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ON MEAN ERGODIC CONVERGENCE IN THE CALKIN ALGEBRAS

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ABSTRACT. In this paper we give a geometric characterization of mean ergodic convergence in the Calkin algebras for Banach spaces that have the bounded compact approximation property.

1. INTRODUCTION

Let X be a real or complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. Suppose that $T \in B(X)$ and consider the sequence

$$M_n(T) := \frac{I + T + \ldots + T^n}{n+1}, \quad n \geq 1.$$

In [3], Dunford considered the norm convergence of $(M_n(T))_n$ and established the following characterizations.

Theorem 1.1. *Suppose that X is a complex Banach space and that $T \in B(X)$ satisfies $\frac{\|T^n\|}{n} \to 0$. Then the following conditions are equivalent.*

1. *$(M_n(T))_n$ converges in norm to an element in $B(X)$.*
2. *1 is a simple pole of the resolvent of T or 1 is in the resolvent set of T.*
3. *$(I - T)^2$ has closed range.*

It was then discovered by Lin in [6] that $I - T$ having closed range is also an equivalent condition. Moreover, Lin's argument worked also for real Banach spaces. This result was later improved by Mbekhta and Zemánek in [9], in which they showed that $(I - T)^m$ having closed range, where $m \geq 1$, are also equivalent conditions. More precisely,

Theorem 1.2. *Let $m \geq 1$. Suppose that X is a real or complex Banach space and that $T \in B(X)$ satisfies $\frac{\|T^n\|}{n} \to 0$. Then the sequence $(M_n(T))_n$ converges in norm to an element in $B(X)$ if and only if $(I - T)^m$ has closed range.*

Let $K(X)$ be the closed ideal of compact operators in $B(X)$. If $T \in B(X)$, then its image in the Calkin algebra $B(X)/K(X)$ is denoted by $\dot{T}$. By Dunford's

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Theorem 1.1 or by an analogous version for Banach algebras (without condition (3)), when X is a complex Banach space and $\frac{\|\dot{T}^n\|}{n} \to 0$, the convergence of $(M_n(\dot{T}))_n$ in the Calkin algebra is equivalent to 1 being a simple pole of the resolvent of $\dot{T}$ or being in the resolvent set of $\dot{T}$. But even if we are given that the limit $\dot{P} \in B(X)/K(X)$ of $(M_n(\dot{T}))_n$ exists, there is no obvious geometric interpretation of $\dot{P}$. In the context of Theorems 1.1 and 1.2, if the limit of $(M_n(T))_n$ exists, then it is a projection onto $\ker(I-T)$. In the context of the Calkin algebra, the limit $\dot{P}$ is still an idempotent in $B(X)/K(X)$; hence by making a compact perturbation, we can assume that P is an idempotent in $B(X)$ (see Lemma 2.6 below).

A natural question to ask is: what is the range of P? Although the range of P is not unique (since P is only unique up to a compact perturbation), it can be thought of as an analog of $\ker(I-T)$ in the Calkin algebra setting. If $T_0 \in B(X)$, then $\ker T_0$ is the maximal subspace of X on which $T_0 = 0$. This suggests that the analog of $\ker T_0$ in the Calkin algebra setting is the maximal subspace of X on which T_0 is compact. But the maximal subspace does not exist unless it is the whole space X. Thus, we introduce the following concept.

Let X be a Banach space and let (P) be a property that a subspace M of X may or may not have. We say that a subspace $M \subset X$ is an *essentially maximal* subspace of X satisfying (P) if it has (P) and if every subspace $M_0 \supset M$ having property (P) satisfies $\dim M_0/M < \infty$.

Then the analog of $\ker T_0$ in the Calkin algebra setting is an essentially maximal subspace of X on which T_0 is compact. It turns that if such an analog for $I-T$ exists, then it is already sufficient for the convergence of $(M_n(\dot{T}))_n$ in the Calkin algebra (at least for a large class of Banach spaces), which is the main result of this paper.

