G')$, the counterpart of Theorem 3.1 for graphs will be the following.

Theorem 3.2. *Let G and G' be two graphs on disjoint sets of vertices and of dimensions $d - 1$ and $d' - 1$, respectively. Then*

- (i) *The graph $G \sqcup G'$ is Cohen-Macaulay if and only if both G and G' are so.*
- (ii) *If G is Cohen-Macaulay and G' is $\text{CM}_{r'}$ for some $r' \geq 1$, then $G \sqcup G'$ is $\text{CM}_{d+r'}$. If G' is not $\text{CM}_{r'-1}$, then $G \sqcup G'$ is not $\text{CM}_{d+r'-1}$.*
- (iii) *If G is CM_r and G' is $\text{CM}_{r'}$ for some $r, r' \geq 1$, then $G \sqcup G'$ is CM_t where $t = \max\{d + r', d' + r\}$. Conversely, if $G \sqcup G'$ is CM_t , then G is $\text{CM}_{t-d'}$ and G' is CM_{t-d} .*

4. TWO CHARACTERIZATIONS OF BIPARTITE CM_t GRAPHS

We now restrict to the case of bipartite graphs. Since Cohen-Macaulay bipartite graphs are characterized by Herzog and Hibi [5, Theorem 3.4], and also in

a different version by Cook and Nagel [1, Proposition 4.8], we consider the non-Cohen-Macaulay case.

Theorem 4.1. *Let G be an unmixed bipartite graph of dimensions $d-1$. Let $K_{n,n}$, with $n \geq 2$, be a maximal complete bipartite subgraph of G of minimum dimension. Then G is CM_{d-n+1} but it is not CM_{d-n} .*

Proof. We prove both assertions by induction on $d \geq 2$. If $d = 2$, then $G = K_{2,2}$ which is CM_1 but it is not Cohen-Macaulay. Assume that $d > 2$. We show that for every $v \in G$, $G \setminus N[v]$ is CM_{n-d} and for some $v \in G$ it is not CM_{n-d-1} . Let $(\{x_1, \dots, x_d\}, \{y_1, \dots, y_d\})$ be a pure order of G . Let x_i be a vertex of some maximal bipartite subgraph $K_{m,m}$ with $m \geq n$. Then $G \setminus N[x_i]$ is a disjoint union of $c \geq m-1$ isolated vertices and an unmixed bipartite graph H of dimension $d-c-2$. The graph H is unmixed because $\text{Ind}(G \setminus N[x_i]) = \text{link}_{x_i}(\text{Ind}(G))$, and any link of a pure complex is pure. But $G \setminus N[x_i] = \{x_{i_1}, \dots, x_{i_c}\} \sqcup H$ is unmixed if and only if H is so. Observe that if y_{j_0} is a vertex of a maximal bipartite subgraph of G and $y_{j_0} \in N[x_i]$, then by Lemma 2.4, all y_j vertices of this subgraph belong to $N[x_i]$. Thus if H has no crosses, by Proposition 2.6 it is Cohen-Macaulay. Otherwise, the minimum dimension of maximal complete bipartite subgraphs of H will not be less than the minimum dimension of such subgraphs in G . Hence by the induction hypothesis, H is CM_{d-c-n} and by Theorem 3.2(ii), $G \setminus N[x_i]$ is CM_{n-d} . If x_i does not belong to any maximal bipartite subgraph of G of positive dimension, then $G \setminus N[x_i]$ is a disjoint union of $c \geq 0$ isolated vertices and an unmixed bipartite graph H of dimension $d-c-2$. Hence H is CM_{d-c-n} and by Theorem 3.2(ii), $G \setminus N[x_i]$ is CM_{d-n} . A similar argument reveals that for any $y_i \in G$, the graph $G \setminus N[y_i]$ is CM_{d-n} . Therefore, by Lemma 2.2, G is CM_{d-n+1} . We now proceed with the induction step to show that this result is sharp. Let $d > 2$ and let $K_{n,n}$, $n \geq 2$, be a maximal bipartite subgraph of G of minimum dimension. Take $x_i \in G \setminus K_{n,n}$. First assume that x_i is not adjacent to any vertex in $K_{n,n}$ and consider $G \setminus N[x_i]$. Let $G \setminus N[x_i]$ be the disjoint union of $c \geq 0$ isolated vertices and an unmixed bipartite graph H of dimension $d-c-2$. Then H contains $K_{n,n}$ and hence by induction hypothesis, H is sharp CM_{d-c-n} and $G \setminus N[x_i]$ is sharp CM_{d-n} . Therefore, G cannot be CM_{d-n} . Now assume that $x_i y_j \in G$ for some j with $y_j \in K_{n,n}$. Then by purity of the order, all $y_k \in K_{n,n}$ is adjacent to x_i . But then y_i is not adjacent to any vertex of $K_{n,n}$, because otherwise, $K_{n,n}$ will not be maximal. In this case, consider $G \setminus N[y_i]$ and proceed in a way similar to the previous case. $\square$

As a second characterization of bipartite CM_t graphs, we show that any CM_t graph is obtained from a Cohen-Macaulay graph H by replacing the perfect matching edges of H by complete bipartite graphs. This statement will be more precise in the next theorem. But first we provide a definition and a lemma.

