

SINGULARITIES OF MEAN CURVATURE FLOW AND ISOPERIMETRIC INEQUALITIES IN \mathbb{H}^3

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ABSTRACT. In this paper, we mainly consider the mean curvature flow of surfaces in hyperbolic 3-space. First, we establish the isoperimetric inequality using the flow, provided the enclosed volume approaches zero at the final time. Second, we construct two singular examples of the flow. More precisely, there exists a torus which must develop a singularity in the flow before the volume it encloses decreases to zero. There also exists a topological sphere in the shape of dumbbell, which must develop a singularity in the flow before its area shrinks to zero.

1. INTRODUCTION

Given a smooth and compact immersion $F_0 : N^{n-1} \rightarrow M^n$, the mean curvature flow $F(x, t) : N \times [0, T) \rightarrow M^n$ with initial data F_0 is defined according to the evolution equation

$$(1.1) \quad \begin{cases} \partial_t F(x, t) = -H\nu, \\ F(x, 0) = F_0, \end{cases}$$

where ν is the outer unit normal field, and the mean curvature H is the mean of principal curvatures κ_i . The flow (1.1) has been a useful topic in the study of geometry problems and there are many good results of the flow, beginning with the G. Huisken's paper [11] in the case $M^n = \mathbb{R}^n$. Recently, Andrews and Chen [1] and Andrews et al. [2] studied the case $M^n = \mathbb{S}^n$ and $M^n = \mathbb{H}^n$. They proved in [1] that the flow Σ_t with initial Gauss curvature $GK > 1$ shrinks to a point and becomes more spherical. In this paper, we consider the hyperbolic case $M^n = \mathbb{H}^3$.

One useful application of the flow is to study the isoperimetric inequality. The classical isoperimetric inequality in Euclidian spaces states that the round balls minimize the perimeter among all domains with the same volume. The isoperimetric inequality in hyperbolic spaces is also true, proved by Schmidt for all dimensions, using symmetrisation (cf. [9, 10]). Isoperimetric inequality for arbitrary Hadamard spaces is still open for higher dimensions. For dimension 3 and 4, the isoperimetric

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inequality was proved by B. Kleiner in [3] and B. Croke in [4] respectively. The inequality in \mathbb{H}^3 is

$$(1.2) \quad \text{Area}(\partial\Omega) \geq \text{Area}(\partial\Omega^*),$$

for any bounded smooth domain Ω in \mathbb{H}^3 , where Ω^* is the geodesic ball with the volume equal to $\text{Vol}(\Omega)$. Moreover, the equality in (1.2) holds if and only if Ω is also a geodesic ball. In this paper we denote the area of $\partial\Omega$ and volume of Ω by $\text{Area}(\partial\Omega)$ and $\text{Vol}(\Omega)$ respectively.

On the one hand, isoperimetric inequalities can be derived from curvature flows, especially in \mathbb{R}^n (cf. [5, 7, 12, 15]). In [12], G. Huisken explained that MCF can be expected to converge to solutions of the isoperimetric problem in \mathbb{R}^n . In [5], the isoperimetric inequality was proved by curvature flows in hyperbolic spaces for some special domains. In mean curvature flow the area of hypersurfaces evolves by the Willmore energy which is defined in Section 2. By that, we can use the Willmore energy to control the area of hypersurfaces in the flow. In this paper, we obtain the following sharp isoperimetric inequality.

Theorem 1.1. *Let Σ_0 be a smooth closed hypersurface in \mathbb{H}^3 . If the flow has no singularity before the volume it encloses shrinks to zero, then*

$$(1.3) \quad \text{Area}(\Sigma_0) \geq \text{Area}(\bar{\Sigma}),$$

where $\bar{\Sigma}$ is a geodesic sphere with enclosed volume equal to that enclosed by Σ_0 . Moreover, if the equality holds, then Σ_0 is a geodesic sphere.

On the other hand, we can use the isoperimetric inequality to study the singularities of the flow. In the case $M = \mathbb{R}^3$, Topping used the isoperimetric inequalities to study singularities and two examples of singularities are developed in [15]. One is a torus and the other is a topological sphere. Based on Topping’s idea, we exhibit examples of singular behaviour under the flow in Hyperbolic case \mathbb{H}^3 .