Before stating this theorem, we recall that a Banach space Z has the *bounded compact approximation property* (BCAP) if there is a uniformly bounded net $(S_\alpha)_{\alpha\in\Lambda}$ in $K(Z)$ converging strongly to the identity operator $I \in B(Z)$. It is always possible to choose Λ to be the set of all finite dimensional subspaces of Z directed by inclusion. If the net $(S_\alpha)_{\alpha\in\Lambda}$ can be chosen so that $\sup_{\alpha\in\Lambda} \|S_\alpha\| \le \lambda$, then we say that Z has the λ-BCAP. It is known that if a reflexive space has the BCAP, then the space has the 1-BCAP. For $T \in B(X)$, the essential norm $\|T\|_e$ is the norm of $\dot{T}$ in $B(X)/K(X)$.

Theorem 1.3. *Let $m \ge 1$. Suppose that X is a real or complex Banach space having the bounded compact approximation property. If $T \in B(X)$ satisfies $\frac{\|T^n\|_e}{n} \to 0$, then the following conditions are equivalent.*

1. *The sequence $(M_n(\dot{T}))_n$ converges in norm to an element in $B(X)/K(X)$.*
2. *There is an essentially maximal subspace of X on which $(I-T)^m$ is compact.*

The idea of the proof is to reduce Theorem 1.3 to Theorem 1.2 by constructing a Banach space $\widehat{X}$ and an embedding $f : B(X)/K(X) \to B(\widehat{X})$ so that if $T \in B(X)$ and there is an essentially maximal subspace M of X on which T is compact, then $f(\dot{T})$ has closed range, and then applying Theorem 1.2 to $f(\dot{T})$. The BCAP of X is used to show that f is an embedding but is not used in the construction of $\widehat{X}$ and f. The construction of f is based on the Calkin representation [1, Theorem 5.5].

2. The Calkin representation for Banach spaces

In this section, X is a fixed infinite dimensional Banach space. Let Λ_0 be the set of all finite dimensional subspaces of X directed by inclusion $\subset$. Then $\{\{\alpha \in \Lambda_0 : \alpha \supset \alpha_0\} : \alpha_0 \in \Lambda_0\}$ is a filter base on Λ_0, so it is contained in an ultrafilter U on Λ_0.

Let Y be an arbitary infinite dimensional Banach space and let $(Y^*)^U$ be the ultrapower (see e.g. [2, Chapter 8]) of Y^* with respect to U. (The ultrafilter U and the directed set Λ_0 do not depend on Y.) If $(y^*_\alpha)_{\alpha\in\Lambda_0}$ is a bounded net in Y^*, then its image in $(Y^*)^U$ is denoted by $(y^*_\alpha)_{\alpha,U}$. Consider the (complemented) subspace

$$\widehat{Y} := \left\{(y^*_\alpha)_{\alpha,U} \in (Y^*)^U : w^*\text{-}\lim_{\alpha,U} y^*_\alpha = 0\right\}$$

of $(Y^*)^U$. Here $w^*\text{-}\lim_{\alpha,U} y^*_\alpha$ is the w^*-limit of $(y^*_\alpha)_{\alpha\in\Lambda_0}$ through U, which exists by the Banach-Alaoglu Theorem.

Whenever $T \in B(X,Y)$, we can define an operator $\widehat{T} \in B(\widehat{Y},\widehat{X})$ by sending $(y^*_\alpha)_{\alpha,U}$ to $(T^*y^*_\alpha)_{\alpha,U}$. Note that if $K \in K(X,Y)$, then $\widehat{K} = 0$, where $K(X,Y)$ denotes the space of all compact operators in $B(X,Y)$.