Definition 4.2. Let H be an unmixed bipartite graph with pure order

$$(\{x_1, \dots, x_r\}, \{y_1, \dots, y_r\}).$$

For a fixed i , by replacing the edge $x_i y_i \in H$ with a complete bipartite graph

$$K_{n_i, n_i} = \{x_{i1}, \dots, x_{in_i}\} \times \{y_{i1}, \dots, y_{in_i}\}$$

we mean a bipartite graph H' with vertex set

$$\begin{aligned} &(\{x_1, \dots, x_{i-1}, x_{i1}, \dots, x_{in_i}, x_{i+1}, \dots, x_r\}, \\ &\{y_1, \dots, y_{i-1}, y_{i1}, \dots, y_{in_i}, y_{i+1}, \dots, y_r\}), \end{aligned}$$

preserving all adjacencies, i.e.,

- (i) $x_s y_t \in H'$ for all $s, t \neq i$ if and only if $x_t y_s \in H$,
- (ii) $x_{i_k} y_j \in H'$ for all k if and only if $x_i y_j \in H$,
- (iii) $x_j y_{i_k} \in H'$ for all k if and only if $x_j y_i \in H$.

Lemma 4.3. *Let G be an unmixed bipartite graph with pure order on the vertex set $V(G) = V \cup W$ where $V = \{x_1, \dots, x_d\}$ and $W = \{y_1, \dots, y_d\}$. Let $n_1, \dots, n_d$ be any positive integers. Let $G' = G(n_1, \dots, n_d)$ be the graph obtained by replacing each edge $x_i y_i$ with the complete bipartite graph $K_{n_i, n_i} = \{x_{i1}, \dots, x_{in_i}\} \times \{y_{i1}, \dots, y_{in_i}\}$ for all $i = 1, \dots, d$. Then G' is also unmixed.*

Proof. Observe that

$$V(G') = (\{x_{11}, \dots, x_{1n_1}, \dots, x_{d1}, \dots, x_{dn_d}\}, \{y_{11}, \dots, y_{1n_1}, \dots, y_{d1}, \dots, y_{dn_d}\})$$

is a pure order of G' . In fact, for all i, r , $x_{ir} y_{ir} \in G'$. Also if $x_{ir} y_{js} \in G'$ and $x_{js} y_{kt} \in G'$, then $x_i y_j \in G$ and $x_j y_k \in G$, and hence, $x_i y_k \in G$. Thus by the construction of G' , $x_{ir} y_{kt} \in G'$. $\square$

Theorem 4.4. *Let G be a Cohen-Macaulay bipartite graph with a Macaulay order on the vertex set $V(G) = V \cup W$ where $V = \{x_1, \dots, x_d\}$ and $W = \{y_1, \dots, y_d\}$. Let $n_1, \dots, n_d$ be any positive integers with $n_i \geq 2$ for at least one value of i . Let $G' = G(n_1, \dots, n_d)$ be the graph obtained by replacing each edge $x_i y_i$ with the complete bipartite graph K_{n_i, n_i} for all $i = 1, \dots, d$. Let $n_{i_0} = \min\{n_i > 1 : i = 1, \dots, d\}$, $n = \sum_{i=1}^d n_i$. Then G' is exclusively a $CM_{n-n_{i_0}+1}$ graph. Furthermore, any bipartite CM_t graph is obtained by such a replacement of complete bipartite graphs in a unique bipartite Cohen-Macaulay graph.*

Proof. The first claim follows by Lemma 4.3 and Theorem 4.1. We settle the second claim. Let G be a bipartite CM_t graph with a pure order of vertices. Let $K_{n_1, n_1}, \dots, K_{n_d, n_d}$ be the maximal bipartite subgraphs of G , where $n_i \geq 1$ for all i . Observe that, by maximality, these complete subgraphs of G are disjoint. Choose one edge $x_{i1} y_{i1}$ from each subgraph K_{n_i, n_i} for all $i = 1, \dots, d$. Let H be the induced subgraph of G on the vertex set $(\{x_{11}, \dots, x_{d1}\}, \{y_{11}, \dots, y_{d1}\})$. By Lemma 2.4, H is independent of the choice of particular edge $x_{i1} y_{i1}$ from K_{n_i, n_i} and hence H is unique. Since the ordering of vertices of G is a pure order, its restriction to H is also pure. Thus, H is an unmixed bipartite graph. But by maximality of the complete bipartite subgraphs K_{n_i, n_i} , and the construction of H , it is cross-free. Therefore, by Proposition 2.6, H is Cohen-Macaulay. Now replace any edge $x_{i1} y_{i1}$ in H with K_{n_i, n_i} for all $i = 1, \dots, d$, preserving all other adjacencies. Let H' be the resulting graph. Then by the construction, $G = H'$, as required. $\square$

Remark 4.5. Let H be a bipartite Cohen-Macaulay graph and let $G = H'$ be a bipartite CM_t graph obtained from H by the replacing process described above. Assume that G is not CM_{t-1} and $t \geq 2$. Using the results of this section, the following observations are immediate.

