GROUPS OF UNITARY COMPOSITION OPERATORS ON HARDY-SMIRNOV SPACES

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ABSTRACT. Let Ω be an open simply connected proper subset of the complex plane. We identify, up to isomorphism, which groups are possible for the group of unitary composition operators of a Hardy-Smirnov space defined on Ω . We also study the relationship between the geometry of Ω and the corresponding group.

1. INTRODUCTION

Let Ω be an open simply connected proper subset of the complex plane. The Hardy-Smirnov space $H^2(\Omega)$ is the Hilbert space of functions F analytic on Ω such that the integrals of $|F|^2$ over the images of the circles $|z| = r$, $0 < r < 1$, under a Riemann map from the unit disc $\mathbb{D}$ onto Ω are uniformly bounded; see [3, Chapter 10] or [7, p. 63]. A different choice of Riemann map onto Ω results in the same space of functions. The classical Hardy space H^2 of analytic functions on the unit disc corresponds to the choice $\Omega = \mathbb{D}$.

If $f \in H^2(\Omega)$ and ϕ is an analytic self map of Ω , then the composition operator induced by ϕ on $H^2(\Omega)$, denoted by C_ϕ , is the linear operator defined by

$$C_\phi(f) = f \circ \phi.$$

Such operators on $H^2(\Omega)$ are studied in [7]. Also, composition operators on a Hardy space of a half-plane are studied in [5, 6]. It is elementary that the collection of all unitary operators on a Hilbert space forms a group with group operation the composition of operators. It is not difficult to show that the set of unitary composition operators on $H^2(\Omega)$ also forms a group, which we denote by $\mathcal{U}_\Omega$ (see Lemma 2.3).

In the case that $\Omega = \mathbb{D}$, it is known that unitary composition operators on H^2 are precisely those induced by rotations of $\mathbb{D}$, i.e. by the maps $\phi_\lambda(z) = \lambda z$, where $|\lambda| = 1$; see, for example, [1, Theorem 6]. Moreover since $C_{\phi_\lambda} C_{\phi_\eta} = C_{\phi_{\lambda\eta}}$, $\mathcal{U}_\mathbb{D}$ is isomorphic to the unit circle with ordinary multiplication of complex numbers.

Our goal in this paper is to show how the geometry of Ω determines $\mathcal{U}_\Omega$ and to identify, up to isomorphism, what groups are possible. In Section 4 it is shown that if $\mathcal{U}_\Omega$ is of finite order n , then Ω has n -fold symmetry and $\mathcal{U}_\Omega$ is isomorphic to $\mathbb{Z}_n$. The case that $\mathcal{U}_\Omega$ is an infinite group is studied in Section 5. In Theorem 5.10

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we show that when $\mathcal{U}_\Omega$ is infinite, it is isomorphic to one of just five groups. Our results about the geometry of Ω leading to these groups are summarized in Theorem 5.11. In particular, we show that $\mathcal{U}_\Omega$ is isomorphic to the unit circle with usual multiplication if and only if Ω is a disc.

2. BACKGROUND MATERIAL

Throughout this paper Ω will represent an open simply connected proper subset of the complex plane. Let γ be a Riemann map that takes $\mathbb{D}$ onto Ω . For $0 < r < 1$ let Γ_r be the curve in Ω defined by $\Gamma_r = \gamma(\{|z| = r\})$. The set of functions analytic on Ω for which

$$\sup_{0 < r < 1} \int_{\Gamma_r} |f(w)|^2 |dw| < \infty$$

is a Hardy-Smirnov space on Ω . We denote this space by $H^2(\Omega)$ and refer to it simply as a Hardy space on Ω . The functional $\|\cdot\|_\Omega$ defined on $H^2(\Omega)$ by $\|f\|_\Omega = \left(\sup_{0 < r < 1} (1/2\pi) \int_{\Gamma_r} |f(w)|^2 |dw| \right)^{1/2}$ is a norm on $H^2(\Omega)$ (see [7, p. 63]).

As mentioned above, $H^2(\mathbb{D})$ is the classical Hardy space H^2 . For an arbitrary Ω it turns out that $H^2(\Omega)$ is isometrically isomorphic to H^2 .

Theorem A. *Suppose f is holomorphic on Ω . Then $f \in H^2(\Omega)$ if and only if $(f \circ \gamma)(\gamma')^{1/2} \in H^2$. The map V given by $V(f) = (f \circ \gamma)(\gamma')^{1/2}$ is a linear isometry from $H^2(\Omega)$ onto H^2 .*

For a proof see [7, p. 63] or [3, p. 169]. Using V , we can define a function $\langle \cdot, \cdot \rangle_\Omega : H^2(\Omega) \times H^2(\Omega) \rightarrow \mathbb{C}$ by

$$\langle f, g \rangle_\Omega = \langle V(f), V(g) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in H^2 . Since V is an isometry it follows that $\langle f, f \rangle_\Omega = \|V(f)\|_{H^2}^2 = \|f\|_\Omega^2$. Thus $H^2(\Omega)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_\Omega$.