Theorem 1.2. *Denote by B_0 a round unit ball in \mathbb{H}^3 , by B_ϵ the torus given by drilling a narrow hole D_ϵ with radius ϵ through B_0 , and by Σ_ϵ the boundary of B_ϵ . For any ϵ satisfying*

$$(1.4) \quad \text{Area}(\partial D_\epsilon) + 2\text{Vol}(D_\epsilon) \leq \int_0^{\text{Area}(\partial B_0) - \text{Area}(\partial D_\epsilon)} \left(\frac{x}{4\pi + x}\right)^{\frac{1}{2}} - \left(\frac{x}{2\pi^2 + x}\right)^{\frac{1}{2}} dx,$$

Σ_ϵ under MCF must develop a singularity before the volume it encloses shrinks to zero.

Recall that the condition $GK > 1$ in [1] for initial hypersurface Σ_0 implies

$$(1.5) \quad \int_{\Sigma_0} (GK - 1) d\sigma > 0.$$

If Σ_0 is a torus, then the Gauss-Bonnet formula gives

$$(1.6) \quad \int_{\Sigma_0} (GK - 1) d\sigma = 0.$$

From this point of view, Theorem 1.2 gives singular examples, which violates the integration condition (1.5).

Theorem 1.3. *There exists an embedded topological sphere M_0 in the shape of a dumbbell in \mathbb{H}^3 enclosing two unit spheres with centres separated by a sufficiently*

large distance d , such that M_0 must develop a singularity under the flow before the area of M_t decreases to zero.

Remark 1.4. If M_0 is a topological sphere, then the Gauss-Bonnet formula gives

$$\int_{M_0} (GK - 1) d\sigma = 4\pi > 0,$$

which satisfies the condition (1.5). Thus the condition $GK > 1$ in [1] may not be replaced by the integration condition (1.5).

This paper is organized as follows. In Section 2, we study the properties of Willmore energy in three dimensional hyperbolic space. In Section 3, we use the flow to prove the isoperimetric inequality in \mathbb{H}^3 . In Section 4 and 5, we give two examples of singularities evolved by the flow.

2. WILLMORE ENERGY

Definition 2.1. For any closed hypersurfaces $\Sigma \subset \mathbb{R}^3$, the Willmore energy is defined by

$$W(\Sigma) \doteq \int_{\Sigma} H^2 d\sigma,$$

where $d\sigma$ denotes the area element of Σ .

For any immersed closed hypersurfaces, a lower bound of Willmore energy was obtained in [14, 15]. More precisely,

$$(2.1) \quad W(\Sigma) = \int_{\Sigma} H^2 d\sigma \geq \int_{\Sigma^+} GK d\sigma \geq 4\pi,$$

where $\Sigma^+ = \{x \in \Sigma, GK(x) \geq 0\}$ and the equality holds if only if Σ is a round sphere.

The problem of minimizing the Willmore energy among the class of immersed tori was proposed by Willmore. The existence of a torus that minimizes the Willmore energy was established by L. Simon in [13]. Together with (2.1), for any torus Σ , we have

$$(2.2) \quad \int_{\Sigma} H^2 d\sigma \geq c_0 > 4\pi,$$

where c_0 is the optimal Willmore energy.

Recently, F. Marques and A. Neves proved that the optimal c_0 is $2\pi^2$ by using the min-max theory of minimal surfaces and the optimal energy is achieved by the Clifford torus. For further details, we refer the readers to the paper [6].

In hyperbolic space \mathbb{H}^3 , we consider the following energy

$$(2.3) \quad \bar{W}(\Sigma) \doteq \int_{\Sigma} (H^2 - 1) d\sigma.$$

This functional comes from the conformal transformation from \mathbb{R}^3 to \mathbb{H}^3 . By the conformal invariant of energy W , we conclude that:

Proposition 2.2. *Let Σ be an immersed closed hypersurface in \mathbb{H}^3 , then we have*

$$(2.4) \quad \int_{\Sigma} (H^2 - 1) d\sigma \geq 4\pi.$$

Moreover, if Σ is an immersed torus in \mathbb{H}^3 , then

$$(2.5) \quad \int_{\Sigma} (H^2 - 1) d\sigma \geq 2\pi^2.$$

For the inequality (2.4), one can see a more direct proof in [8].

3. AN ISOPERIMETRIC INEQUALITY IN \mathbb{H}^3

In this section, we consider the relationships between the mean curvature flow Σ_t and isoperimetric inequalities in \mathbb{H}^3 . The main idea in this section is based on Topping’s paper [15].