Theorem 2.1. *Suppose that X has the λ-BCAP. Then the operator*

$$f : B(X)/K(X) \to B(\widehat{X}),\ \dot{T} \mapsto \widehat{T},$$

is a conjugate linear norm one $(\lambda+1)$-embedding into $B(\widehat{X})$ satisfying

$$f(\dot{I}) = I \text{ and } f(\dot{T}_1\dot{T}_2) = f(\dot{T}_2)f(\dot{T}_1), \qquad T_1, T_2 \in B(X).$$

Proof. It is easy to verify that f is a conjugate linear map, $f(\dot{I}) = I$, and $f(\dot{T}_1\dot{T}_2) = f(\dot{T}_2)f(\dot{T}_1)$ for T_1, $T_2 \in B(X)$. If $T \in B(X)$, then clearly $\|f(\dot{T})\| \le \|T\|$, and thus we also have $\|f(\dot{T})\| \le \|T\|_e$. Hence $\|f\| \le 1$. It remains to show that f is a $(\lambda+1)$-embedding (i.e., $\inf_{\|T\|_e>1} \|f(\dot{T})\| \ge (\lambda+1)^{-1}$).

To do this, let $T \in B(X)$ satisfy $\|T\|_e > 1$. Since X has the λ-BCAP, we can find a net of operators $(S_\alpha)_{\alpha\in\Lambda_0} \subset K(X)$ converging strongly to I such that $\sup_{\alpha\in\Lambda_0} \|S_\alpha\| \le \lambda$. Then $\|T^*(I-S_\alpha)^*\| = \|(I-S_\alpha)T\| \ge \|T\|_e > 1,\ \alpha \in \Lambda_0$. Thus, there exists $(x^*_\alpha)_{\alpha\in\Lambda_0} \subset X^*$ such that $\|x^*_\alpha\| = 1$ and $\|T^*(I-S_\alpha)^*x^*_\alpha\| > 1$ for $\alpha \in \Lambda_0$.

Note that for every $x \in X$,

$$\limsup_{\alpha\in\Lambda_0} |\langle (I-S_\alpha)^*x^*_\alpha, x\rangle| = \limsup_{\alpha\in\Lambda_0} |\langle x^*_\alpha, (I-S_\alpha)x\rangle| \le \limsup_{\alpha\in\Lambda_0} \|(I-S_\alpha)x\| = 0,$$

and so the net $((I-S_\alpha)^*x^*_\alpha)_{\alpha\in\Lambda_0}$ converges in the w^*-topology to 0. By the construction of U, this implies that

$$w^*\text{-}\lim_{\alpha,U}(I-S_\alpha)^*x^*_\alpha = 0.$$

Therefore, due to the definition $f(\dot{T}) = \widehat{T}$, we obtain

$$\begin{aligned}(1+\lambda)\|f(\dot{T})\| \ge \|f(\dot{T})\| \lim_{\alpha,U} \|(I-S_\alpha)^*x^*_\alpha\| &= \|f(\dot{T})\| \|((I-S_\alpha)^*x^*_\alpha)_{\alpha,U}\| \\ &\ge \|f(\dot{T})((I-S_\alpha)^*x^*_\alpha)_{\alpha,U}\| \\ &= \lim_{\alpha,U} \|T^*(I-S_\alpha)^*x^*_\alpha\| \ge 1.\end{aligned}$$

It follows that $\|f(\dot{T})\| \geq (1+\lambda)^{-1}$ whenever $\|T\|_e > 1$. $\square$

Remark 1. We do not know whether Theorem 2.1 is true without the hypothesis that X has the BCAP.

Remark 2. The embedding in Theorem 2.1 is an isometry if the approximating net can be chosen so that $\|I - S_\alpha\| = 1$ for every α. This is the case if, for example, the space X has a 1-unconditional basis. However, we do not know whether the embedding is an isometry if $X = L_p(0,1)$ with $p \neq 2$.