Suppose C_ϕ is a composition operator on $H^2(\Omega)$. If $g \in H^2$ and $z \in \mathbb{D}$, then

$$VC_\phi V^{-1}(g)(z) = \left(\frac{\gamma'(z)}{\gamma'(\phi(z))} \right)^{1/2} g(\phi(z)),$$

where $\phi = \gamma^{-1} \circ \phi \circ \gamma$. Let $\psi(z) = (\gamma'(z)/\gamma'(\phi(z)))^{1/2}$. Then $VC_\phi V^{-1}(g) = C_{\psi, \phi}(g)$, where $C_{\psi, \phi}(g) = \psi \cdot g \circ \phi$. Such an operator is called a weighted composition operator. Moreover, C_ϕ on $H^2(\Omega)$ is isometrically similar to the weighted composition operator $C_{\psi, \phi}$ on H^2 (see [7, p. 66]).

Lemma 2.1. *If C_ϕ is invertible, then ϕ is an automorphism of Ω and $C_\phi^{-1} = C_{\phi^{-1}}$.*

Proof. Clearly, C_ϕ is invertible if and only if $C_{\psi, \phi} = VC_\phi V^{-1}$ is invertible on H^2 , where $\phi = \gamma^{-1} \circ \phi \circ \gamma$ and $\psi(z) = (\gamma'(z)/\gamma'(\phi(z)))^{1/2}$. From Theorem 2.0.1 of [4] it follows that $C_{\psi, \phi}$ is invertible if and only if ϕ is an automorphism of $\mathbb{D}$ and ψ and $1/\psi$ are in H^∞ . Moreover $C_{\psi, \phi}^{-1} = C_{1/\psi \circ \phi^{-1}, \phi^{-1}}$. It follows from $\phi = \gamma^{-1} \circ \phi \circ \gamma$ that ϕ is an automorphism of Ω . If $g \in H^2(\Omega)$, then a direct computation yields

$$V^{-1}C_{\psi, \phi}^{-1}V(g) = g \circ \phi^{-1}.$$

Thus $C_{\phi^{-1}}$ is a bounded operator on $H^2(\Omega)$ and $C_\phi^{-1} = C_{\phi^{-1}}$. □

Interestingly, for certain Ω it is possible for an automorphism of Ω to induce a bounded composition operator that is not invertible [8].

Lemma 2.2. *Suppose that U is a bounded operator on $H^2(\Omega)$. Then U is unitary if and only if VUV^{-1} is unitary on H^2 .*

Proof. Let $T = VUV^{-1}$. Since V is an isometry it is clear that T is an isometry if and only if U is an isometry. Also, T is invertible if and only if U is invertible. $\square$

Lemma 2.3. *The set of all unitary composition operators of $H^2(\Omega)$ is a group under the composition of operators.*

Proof. Since $C_\phi^{-1} = C_{\phi^{-1}}$ and $C_f \circ C_g = C_{g \circ f}$ the result follows. $\square$

Notation

- The group of unitary composition operators of $H^2(\Omega)$ is denoted by $\mathcal{U}_\Omega$.
- The group of the unit circle with usual multiplication is denoted by $(\mathbb{T}, \cdot)$.
- The group of real numbers with usual addition is denoted by $(\mathbb{R}, +)$.
- The group of integers with usual addition is denoted by $(\mathbb{Z}, +)$.
- $\{1, -1\} \rtimes \mathbb{R}$ is used to denote the semidirect product of the multiplicative group $\{1, -1\}$ and $(\mathbb{R}, +)$ with group operation

$$(a_1, t_1) * (a_2, t_2) = (a_1 a_2, a_2 t_1 + t_2)$$

where $a_1, a_2 \in \{1, -1\}$ and $t_1, t_2 \in \mathbb{R}$.

- $\{1, -1\} \rtimes \mathbb{Z}$ denotes the subgroup of $\{1, -1\} \rtimes \mathbb{R}$ where the second coordinate is in $\mathbb{Z}$.

- Suppose that f is a self map of the set Ω . If $n \in \mathbb{N}$, then f_n is the composition of f with itself n times. If f is invertible and m is a negative integer, then $f_m = f_{|m|}^{-1}$. Also f_0 denotes the identity map.

We leave it to the reader as an easy exercise to show that each of the five groups $(\mathbb{T}, \cdot)$, $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, $\{1, -1\} \rtimes \mathbb{R}$ and $\{1, -1\} \rtimes \mathbb{Z}$ belongs to a different isomorphism class.

3. UNITARY OPERATORS

We begin our work by classifying those symbols ϕ which induce unitary composition operators on $H^2(\Omega)$.

Definition 3.1. Let $\mathcal{UA}_\Omega = \{\phi : C_\phi \text{ is unitary on } H^2(\Omega)\}$.

Theorem 3.2. *Suppose that ϕ is an analytic self map of Ω . Then $\phi \in \mathcal{UA}_\Omega$ if and only if there exist $\mu, c \in \mathbb{C}$, $|\mu| = 1$, such that $\phi(w) = \mu w + c$ is an automorphism of Ω .*

Proof. First suppose that $\phi(w) = \mu w + c$, $|\mu| = 1$, $c \in \mathbb{C}$, is an automorphism of Ω . Let $\varphi = \gamma^{-1} \circ \phi \circ \gamma$. Then

$$\phi \circ \gamma = \gamma \circ \varphi.$$

Therefore

$$\mu \gamma(z) + c = \gamma(\varphi(z))$$

for $z \in \mathbb{D}$. Now it follows that

$$\mu \gamma'(z) = \gamma'(\varphi(z)) \cdot \varphi'(z).$$

Since ϕ is an automorphism of Ω , φ is an automorphism of $\mathbb{D}$. Let $\beta = \varphi^{-1}(0)$. Then $\varphi(z) = \kappa(z - \beta)/(1 - \bar{\beta}z)$ for some $|\kappa| = 1$. Hence

$$\frac{\gamma'(z)}{\gamma'(\varphi(z))} = \bar{\mu}\kappa \frac{1 - |\beta|^2}{(1 - \bar{\beta}z)^2}.$$