To begin with, we give some notations. $A(t)$ denotes the area of Σ_t and $V(t)$ denotes the volume of the domain bounded by Σ_t . The evolution equations of $A(t)$ and $V(t)$ are given by

$$(3.1) \quad \frac{d}{dt} A(t) = -2 \int_{\Sigma_t} H^2 d\sigma_t,$$

and

$$(3.2) \quad \frac{d}{dt} V(t) = - \int_{\Sigma_t} H d\sigma_t.$$

Using the Willmore energy, we prove the isoperimetric inequality (1.3).

Proof of Theorem 1.1. From equation (3.2) and the Cauchy-Schwartz inequality, we derive

$$(3.3) \quad -\frac{d}{dt} V(t) \leq \left(\int_{\Sigma_t} H^2 d\sigma_t \right)^{\frac{1}{2}} A^{\frac{1}{2}}(t).$$

The inequality (2.4) gives that

$$(3.4) \quad \int_{\Sigma_t} H^2 d\sigma \geq A(t) + 4\pi,$$

then we have

$$-\frac{d}{dt} V(t) \leq \left(\int_{\Sigma_t} H^2 d\sigma \right)^{\frac{1}{2}} A^{\frac{1}{2}}(t) \left(\frac{\int_{\Sigma_t} H^2 d\sigma}{4\pi + A(t)} \right)^{\frac{1}{2}} = -\frac{1}{2} \frac{dA(t)}{dt} \left(\frac{A(t)}{4\pi + A(t)} \right)^{\frac{1}{2}}.$$

Denote by T , the time at which the volume enclosed converges to 0, which exists by assumption. Integrating the above inequality from $t = 0$ to $t = T$, we obtain

$$(3.5) \quad V_0 \leq \frac{1}{2} \int_{A(T)}^{A(0)} \left(\frac{x}{4\pi + x} \right)^{\frac{1}{2}} dx,$$

where V_0 denotes the volume of the domain enclosed by Σ_0 . Let A_0 be the area of the geodesic spheres with the enclosed volume equal to V_0 , and then direct calculation shows that

$$(3.6) \quad V_0 = \frac{1}{2} \int_0^{A_0} \left(\frac{x}{4\pi + x} \right)^{\frac{1}{2}} dx.$$

By (3.5) and the above equality, we conclude

$$\frac{1}{2} \int_0^{A_0} \left(\frac{x}{4\pi + x} \right)^{\frac{1}{2}} dx \leq \frac{1}{2} \int_{A(T)}^{A(0)} \left(\frac{x}{4\pi + x} \right)^{\frac{1}{2}} dx,$$

then (1.3) holds.

If equality holds, from the equality case of the Cauchy-Schwarz inequality in (3.3), we know that the mean curvature H of Σ_0 is a constant, and by (3.4) we know $H > 1$. For the same reason in Section 4.4 of [8], we know that Σ_0 is a round sphere. \square

4. SINGULARITIES OF 2-TORI

From the previous section, we know that using the mean curvature flow to prove isoperimetric inequality may be limited by the possibility of singularity formation in flows. Our aim in this section is to prove Theorem 1.2 by using the isoperimetric inequality.

Proof of Theorem 1.2. For any small ϵ , Σ_ϵ is a torus. Then from the inequality (2.5), we have

$$\int_{\Sigma_\epsilon} H^2 d\sigma \geq 2\pi^2 + \text{Area}(\Sigma_\epsilon).$$

Denote the right hand side of inequality (1.4) by C_ϵ . Suppose there is no singularity before the volume it encloses shrinks to zero. Then following the same process in Section 3, we obtain

$$\begin{aligned} \text{Vol}(B_\epsilon) &\leq \frac{1}{2} \int_0^{\text{Area}(\Sigma_\epsilon)} \left(\frac{x}{2\pi^2 + x}\right)^{\frac{1}{2}} dx \\ (4.1) \quad &\leq \frac{1}{2} \int_0^{\text{Area}(\Sigma_\epsilon)} \left(\frac{x}{4\pi + x}\right)^{\frac{1}{2}} dx - \frac{1}{2} C_\epsilon. \end{aligned}$$