If N is a subset of Y^*, then we can define a subset N' of $\widehat{Y}$ by

$$N' := \left\{ (y_\alpha^*)_{\alpha,U} \in \widehat{Y} : \lim_{\alpha,U} d(y_\alpha^*, N) = 0 \right\},$$

where

$$d(y_\alpha^*, N) := \inf_{z^* \in N} \|y_\alpha^* - z^*\|.$$

Lemma 2.2. *If N is a w^*-closed subspace of Y^*, then for every $(y_\alpha^*)_{\alpha,U} \in \widehat{Y}$,*

$$d((y_\alpha^*)_{\alpha,U}, N') \leq 2 \lim_{\alpha,U} d(y_\alpha^*, N).$$

Proof. Let $a = \lim_{\alpha,U} d(y_\alpha^*, N)$. Let $\delta > 0$. Then

$$A := \{\alpha \in \Lambda : d(y_\alpha^*, N) < a + \delta\} \in U.$$

Whenever $\alpha \in A$, $\|y_\alpha^* - z_\alpha^*\| < a + \delta$ for some $z_\alpha^* \in N$. If we take $z_\alpha^* = 0$ for $\alpha \notin A$, then, since $\sup_{\alpha \in \Lambda} \|y_\alpha^*\| < \infty$,

$$\sup_{\alpha \in \Lambda} \|z_\alpha^*\| = \sup_{\alpha \in A} \|z_\alpha^*\| \leq (a+\delta) + \sup_{\alpha \in A} \|y_\alpha^*\| < \infty.$$

As a consequence, $\left(z_\alpha^* - w^*\text{-}\lim_{\beta,U} z_\beta^* \right)_{\alpha,U} \in N'$, since N is w^*-closed. Therefore,

$$\begin{aligned} d\left((y_\alpha^*)_{\alpha,U}, N'\right) &\leq d\left((y_\alpha^*)_{\alpha,U}, \left(z_\alpha^* - w^*\text{-}\lim_{\beta,U} z_\beta^* \right)_{\alpha,U} \right) \\ &= \lim_{\alpha,U} \left\| y_\alpha^* - z_\alpha^* + w^*\text{-}\lim_{\beta,U} z_\beta^* \right\| \\ &\leq \lim_{\alpha,U} \|y_\alpha^* - z_\alpha^*\| + \left\| w^*\text{-}\lim_{\beta,U} z_\beta^* \right\| \\ &\leq (a+\delta) + \left\| w^*\text{-}\lim_{\beta,U} (z_\beta^* - y_\beta^*) \right\| \\ &\leq (a+\delta) + \lim_{\beta,U} \|z_\beta^* - y_\beta^*\| \leq 2(a+\delta). \end{aligned}$$

But δ can be arbitarily close to 0, so $d\left((y_\alpha^*)_{\alpha,U}, N'\right) \leq 2a = 2 \lim_{\alpha,U} d(y_\alpha^*, N)$. $\square$

Proposition 2.3. *If X and Y are infinite dimensional Banach spaces and if $T \in B(X,Y)$ has closed range, then $\widehat{T} \in B(\widehat{Y}, \widehat{X})$ also has closed range.*

Proof. The operator T has closed range, so T^* also has closed range. Let $c = \inf\{\|T^*y^*\| : y^* \in Y^*,\ d(y^*, \ker T^*) = 1\} > 0$. Then by Lemma 2.2, for every $(y^*_\alpha)_{\alpha,U} \in \widehat{Y}$,

$$\|\widehat{T}(y^*_\alpha)_{\alpha,U}\| = \lim_{\alpha,U} \|T^*y^*_\alpha\| \geq c \lim_{\alpha,U} d(y^*_\alpha, \ker T^*) \geq \frac{c}{2} d((y^*_\alpha)_{\alpha,U}, (\ker T^*)').$$

But obviously $(\ker T^*)' \subset \ker \widehat{T}$, and so

$$\|\widehat{T}(y^*_\alpha)_{\alpha,U}\| \geq \frac{c}{2} d((y^*_\alpha)_{\alpha,U}, \ker \widehat{T}), \quad (y^*_\alpha)_{\alpha,U} \in \widehat{Y}.$$