If $\psi(z) = \left(\frac{\gamma'(z)}{\gamma'(\varphi(z))}\right)^{1/2}$, then $\psi = \lambda \frac{K_\beta}{\|K_\beta\|}$, where $|\lambda| = 1$ and $K_\beta(z) = 1/(1 - \bar{\beta}z)$ is the reproducing kernel for H^2 ; i.e. if $f \in H^2$, then $\langle f, K_\beta \rangle = f(\beta)$. Thus $C_{\psi, \varphi}$ is a unitary operator on H^2 [1, Theorem 6], and by Lemma 2.2, C_ϕ is unitary on $H^2(\Omega)$.

Now suppose that C_ϕ is unitary on $H^2(\Omega)$. Then $C_{\psi, \varphi}$ is unitary on H^2 , where

$$\psi(z) = \left(\frac{\gamma'(z)}{\gamma'(\varphi(z))}\right)^{1/2}$$

and $\varphi = \gamma^{-1} \circ \phi \circ \gamma$. If C_ϕ is invertible, then ϕ is an automorphism of Ω ; hence φ is an automorphism of $\mathbb{D}$. If $\beta = \varphi^{-1}(0)$, then $\varphi(z) = \zeta(z - \beta)/(1 - \bar{\beta}z)$ for some unimodular ζ . Since $C_{\psi, \varphi}$ is unitary, $\psi = \lambda K_\beta / \|K_\beta\|$ for some unimodular λ [1, Theorem 6]. Therefore

$$\frac{\gamma'(z)}{\gamma'(\varphi(z))} = \lambda^2 \frac{1 - |\beta|^2}{(1 - \bar{\beta}z)^2}.$$

But $\bar{\zeta}\varphi'(z) = (1 - |\beta|^2)/(1 - \bar{\beta}z)^2$, and thus

$$\gamma'(z) = \lambda^2 \bar{\zeta} \gamma'(\varphi(z)) \varphi'(z).$$

Therefore $\gamma'(z) = \lambda^2 \bar{\zeta} (\gamma(\varphi(z)))'$, and by integrating both sides

$$\gamma(z) = \lambda^2 \bar{\zeta} \gamma(\varphi(z)) + k.$$

Since $\gamma \circ \varphi = \phi \circ \gamma$, we have

$$\gamma(z) = \lambda^2 \bar{\zeta} \phi(\gamma(z)) + k.$$

The desired result follows by letting $w = \gamma(z)$. □

4. FINITE ORDER

Lemma 4.1. *Suppose that $\mathcal{U}_\Omega$ is a finite group. Further, suppose that $\phi(w) = \mu w + c$ and $\psi(w) = \mu w + d$. If $\phi, \psi \in \mathcal{UA}_\Omega$, then $c = d$.*

Proof. It is readily seen that

$$\phi^{-1}(w) = \mu^{-1}w - c\mu^{-1}.$$

If $\varphi = \psi \circ \phi^{-1}$, it follows that $\varphi \in \mathcal{UA}_\Omega$ since $C_{\phi^{-1}} \circ C_\psi \in \mathcal{U}_\Omega$. Now

$$\varphi(w) = \mu(\mu^{-1}w - c\mu^{-1}) + d = w - c + d.$$

It is easy to see that $\varphi_m(w) = w + m(d - c)$ for any positive integer m . If $d - c \neq 0$, then $\{\varphi_m : m = 1, 2, \dots\}$ is infinite and thus $G = \{C_{\varphi_m} : m = 1, 2, \dots\}$ is also infinite. But $G \subseteq \mathcal{U}_\Omega$, and hence $d - c = 0$. □

Suppose that $n > 1$ is an integer. The domain Ω is said to have n -fold symmetry about the point p if $e^{2\pi i/n}(\Omega - p) = \Omega - p$.

Theorem 4.2. *If $\mathcal{U}_\Omega$ has finite order $n > 1$, then Ω has n -fold symmetry and $\mathcal{U}_\Omega$ is isomorphic to $\mathbb{Z}_n$.*

Before giving the proof, we note that n -fold symmetry of Ω does not imply that $\mathcal{U}_\Omega$ has finite order. A simple example is when Ω is the unit disc $\mathbb{D}$, which has n -fold symmetry for every integer n . As previously noted, $\mathcal{U}_\mathbb{D} \cong (\mathbb{T}, \cdot)$. Another example is when Ω is a strip which has 2-fold symmetry, but no higher symmetry. See Proposition 5.8 below.

Proof. If $C_\phi \in \mathcal{U}_\Omega$, then it follows from Theorem 3.2 that $\phi(w) = \mu w + c$, $|\mu| = 1$. Since the order of $\mathcal{U}_\Omega$ is n , C_ϕ^n is the identity. But $C_\phi^n = C_{\phi_n}$ and $\phi_n(w) = \mu^n w + (1 + \mu + \cdots + \mu^{n-1})c$. Thus $\mu^n w + (1 + \mu + \cdots + \mu^{n-1})c = w$, which gives $\mu^n = 1$. Since ϕ is an automorphism, it follows that Ω has n -fold symmetry about $c/(1 - \mu)$, the fixed point of ϕ .