Here we used the fact that $\text{Area}(\Sigma_\epsilon) \geq \text{Area}(\partial B_0) - \text{Area}(\partial D_\epsilon)$. From (4.1), we deduce

$$\begin{aligned} -\text{Vol}(D_\epsilon) &\leq \frac{1}{2} \int_0^{\text{Area}(\partial B_0)} \left(\frac{x}{4\pi + x}\right)^{\frac{1}{2}} dx - \text{Vol}(B_0) \\ &\quad + \frac{1}{2} \int_{\text{Area}(\partial B_0)}^{\text{Area}(\partial B_0) + \text{Area}(\partial D_\epsilon)} \left(\frac{x}{4\pi + x}\right)^{\frac{1}{2}} dx - \frac{1}{2} C_\epsilon \\ &< \frac{1}{2} \text{Area}(\partial D_\epsilon) - \frac{1}{2} C_\epsilon, \end{aligned}$$

by equality (3.6) and since $(\frac{x}{4\pi+x})^2 < 1$. Therefore we get

$$-2\text{Vol}(D_\epsilon) < \text{Area}(\partial D_\epsilon) - C_\epsilon,$$

which is a contradiction with (1.4) and therefore the theorem follows. \square

5. SINGULARITIES OF 2-SPHERES

To prove Theorem 1.3, we need an estimate of external diameter of closed hypersurfaces in terms of its area and Willmore energy in \mathbb{H}^3 . The estimate in Euclidean space was obtained by L. Simon in [13] with some universal constant, and another proof was given by Topping. In [15], Topping proved that in \mathbb{R}^3

$$\text{diam}(M) < \frac{2}{\pi} A^{\frac{1}{2}}(M) W^{\frac{1}{2}}(M).$$

By following the same strategy as in [15], we prove a similar estimate in the Hyperbolic space.

Lemma 5.1. *For a closed hypersurface M in \mathbb{H}^3 , we have the estimate*

$$(5.1) \quad \text{diam}(M) \leq \frac{7}{2\pi} A^{\frac{1}{2}}(M)W^{\frac{1}{2}}(M),$$

where $W(M) = \int_M H^2 d\sigma$ and $A(M)$ denotes the area of M .

Proof. For any fixed point $x_0 \in M \subset \mathbb{H}^3$, $0 < \rho < \rho_0$, we denote $d(x) \doteq d(x, x_0)$ and set

$$X \doteq \nabla \cosh d, \text{ and } |X|_\rho = \max\{|X|, \rho\},$$

$$M_\rho = \{x \in M : d(x) \leq \rho\},$$

$$\widetilde{M}_\rho = \{x \in M : \sinh d(x) \leq \rho\}.$$

Note that $|X| = |\sinh(d)\nabla d| = \sinh(d)$ so that $M_\rho = \{x \in M : |X| \leq \rho\}$.

Define a vector field on \mathbb{H}^3 by

$$\Phi(x) \doteq \left(\frac{1}{|X(x)|_\rho^2} - \frac{1}{\rho_0^2}\right)_+ X(x),$$

where $\left(\frac{1}{|X(x)|_\rho^2} - \frac{1}{\rho_0^2}\right)_+ = \max\{0, \frac{1}{|X(x)|_\rho^2} - \frac{1}{\rho_0^2}\}$.

Choose an orthogonal normal frame $\{e_1, e_2\}$ of TM , then direct calculation shows

$$\begin{aligned} \sum_{i=1}^2 \int_M \langle \nabla_{e_i} \Phi, e_i \rangle d\sigma_M &= \sum_{i=1}^2 \int_M e_i \langle \Phi, e_i \rangle - \langle \Phi, \nabla_{e_i} e_i \rangle d\sigma_M \\ &= \sum_{i=1}^2 \int_M e_i \langle \Phi, e_i \rangle - \langle \Phi^\perp, \nabla_{e_i} e_i \rangle - \langle \Phi^T, \nabla_{e_i}^M e_i \rangle d\sigma_M \\ &= \int_M \text{div}^M \Phi^T + 2\langle \Phi, H\nu \rangle d\sigma_M \\ (5.2) \quad &= 2 \int_M \langle \Phi, H\nu \rangle d\sigma_M, \end{aligned}$$

where ∇ denotes the covariant derivative with respect to the standard metric in \mathbb{H}^3 and ν is the outer normal direction of M .