Hence $\widehat{T}$ has closed range. □

Lemma 2.4. *Suppose that $X \subset Y$ and that $T \in B(X)$. Let $T_0 \in B(X,Y)$, $x \mapsto Tx$. Then $\widehat{T_0}\widehat{Y} = \widehat{T}\widehat{X}$.*

Proof. If $(y^*_\alpha)_{\alpha,U} \in \widehat{Y}$, then for each $\alpha \in \Lambda$, we have $T_0^* y^*_\alpha = T^*(y^*_{\alpha|X})$ and $(y^*_{\alpha|X})_{\alpha,U} \in \widehat{X}$. Thus $\widehat{T_0}(y^*_\alpha)_{\alpha,U} = (T_0^* y^*_\alpha)_{\alpha,U} = (T^*(y^*_{\alpha|X}))_{\alpha,U} = \widehat{T}(y^*_{\alpha|X})_{\alpha,U} \in \widehat{T}\widehat{X}$. Hence $\widehat{T_0}\widehat{Y} \subset \widehat{T}\widehat{X}$.

Conversely, if $(x^*_\alpha)_{\alpha,U} \in \widehat{X}$, then we can extend each x^*_α to an element $y^*_\alpha \in Y^*$ such that $\|y^*_\alpha\| = \|x^*_\alpha\|$. Thus we have $\left(y^*_\alpha - w^*\text{-}\lim_{\beta,U} y^*_\beta\right)_{\alpha,U} \in \widehat{Y}$. Note that

$$T_0^*\left(w^*\text{-}\lim_{\beta,U} y^*_\beta\right) = w^*\text{-}\lim_{\beta,U} T_0^* y^*_\beta = w^*\text{-}\lim_{\beta,U} T^* x^*_\beta = T^*\left(w^*\text{-}\lim_{\beta,U} x^*_\beta\right) = 0.$$

This implies that

$$\begin{aligned}\widehat{T}(x^*_\alpha)_{\alpha,U} = (T^*x^*_\alpha)_{\alpha,U} &= (T_0^* y^*_\alpha)_{\alpha,U} \\ &= \left(T_0^*\left(y^*_\alpha - w^*\text{-}\lim_{\beta,U} y^*_\beta\right)\right)_{\alpha,U} \\ &= \widehat{T_0}\left(y^*_\alpha - w^*\text{-}\lim_{\beta,U} y^*_\beta\right)_{\alpha,U} \in \widehat{T_0}\widehat{Y}.\end{aligned}$$

Therefore $\widehat{T}\widehat{X} \subset \widehat{T_0}\widehat{Y}$. □

Proposition 2.5. *Suppose that $T \in B(X)$ and that there exists an essentially maximal subspace M of X on which T is compact. Then $\widehat{T}$ has closed range.*

Proof. Without loss of generality, we may assume that X is a subspace of $Y = \ell_\infty(J)$ for some set J. Define $T_0 \in B(X, \ell_\infty(J))$, $x \mapsto Tx$. Then by assumption, there is an essentially maximal subspace M of X on which T_0 is compact. By [7, Theorem 3.3], there exists $K \in K(X, \ell_\infty(J))$ such that $K_{|M} = T_{0|M}$.

We now show that $T_0 - K \in B(X, \ell_\infty(J))$ has closed range. Since $M \subset \ker(T_0 - K)$ and M is an essentially maximal subspace of X on which $T_0 - K$ is compact, $\ker(T_0 - K)$ is an essentially maximal subspace of X on which $T_0 - K$ is compact.

Let π be the quotient map from X onto $X/\ker(T_0 - K)$. Define the (one-to-one) operator $R : X/\ker(T_0 - K) \to \ell_\infty(J)$, $\pi x \mapsto (T_0 - K)x$. If R does not have closed range, then by [8, Proposition 2.c.4], R is compact on an infinite dimensional subspace V of $X/\ker(T_0 - K)$. Hence, $T_0 - K$ is compact on $\pi^{-1}V$ and so by the essential maximality of $\ker(T_0 - K)$, we have $\dim \pi^{-1}V/\ker(T_0 - K) < \infty$. Thus, $V = \pi^{-1}V/\ker(T_0 - K)$ is finite dimensional, which contradicts the definition of V.