FIGURE 1

First of all, $1 \leq \dim H \leq t - 1$. Because if $\dim H \geq t$ and we replace just one edge with $K_{n,n}$ where $n \geq 2$, then G is strictly CM_r with $r \geq t + 1$. On the other hand, if $\dim H = 0$, then G is CM_1 .

If $\dim H = t - 1$, then only one edge can be replaced with $K_{n,n}$ where $n \geq 2$. Because replacing at least two edges with $K_{n,n}$'s, $n \geq 2$, G will be strictly CM_r where $r \geq t + 1$.

If $\dim H = t - 1$, for replacing just one edge with $K_{n,n}$, n can be arbitrary and hence G will be of dimension $n + t - 2$.

If $\dim H \leq t - 2$, the number of replacements should be at least 2. Again because if we replace one edge with $K_{n,n}$, $n \geq 2$, then G would be CM_r for $r \leq t - 1$.

When $\dim H \leq t - 2$, the maximum number of replacements of edges with $K_{n,n}$, $n \geq 2$, is at most $t - \dim H$ which may occur replacing $K_{2,2}$'s.

For $\dim H \leq t - 2$, the maximum size of $K_{n,n}$ to be used for replacements is also $n = t - \dim H$ which may occur when we have two replacements.

Using these remarks we may easily distinguish all bipartite CM_t graphs for $t = 2, 3, 4$.

Example 4.6. Bipartite CM_2 graphs which are not Buchsbaum. Using the notation of Remark 4.5 we have $\dim H = 1$. There are just two nonisomorphic bipartite Cohen-Macaulay graphs of dimension one. By the replacing process, they produce two types of bipartite CM_2 graphs which are not Buchsbaum. They are of arbitrary dimensions. More precisely, one such graph is the disjoint union of an edge $x_1 y_1$ with $K_{n_2, n_2} = \{x_{21}, \dots, x_{2n_2}\} \times \{y_{21}, \dots, y_{2n_2}\}$, $n_2 \geq 2$, and the other one consists of the first graph together with the edges $x_1 y_{2i}$ for all $i = 1, \dots, n_2$. The second graph with $n_2 = 3$ could be depicted in Figure 1.

Example 4.7. Bipartite CM_3 graphs which are not CM_2 . For these graphs $\dim H = 1, 2$.

If $\dim H = 1$, by Example 4.6, there are just two bipartite CM_3 graphs obtained by replacing two edges of a perfect matching by $K_{2,2}$'s. In this case, $\dim G = 3$ (see Figure 2 and Figure 3).

If $\dim H = 2$, then there are 4 non-isomorphic bipartite Cohen-Macaulay graphs of dimension 2. By replacing one perfect matching edge with $K_{n,n}$ of arbitrary size in each Cohen-Macaulay graph, they produce 7 types of bipartite CM_3 graphs which are not CM_2 . Note that depending on the choice of the edge to be replaced in each case, we may get nonisomorphic bipartite graphs. In this case $\dim G = n + 1$.

Example 4.8. Bipartite CM_4 graphs which are not CM_3 . For these graphs $\dim H = 1, 2, 3$.

FIGURE 2

FIGURE 3

If $\dim H = 1$, there are two bipartite CM_4 graphs obtained by replacing two edges of a perfect matching by $K_{3,3}$'s. And, similarly, there are two others obtained by replacing one edge with $K_{2,2}$ and another edge with $K_{3,3}$. In both cases, $\dim G = 5$.

If $\dim H = 2$, then while there are 4 nonisomorphic bipartite Cohen-Macaulay graphs of dimension 2, by replacing two perfect matching edges with $K_{2,2}$'s in each Cohen-Macaulay graph, they produce 7 bipartite CM_4 graphs which are not CM_3 . They all have dimension 4.

If $\dim H = 3$, then there are 10 nonisomorphic bipartite Cohen-Macaulay graphs of dimension 3. Replacing one perfect matching edge with $K_{n,n}$, $n \geq 2$, in each Cohen-Macaulay graph, they produce 25 bipartite CM_4 graphs which are not CM_3 . They all have dimension $n + 2$. Out of all 36 bipartite CM_4 graphs, 21 graphs are connected.

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DEPARTMENT OF MATHEMATICS, K. N. TOOSI UNIVERSITY OF TECHNOLOGY, TEHRAN, IRAN
E-mail address: `haghighi@kntu.ac.ir`

SCHOOL OF MATHEMATICS, STATISTICS & COMPUTER SCIENCE, UNIVERSITY OF TEHRAN, TEHRAN IRAN – AND – SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O. BOX 19395-5746, TEHRAN, IRAN

E-mail address: `yassemi@ipm.ir`

E-mail address: `yassemi@ut.ac.ir`

SCHOOL OF MATHEMATICS, STATISTICS & COMPUTER SCIENCE, UNIVERSITY OF TEHRAN, TEHRAN IRAN

E-mail address: `rahimzn@ut.ac.ir`