Let G_n denote the group $\{e^{2\pi i k/n} : k = 0, 1, \dots, n-1\}$ under usual complex number multiplication. Define $\Pi : \mathcal{U}_\Omega \rightarrow G_n$ by

$$\Pi(C_\phi) = \phi'(0).$$

A straightforward computation shows that

$$\Pi(C_{\phi_1} \circ C_{\phi_2}) = \mu_1 \mu_2 = \Pi(C_{\phi_1}) \Pi(C_{\phi_2}).$$

Thus Π is a group homomorphism. From Lemma 4.1 it follows that Π is one-to-one. Since $\mathcal{U}_\Omega$ and G_n contain the same number of elements, Π is onto as well. Hence Π is an isomorphism. Clearly G_n is isomorphic to $\mathbb{Z}_n$, and the result follows. $\square$

Example 4.3. Let $n > 1$ be an integer. Let Ω be the interior of the regular n -sided polygon whose vertices are $1, e^{2\pi i/n}, e^{2\pi i 2/n}, \dots, e^{2\pi i(n-1)/n}$. Then $\mathcal{U}_\Omega$ is isomorphic to $\mathbb{Z}_n$.

Proof. It is easy to see that $f(w) = e^{2\pi i \theta} w + c$ is an automorphism of Ω if and only if $\theta = m/n, m \in \mathbb{Z}$ and $c = 0$. The desired result now follows from Theorem 3.2. $\square$

The following example gives a domain Ω with $\mathcal{U}_\Omega$ the trivial group, showing that order one is possible.

Example 4.4. Let $\Omega = \{x + iy, x > 0, y > 0\}$. Then $\mathcal{U}_\Omega$ is the trivial group.

It is easily seen that $f(w) = e^{2\pi i \theta} w + c$ is an automorphism of Ω if and only if $f(w) = w$. Thus from Theorem 3.2 it follows that $\mathcal{U}_\Omega$ is the trivial group.

5. INFINITE ORDER

In this section we work under the assumption that $\mathcal{U}_\Omega$ is of infinite order.

Lemma 5.1. *If $f(w) = w + \alpha$ and $g(w) = w + \beta$ are distinct automorphisms of Ω , then the line through α and β contains the origin.*

Proof. Assume, contrary to the statement, that α and β do not lie on the same line through the origin. Note that this implies $\alpha\beta \neq 0$. Next, observe that f and g are automorphisms of Ω if and only if $\hat{f}(z) = z + 1$ and $\hat{g}(z) = z + \beta/\alpha$ are automorphisms of $\alpha^{-1}\Omega$. Thus we may assume that $\alpha = 1$ and $\beta \notin \mathbb{R}$. We further assume that $\text{Im}\beta > 0$; if $\text{Im}\beta < 0$ the proof will be similar, except that the orientation of the curve Γ_N below will be reversed.

Let $w_0 \in \Omega$, so $f(w_0) = w_0 + 1 \in \Omega$ and $g(w_0) = w_0 + \beta \in \Omega$ as well. Since Ω is path connected, there are smooth curves γ from w_0 to $w_0 + 1$ and η from w_0 to $w_0 + \beta$ that lie in Ω . Now fix

$$(5.1) \quad R > \max\{\text{diam}(\gamma), \text{diam}(\eta)\},$$

and choose an integer N large enough so that the disc $B(0, 2R)$ with center 0 and radius $2R$ is contained in the interior of the parallelogram P_N with vertices $w_0 \pm N \pm N\beta$.

Next recall from the end of §2 the definition of the iterates $\{f_n\}_{n \in \mathbb{Z}}$ and $\{g_n\}_{n \in \mathbb{Z}}$, and set

$$\sigma = \sum_{n=-N}^{N-1} f_n(\gamma) \quad \text{and} \quad \lambda = \sum_{n=-N}^{N-1} g_n(\eta).$$

Then the curve

$$\Gamma_N = g_{-N}(\sigma) + f_N(\lambda) - g_N(\sigma) - f_{-N}(\lambda) = \sum_{j=1}^4 \Gamma_{Nj}$$

is a closed curve in Ω that passes through the vertices of P_N .

Now let $\zeta \in B(0, R)$. Note that since $B(0, 2R)$ is in the interior of P_N , it follows from (5.1) that each Γ_{Nj} is contained in the complement of a half-plane that contains $B(0, R)$. Thus the increment of the argument of $z - \zeta$ over Γ_{Nj} satisfies

$$0 < \Delta \arg(z - \zeta, \Gamma_{Nj}) < \pi.$$

Therefore $0 < \Delta \arg(z - \zeta, \Gamma_N) < 4\pi$, and hence $\Delta \arg(z - \zeta, \Gamma_N) = 2\pi$ since Γ_N is a closed curve. Thus the winding number of Γ_N about each point $\zeta \in B(0, R)$ is 1, and it follows that $B(0, R) \subset \Omega$. Since the only assumption on R was (5.1), this contradicts the standing assumption that $\Omega \neq \mathbb{C}$ and completes the proof. $\square$

Lemma 5.2. *Suppose that $f(w) = \mu w + c$ and $g(w) = \mu w + d$ are both automorphisms of Ω , where μ is unimodular. If $c \neq d$, then μ is 1 or -1 .*

Proof. Let $w \in \Omega$. Then

$$h(w) = f \circ g^{-1}(w) = w + c - d.$$

Let

$$p(w) = f \circ h \circ f^{-1}(w) = w + \mu(c - d).$$

Then p and h are automorphisms of Ω . By Lemma 5.1 $c - d$ and $\mu(c - d)$ lie on the same line through the origin. Hence $\mu^2 = 1$. $\square$

We saw in Theorem 3.2 that $f \in \mathcal{UA}_\Omega$ has the form $f(w) = \mu w + c$, and so extends to be an entire function. We will always consider f extended in this way, so that even if $0 \notin \Omega$ we can write $f(0) = c$ and $f'(0) = \mu$. This convention is used in the next definition and subsequently.