Therefore, R has closed range and so $T_0 - K$ also has closed range. By Proposition 2.3, $\widehat{T_0 - K}$ has closed range. But $\widehat{K} = 0$, so $\widehat{T}_0$ has closed range and by Lemma 2.4, $\widehat{T}$ has closed range. $\square$

Lemma 2.6. *Suppose that $P \in B(X)$ and that $\dot{P}$ is an idempotent in $B(X)/K(X)$. Then P is the sum of an idempotent in $B(X)$ and a compact operator on X.*

Proof. We first treat the case where the scalar field is $\mathbb{C}$. From Fredholm theory (see e.g. [5, Chapters XI and XVII]), we know that since $\sigma(\dot{P}) \subset \{0,1\}$, the only possible cluster points of $\sigma(P)$ are 0 and 1. Thus, there exists $0 < r < 1$ such that $\{z \in \mathbb{C} : |z-1| = r\} \cap \sigma(P) = \emptyset$. Then $\dot{P} = \frac{1}{2\pi i}\oint_{|z-1|=r}(z\dot{I} - \dot{P})^{-1}\,dz$ and so $P - \frac{1}{2\pi i}\oint_{|z-1|=r}(zI - P)^{-1}\,dz \in K(X)$. But $\frac{1}{2\pi i}\oint_{|z-1|=r}(zI-P)^{-1}\,dz$ is an idempotent in $B(X)$ (see e.g. [10, Theorem 2.7]). This completes the proof in the complex case.

If X is a real Banach space, then let X_C and P_C be the complexifications (see [4, page 266]) of X and P, respectively. Thus, $\dot{P}_C$ is an idempotent in $B(X_C)/K(X_C)$. Since the only possible cluster points of $\sigma(P_C)$ are 0 and 1, there exists a closed rectangle R in the complex plane symmetric with respect to the real axis such that 1 is in the interior of R, 0 is in the exterior of R, and $\sigma(P_C)$ is disjoint from the boundary ∂R of R. By [4, Lemma 3.4], the idempotent $\frac{1}{2\pi i}\oint_{\partial R}(zI - P_C)^{-1}\,dz$ in $B(X_C)$ is induced by an idempotent P_0 in $B(X)$. Since $P_C - \frac{1}{2\pi i}\oint_{\partial R}(zI - P_C)^{-1}\,dz \in K(X_C)$, we see that $P - P_0 \in K(X)$. $\square$

Proof of Theorem 1.3. "(1)⇒(2)": Let $\dot{P} := \lim_{n\to\infty} \dfrac{\dot{I} + \dot{T} + \ldots + \dot{T}^n}{n+1}$.
Since $\lim_{n\to\infty} \dfrac{\|\dot{T}^n\|}{n} = 0$,

$$(\dot{I} - \dot{T})\dot{P} = \lim_{n\to\infty}(\dot{I} - \dot{T})\frac{\dot{I} + \dot{T} + \ldots + \dot{T}^n}{n+1} = \lim_{n\to\infty}\frac{\dot{I} - \dot{T}^{n+1}}{n+1} = 0. \tag{2.1}$$

Thus $\dot{T}\dot{P} = \dot{P}$, and so

$$\dot{P}^2 = \lim_{n\to\infty}\frac{\dot{P} + \dot{T}\dot{P} + \ldots + \dot{T}^n\dot{P}}{n+1} = \lim_{n\to\infty}\frac{(n+1)\dot{P}}{n+1} = \dot{P}.$$

Hence $\dot{P}$ is an idempotent in $B(X)/K(X)$. By Lemma 2.6, there exists an idempotent P_0 in $B(X)$ such that $P - P_0 \in K(X)$. Replacing P with P_0, we can assume without loss of generality that P is an idempotent in $B(X)$. Equation (2.1) also implies that $(I-T)P \in K(X)$, which means that $I - T$ is compact on PX. Hence $(I-T)^m$ is compact on PX.