Definition 5.3. Let

$$(5.2) \quad R_\Omega = \{f'(0) : f \in \mathcal{UA}_\Omega\}.$$

Also, $R_\Omega = \{\mu : f(w) = \mu w + c, f \text{ is an automorphism of } \Omega, |\mu| = 1\}$. Since the identity map is always in $\mathcal{UA}_\Omega$ it is clear that $1 \in R_\Omega$. If $\lambda, \mu \in R_\Omega$, then there are $f, g \in \mathcal{UA}_\Omega$ so that $f(w) = \lambda w + c$ and $g(w) = \mu w + d$. Since both $f \circ g, f^{-1} \in \mathcal{UA}_\Omega$, it follows that $\lambda\mu \in R_\Omega$ and $\lambda^{-1} \in R_\Omega$. Hence R_Ω is a subgroup of $(\mathbb{T}, \cdot)$.

The set R_Ω can be infinite or finite. We leave the proof of the next lemma as an easy exercise.

Lemma 5.4. *If R_Ω is infinite, then it is dense in the unit circle.*

Lemma 5.5. *Suppose that $\mathcal{U}_\Omega$ is infinite. Then*

- (1) *The following are equivalent:*
 - (a) R_Ω is infinite.
 - (b) Ω is a disc.
 - (c) $R_\Omega = \mathbb{T}$.
- (2) *If Ω is not a disc, then $R_\Omega \subseteq \{1, -1\}$.*

Proof. (a) $\implies$ (b): Suppose that R_Ω is infinite. Choose $\mu_1, \mu_2 \in R_\Omega$ such that $(\mu_1\mu_2)^2 \neq 1$. Then there are $f, g \in \mathcal{UA}_\Omega$ so that $f(w) = \mu_1w + c_1$ and $g(w) = \mu_2w + c_2$. Now

$$f \circ g(w) = \mu_1\mu_2w + (\mu_1c_2 + c_1) \text{ and } g \circ f(w) = \mu_2\mu_1w + (\mu_2c_1 + c_2).$$

Since $f \circ g, g \circ f \in \mathcal{UA}_\Omega$ and $\mu_1\mu_2 \neq \pm 1$, it follows from Lemma 5.2 that $\mu_1c_2 + c_1 = \mu_2c_1 + c_2$. Thus

$$(5.3) \quad (1 - \mu_2)c_1 = (1 - \mu_1)c_2.$$

Since R_Ω is dense in $\mathbb{T}$ we can choose a dense sequence $\{\mu_n\}_{n=1}^\infty$ from R_Ω such that $(\mu_i)^2 \neq 1$ and $(\mu_i\mu_{i+1})^2 \neq 1$. For each μ_i there is an automorphism $f(w) = \mu_iw + c_i$. Since $(\mu_i\mu_{i+1})^2 \neq 1$, it follows from (5.3) that $c_i/(1 - \mu_i) = c_1/(1 - \mu_1)$ for $i = 2, 3, \dots$. Let $p = c_1/(1 - \mu_1)$ and

$$E = \Omega - p.$$

We first show that $\mu_j E \subseteq E$. If $v \in E$, then $v = w - p$ for some $w \in \Omega$. Now

$$\mu_j v + p = \mu_j w - \frac{\mu_j c_j}{1 - \mu_j} + \frac{c_j}{1 - \mu_j} = \mu_j w + c_j.$$

Since $\mu_j v \in E$ if and only if $\mu_j v + p \in \Omega$ it is readily seen that $\mu_j v \in E$.

Let $z \in \mathbb{C} \setminus E$. Since $z + p \in \mathbb{C} \setminus \Omega$ and $h(w) = \mu_j w + c_j$ is an automorphism of Ω that extends to $\mathbb{C}$, it follows that $\mu_j(z + p) + c_j \in \mathbb{C} \setminus \Omega$. Since $\mu_j z = \mu_j(z + p) + c_j - p$ it follows that $\mu_j z \in \mathbb{C} \setminus E$. Thus

$$\{\mu_j z | j = 1, 2, \dots\} \subseteq \mathbb{C} \setminus E.$$

Note that $\mathbb{C} \setminus E$ is closed. The closure of $\{\mu_j z | j = 1, 2, \dots\}$ is a circle centered at 0 which is contained in $\mathbb{C} \setminus E$. Also, E is open and connected; hence E is a disc centered at the origin. Thus Ω is a disc centered at p .

(b) $\implies$ (c): If Ω is a disc with center p , then $w \rightarrow e^{i\theta}(w - p) + p$ is an automorphism of Ω for all real θ . Thus $R_\Omega = \mathbb{T}$.

The proof of (c) $\implies$ (a) is trivial.

Proof of (2): Since Ω is not a disc, from part (1) it follows that R_Ω is finite. If $|\lambda| = 1, \lambda^2 \neq 1$, then from Lemma 5.2 there is at most one c' such that $k(w) = \lambda w + c'$ is an automorphism of Ω . But $\mathcal{U}_\Omega$ is infinite; thus there are infinitely many automorphisms of the form $f(w) = w + d$ or $g(w) = -w + k$. Let $\mu \in R_\Omega$. Then there is $h(w) = \mu w + c$ so that $h \in \mathcal{UA}_\Omega$. Since

$$h \circ f(w) = \mu w + (\mu d + c) \text{ and } h \circ g(w) = -\mu w + (\mu k + c)$$

it follows from Lemma 5.2 that $\mu^2 = 1$. □

Part (1) of the previous lemma gives a complete description of Ω when R_Ω is infinite. Next we turn to the case that R_Ω is finite, so $R_\Omega \subseteq \{1, -1\}$ by Lemma 5.5.

From Lemma 5.1 there is a unimodular constant λ such that if $f(w) = w + c$ is in $\mathcal{UA}_\Omega$, then $\bar{\lambda}c$ is real. Define

$$S_\Omega = \{t \in \mathbb{R} : f_t(w) = w + t\lambda \text{ is in } \mathcal{UA}_\Omega\}.$$

Since $f_s \circ f_t = f_{s+t}$, $(S_\Omega, +)$ is a subgroup of $(\mathbb{R}, +)$.

Since we have assumed in this section that $\mathcal{U}_\Omega$ is infinite, there are infinitely many automorphisms of the form $f(w) = w + c$ or $g(w) = -w + d$. If there are infinitely many of the former, then it is clear that there exists $t \in S_\Omega$ with $t > 0$. If there are infinitely many of the latter, take g and h to be two such automorphisms. Then

$$g(w) = -w + d_1 \text{ and } h(w) = -w + d_2$$

where $d_1 \neq d_2$. But

$$g \circ h(w) = w + (d_1 - d_2),$$

and again the set of $t > 0$ with $t \in S_\Omega$ is nonempty. Let

$$(5.4) \quad c_\Omega = \inf\{t > 0 : t \in S_\Omega\}.$$

Lemma 5.6. *Suppose that $R_\Omega \subseteq \{1, -1\}$.*

- (1) *Suppose that $c_\Omega = 0$. Then Ω is a half-plane or a strip and $S_\Omega = \mathbb{R}$. Moreover, $-1 \in R_\Omega$ if and only if Ω is a strip.*
- (2) *$c_\Omega > 0$ if and only if $S_\Omega = \{n \cdot c_\Omega : n \in \mathbb{Z}\}$.*

Proof. Suppose that $c_\Omega = 0$ and let λ be as in the definition of S_Ω . Let l_λ be the line $\{\lambda t : t \in \mathbb{R}\}$. If $w \in \Omega$, then there is a $\delta > 0$ such that Ω contains the line segment

$$L_{w,\delta} = \{w + \lambda t : -\delta < t < \delta\}.$$

Since $c_\Omega = 0$, there is $t_w \in \mathbb{R}$ such that $0 < |t_w| < \delta$ and $\phi(w) = w + \lambda t_w$ is an automorphism of Ω . If $n \in \mathbb{Z}$, then $\phi_n \in \mathcal{UA}_\Omega$; hence

$$\bigcup_{n \in \mathbb{Z}} \phi_n(L_{w,\delta}) \subseteq \Omega.$$

But $\phi_n(L_{w,\delta}) = L_{w,\delta} + n\lambda t_w$. Thus $L_w = \bigcup_{n \in \mathbb{Z}} \phi_n(L_{w,\delta})$ is a line that goes through w and is parallel to l_λ .

Since w is an arbitrary point in Ω , it follows that $\bigcup_{w \in \Omega} L_w \subseteq \Omega$. Clearly $\Omega \subseteq \bigcup_{w \in \Omega} L_w$; thus

$$\Omega = \bigcup_{w \in \Omega} L_w.$$

Since $\Omega \neq \mathbb{C}$ and is connected, it must be a strip or a half-plane. Thus $S_\Omega = \mathbb{R}$. Clearly $-1 \in R_\Omega$ implies that Ω is not a half-plane, and so must be a strip. Finally, the 2-fold symmetry of a strip implies that $-1 \in R_\Omega$ if Ω is a strip.

Proof of (2): First suppose that $c_\Omega > 0$. Since $(S_\Omega, +)$ is a subgroup of $(\mathbb{R}, +)$, if $c_\Omega > 0$, then S_Ω has no point of accumulation and hence $c_\Omega \in S_\Omega$. It follows that $S_\Omega = \{n \cdot c_\Omega : n \in \mathbb{Z}\}$. Sufficiency easily follows since S_Ω is a nontrivial group. $\square$

If Ω is the upper half-plane, then $R_\Omega = \{1\}$. Next we will look at such regions.

Proposition 5.7. *Suppose that $R_\Omega = \{1\}$. Then*

- (1) *The following are equivalent:*
 - (a) $c_\Omega = 0$.
 - (b) Ω is a half-plane.
 - (c) $\mathcal{U}_\Omega$ is isomorphic to $(\mathbb{R}, +)$.
- (2) *If $c_\Omega > 0$, then $\mathcal{U}_\Omega$ is isomorphic to $(\mathbb{Z}, +)$.*

Proof. (a) $\implies$ (b) : It follows from (1) of Lemma 5.6 that Ω is a half-plane.