We now show that PX is an essentially maximal subspace of X on which $(I-T)^m$ is compact. Suppose that $(I-T)^m$ is compact on a subspace M_0 of X containing PX. Let

$$f_n(z) := \frac{n + (n-1)z + (n-2)z^2 + \ldots + z^{n-1}}{n+1}, \quad z \in \mathbb{C},\, n \geq 1.$$

Note that $\dot{I} - \frac{\dot{I}+\dot{T}+\ldots+\dot{T}^n}{n+1} = (\dot{I} - \dot{T})f_n(\dot{T})$. Therefore,

$$\dot{I} - \dot{P} = (\dot{I} - \dot{P})^m = \lim_{n\to\infty} f_n(\dot{T})^m(\dot{I} - \dot{T})^m,$$

and so

$$\lim_{n\to\infty} \|(I-P)-(f_n(T)^m(I-T)^m+K_n)\|=0,$$

for some $K_1, K_2, \ldots \in K(X)$.

Since $(I-T)^m$ is compact on M_0, the operator $f_n(T)^m(I-T)^m$ is compact on M_0 and so is $f_n(T)^m(I-T)^m+K_n$ on M_0. Thus $(I-P)_{|M_0}$ is the norm limit of a sequence in $K(M_0, X)$, and so $I-P$ is compact on M_0. Since $PX \subset M_0$, we have that $(I-P)M_0 \subset M_0$. Therefore, $(I-P)_{|(I-P)M_0} = I_{|(I-P)M_0}$ is compact, and so $(I-P)M_0$ is finite dimensional. In other words, $\dim M_0/PX < \infty$.

"(2)⇒(1)": By Proposition 2.5, $\widehat{(I-T)}^m = (I-\widehat{T})^m$ has closed range. Since by assumption $\lim_{n\to\infty} \frac{\|T^n\|_e}{n} = 0$, $\lim_{n\to\infty} \frac{\|\widehat{T}^n\|}{n} = \lim_{n\to\infty} \frac{\|\widehat{T^n}\|}{n} = 0$. By Mbekhta-Zemánek's Theorem 1.2, the sequence $(M_n(\widehat{T}))_n$ converges in norm to an element in $B(\widehat{X})$. By Theorem 2.1, the result follows. □

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References

[1] J. W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. (2) **42** (1941), 839–873. MR0005790 (3,208c)

[2] Joe Diestel, Hans Jarchow, and Andrew Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995. MR1342297 (96i:46001)

[3] Nelson Dunford, *Spectral theory. I. Convergence to projections*, Trans. Amer. Math. Soc. **54** (1943), 185–217. MR0008642 (5,39c)

[4] I. S. Èdel′šteĭn and P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces*, Studia Math. **56** (1976), no. 3, 263–276. MR0425585 (54 #13539)

[5] Israel Gohberg, Seymour Goldberg, and Marinus A. Kaashoek, *Classes of linear operators. Vol. I*, Operator Theory: Advances and Applications, vol. 49, Birkhäuser Verlag, Basel, 1990. MR1130394 (93d:47002)

[6] Michael Lin, *On the uniform ergodic theorem*, Proc. Amer. Math. Soc. **43** (1974), 337–340. MR0417821 (54 #5869)

[7] Joram Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. No. **48** (1964), 112 pp. MR0179580 (31 #3828)

[8] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces. I*, Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92, Springer-Verlag, Berlin, 1977. MR0500056 (58 #17766)

[9] Mostafa Mbekhta and Jaroslav Zemánek, *Sur le théorème ergodique uniforme et le spectre* (French, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 12, 1155–1158. MR1257230 (95b:47010)

[10] Heydar Radjavi and Peter Rosenthal, *Invariant subspaces*, 2nd ed., Dover Publications Inc., Mineola, NY, 2003. MR2003221 (2004e:47010)

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