(b) $\implies$ (c) : Since Ω is a half-plane there is a $\lambda \in \mathbb{C}$ so that the boundary of Ω is parallel to the line $l_\lambda = \{\lambda t : t \in \mathbb{R}\}$. If $t \in \mathbb{R}$, and $\phi(w) = w + t\lambda$, then $\phi \in \mathcal{UA}_\Omega$. Define $\Pi : \mathcal{U}_\Omega \rightarrow (\mathbb{R}, +)$ by

$$\Pi(C_\phi) = \lambda^{-1}\phi(0).$$

It is readily seen that Π is a bijection. If $f(w) = w + \lambda t_1, g(w) = w + \lambda t_2$, then $g \circ f(w) = w + \lambda(t_1 + t_2)$. Since $C_f \circ C_g = C_{g \circ f}$ we get

$$\Pi(C_f \circ C_g) = t_1 + t_2 = \Pi(C_f) + \Pi(C_g).$$

Thus $\mathcal{U}_\Omega$ is isomorphic to $(\mathbb{R}, +)$.

(c) $\implies$ (a) : Suppose that $\mathcal{U}_\Omega \cong (\mathbb{R}, +)$. Clearly $c_\Omega \geq 0$, and if $c_\Omega > 0$, then part (2) yields that $\mathcal{U}_\Omega \cong (\mathbb{Z}, +)$. Thus $c_\Omega = 0$.

Proof of (2): This is clear from (2) of Lemma 5.6. $\square$

If $\Omega = \{z : -1 < \text{Im}(z) < 1\}$, then $R_\Omega = \{1, -1\}$. Next we look at such regions.

Proposition 5.8. *Suppose that $R_\Omega = \{1, -1\}$. Then*

- (1) *The following are equivalent:*
 - (a) $c_\Omega = 0$.
 - (b) Ω is a strip.
 - (c) $\mathcal{U}_\Omega$ is isomorphic to $\{1, -1\} \rtimes \mathbb{R}$.
- (2) *If $c_\Omega > 0$, then $\mathcal{U}_\Omega$ is isomorphic to $\{1, -1\} \rtimes \mathbb{Z}$.*

Proof. (a) $\implies$ (b) : The result follows from (1) of Lemma 5.6.

(b) $\implies$ (c) : Since Ω is a strip there is some $\lambda \in \mathbb{C}$ so that Ω is parallel to the line $l_\lambda = \{\lambda t : t \in \mathbb{R}\}$. Let z_0 be the point with minimum modulus such that

$$-\Omega + z_0 = \Omega.$$

Therefore $s(w) = -w + z_0 \in \mathcal{UA}_\Omega$. For any $r(w) = -w + d \in \mathcal{UA}_\Omega$ we have $r \circ s(w) = w + d - z_0 \in \mathcal{UA}_\Omega$; thus $d - z_0 = \lambda t$ for some $t \in \mathbb{R}$. Hence if $f \in \mathcal{UA}_\Omega$ and $f'(0) = -1$, then $f(0) = z_0 + \lambda t$. Thus, $C_\phi \in \mathcal{U}_\Omega$ if and only if $\phi(w) = w + \lambda t$ or $\phi(w) = -w + z_0 + \lambda t, t \in \mathbb{R}$. Let $\Pi : \mathcal{U}_\Omega \rightarrow \{1, -1\} \rtimes \mathbb{R}$ be defined by

$$\Pi(C_\phi) = \left(\phi'(0), \lambda^{-1} \left(\phi(0) - \frac{z_0}{2}(1 - \phi'(0)) \right) \right).$$

It is easy to see that Π is a bijection. If $\phi \in \mathcal{UA}_\Omega$, then it can be written in the form $\phi(w) = \mu w + z_0(1 - \mu)/2 + \lambda t$. Suppose that $h, s \in \mathcal{UA}_\Omega$ with $h(w) = \mu_1 w + z_0(1 - \mu_1)/2 + \lambda t_1$ and $s(w) = \mu_2 w + z_0(1 - \mu_2)/2 + \lambda t_2$. A straightforward computation yields that

$$\Pi(C_h \circ C_s) = (\mu_2 \mu_1, \mu_2 t_1 + t_2) = \Pi(C_h) * \Pi(C_s).$$

Thus, $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{R}$.

(c) $\implies$ (a) : Suppose that $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{R}$. Clearly $c_\Omega \geq 0$, and if $c_\Omega > 0$, then part (2) yields that $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{Z}$. Thus $c_\Omega = 0$.

Proof of (2): If $c_\Omega > 0$, then from (2) of Lemma 5.6 we see that $S_\Omega = \{n \cdot c_\Omega : n \in \mathbb{Z}\}$. Recall that there is a unimodular λ such that if $f \in \mathcal{UA}_\Omega$ and $f'(0) = 1$, then $f(w) = w + \lambda t$ for some $t \in S_\Omega$. If $g(w) = -w + c$ and $h(w) = -w + d$ are automorphisms of Ω , then $g \circ h(w) = w + c - d$ is also an automorphism of Ω ; hence $c - d = n \cdot c_\Omega \lambda$ for some $n \in \mathbb{Z}$. Thus, there is a $d_0 \in \mathbb{C}$ such that if

$h \in \mathcal{UA}_\Omega$ and $h'(0) = -1$, then $h(w) = -w + d_0 + m \cdot c_\Omega \lambda$ for some $m \in \mathbb{Z}$. Let $\Pi : \mathcal{U}_\Omega \rightarrow \{1, -1\} \rtimes \mathbb{Z}$ be defined by

$$\Pi(C_\phi) = \left(\phi'(0), (c_\Omega \lambda)^{-1} \left(\phi(0) - \frac{d_0}{2}(1 - \phi'(0)) \right) \right).$$

Then it is easy to see that Π is an isomorphism. $\square$

Proposition 5.9. *The group $(\mathbb{T}, \cdot)$ is isomorphic to $\mathcal{U}_\Omega$ if and only if Ω is a disc.*

Proof. First suppose that Ω is a disc centered at the origin. It is readily seen that $f \in \mathcal{UA}_\Omega$ if and only if $f(w) = e^{2\pi i \theta} w$ for some $\theta \in \mathbb{R}$ (Theorem 3.2). The map $\Pi : \mathcal{U}_\Omega \rightarrow (\mathbb{T}, \cdot)$ defined by

$$\Pi(C_\phi) = \phi'(0)$$

shows that $\mathcal{U}_\Omega$ is isomorphic to $(\mathbb{T}, \cdot)$. If Ω is a disc centered at p , then the same proof works with the domain $\Omega - p$.

Next we prove the other direction. Suppose that $\mathcal{U}_\Omega$ is isomorphic to $(\mathbb{T}, \cdot)$. From Proposition 5.7 and Proposition 5.8 it follows that $R_\Omega \not\subseteq \{1, -1\}$. But $\mathcal{U}_\Omega$ is infinite; hence the desired result follows from Lemma 5.5. $\square$

We summarize all the results in the theorem below.

Theorem 5.10. *If $\mathcal{U}_\Omega$ is infinite, then it is isomorphic to one of the following groups:*

- (1) $(\mathbb{T}, \cdot)$,
- (2) $(\mathbb{R}, +)$,
- (3) $(\mathbb{Z}, +)$,
- (4) $\{1, -1\} \rtimes \mathbb{R}$,
- (5) $\{1, -1\} \rtimes \mathbb{Z}$.

Proof. If R_Ω is infinite, then from Lemma 5.5 and Proposition 5.9 it follows that $\mathcal{U}_\Omega \cong (\mathbb{T}, \cdot)$.

If R_Ω is not infinite, then from Lemma 5.5 it is contained in $\{1, -1\}$. Notice that 1 is always contained in R_Ω . Thus $R_\Omega = \{1\}$ or $R_\Omega = \{1, -1\}$. If $R_\Omega = \{1\}$, then Proposition 5.7 yields that $\mathcal{U}_\Omega$ is isomorphic to either $(\mathbb{R}, +)$ or $(\mathbb{Z}, +)$. If $R_\Omega = \{1, -1\}$, then from Proposition 5.8 it follows that $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{R}$ or $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{Z}$. $\square$

In some of these cases we can give a precise description of Ω .

Theorem 5.11. *If $\mathcal{U}_\Omega$ is infinite, then the following are true.*

- (1) $\mathcal{U}_\Omega \cong (\mathbb{T}, \cdot)$ if and only if Ω is a disc.
- (2) $\mathcal{U}_\Omega \cong (\mathbb{R}, +)$ if and only if Ω is a half-plane.
- (3) $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{R}$ if and only if Ω is a strip.

Proof. Proposition 5.9 yields (1); (2) follows from Proposition 5.7, and Proposition 5.8 gives (3). $\square$

However, many different Ω can lead to $(\mathbb{Z}, +)$ or $\{1, -1\} \rtimes \mathbb{Z}$. We close the paper with a few examples.

Example 5.12. Let

$$\Omega = \{x + iy : y > \sin(x)\}.$$

Then $\mathcal{U}_\Omega \cong (\mathbb{Z}, +)$.

It is easy to see that $f \in \mathcal{UA}_\Omega$ if and only if $f(w) = w + 2\pi n, n \in \mathbb{Z}$. Thus $R_\Omega = \{1\}$, and the result follows from Proposition 5.7.

Example 5.13. Let

$$\Omega = \{x + iy : 1 + \sin(x) > y > \sin(x)\}.$$

Then $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{Z}$.

It is readily seen that $f \in \mathcal{UA}_\Omega$ if and only if $f(w) = w + 2\pi n, n \in \mathbb{Z}$, or $f(w) = -w + i + (2n)\pi, n \in \mathbb{Z}$. Hence $R_\Omega = \{1, -1\}$, and Proposition 5.8 yields the desired result.

Example 5.14. For $k \in \mathbb{Z}$ define $N_k^+ = \{2k + yi : 1/2 \leq y < 1\}$ and $N_k^- = \{(2k + 1) + yi : -1 < y \leq -1/2\}$. Let

$$\Omega = \{z : -1 < \operatorname{Im}(z) < 1\} \setminus \bigcup_{k \in \mathbb{Z}} (N_k^+ \cup N_k^-).$$

Then $\mathcal{U}_\Omega \cong \{1, -1\} \rtimes \mathbb{Z}$.

Clearly $f \in \mathcal{UA}_\Omega$ if and only if $f(w) = w + 2\pi n$ or $f(w) = -w + 1 + 2\pi n, n \in \mathbb{Z}$. Thus $R_\Omega = \{1, -1\}$, and the result follows from Proposition 5.8.

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