Recall that in standard hyperbolic spaces, we have

$$\nabla^2 \cosh(d) = \cosh(d(x))g_{\mathbb{H}^3},$$

which implies

$$\nabla_{e_i} X = \cosh(d(x))e_i.$$

A short calculation reveals that

$$\sum_{i=1}^2 \langle \nabla_{e_i} \Phi, e_i \rangle = \begin{cases} 2 \cosh(d)(\rho^{-2} - \rho_0^{-2}) & |X| < \rho \\ \frac{2 \cosh(d)|X^\perp|^2}{|X|^4} - \frac{2 \cosh(d)}{\rho_0^2} & \rho < |X| < \rho_0, \\ 0 & \rho_0 < |X| \end{cases}$$

where $X^\perp = \langle X, \nu \rangle \nu$. Hence by integrating the above equality on M and substituting it to (5.2), we find

$$\begin{aligned}
 & \rho^{-2} \int_{\widetilde{M}_\rho} \cosh(d) d\sigma_M - \rho_0^{-2} \int_{\widetilde{M}_{\rho_0}} \cosh(d) d\sigma_M \\
 &= - \int_{\widetilde{M}_{\rho_0, \rho}} \frac{|X^\perp|^2}{|X|^4} \cosh(d) d\sigma_M \\
 (5.3) \quad &+ \int_{\widetilde{M}_{\rho_0, \rho}} (|X|^{-2} - \rho_0^{-2}) \langle X, H\nu \rangle d\sigma_M + \int_{\widetilde{M}_\rho} (\rho^{-2} - \rho_0^{-2}) \langle X, H\nu \rangle d\sigma_M,
 \end{aligned}$$

where $\widetilde{M}_{\rho_0, \rho} \doteq \widetilde{M}_{\rho_0} - \widetilde{M}_\rho$.

For any $x \in \widetilde{M}_{\rho_0, \rho}$, we claim

$$(5.4) \quad -\cosh(d) \frac{|X^\perp|^2}{|X|^4} + \frac{\langle X, H\nu \rangle}{|X|^2} - \rho_0^{-2} \langle X, H\nu \rangle \leq \frac{H^2}{4}.$$

In fact, if $\langle X, H\nu \rangle \leq 0$, by $|X|^{-2} - \rho_0^{-2} \geq 0$ in $\widetilde{M}_{\rho_0, \rho}$, we deduce the left hand side of (5.4) is non-positive, and then inequality (5.4) is obvious. If $\langle X, H\nu \rangle > 0$, by $\cosh(d) \geq 1$, we estimate

$$-\cosh(d) \frac{|X^\perp|^2}{|X|^4} + \frac{\langle X, H\nu \rangle}{|X|^2} - \rho_0^{-2} \langle X, H\nu \rangle \leq -\frac{|X^\perp|^2}{|X|^4} + \frac{\langle X, H\nu \rangle}{|X|^2} \leq \frac{H^2}{4}.$$

Combining (5.4) with (5.3), we get

$$\begin{aligned}
 \rho^{-2} A(\widetilde{M}_\rho) &\leq \rho_0^{-2} A(\widetilde{M}_{\rho_0}) + \int_{\widetilde{M}_{\rho_0, \rho}} \frac{H^2}{4} d\sigma_M + \int_{\widetilde{M}_\rho} (\rho^{-2} - \rho_0^{-2}) \langle X, H\nu \rangle d\sigma_M \\
 &+ \int_{\widetilde{M}_{\rho_0}} \frac{\cosh(d) - 1}{\rho_0^2} d\sigma_M \\
 &\leq \rho_0^{-2} A(\widetilde{M}_{\rho_0}) + \int_{\widetilde{M}_{\rho_0, \rho}} \frac{H^2}{4} d\sigma_M + \frac{1}{2} A(\widetilde{M}_{\rho_0}) \\
 (5.5) \quad &+ \int_{\widetilde{M}_\rho} (\rho^{-2} - \rho_0^{-2}) \langle X, H\nu \rangle d\sigma_M,
 \end{aligned}$$

where we have used that $\frac{\cosh(d)-1}{\rho_0^2} \leq \frac{1}{2}$ in \widetilde{M}_{ρ_0} .

Letting $\rho \rightarrow 0$, the left hand side is π , since M is locally Euclidean and for the same reason as in [15], the limit of the last term in (5.5) vanishes. Thus we get

$$\pi \leq (\rho_0^{-2} + \frac{1}{2}) A(\widetilde{M}_{\rho_0}) + \frac{1}{4} \int_{\widetilde{M}_{\rho_0}} H^2 d\sigma_M.$$

Since $\widetilde{M}_\rho \subset M_\rho$, we have

$$(5.6) \quad \pi \leq (\rho_0^{-2} + \frac{1}{2}) A(M_{\rho_0}) + \frac{1}{4} \int_{M_{\rho_0}} H^2 d\sigma_M.$$

Now we prove Lemma 5.1 by using (5.6). Let $\rho_0 \doteq (\frac{A(M)}{W(M)})^{\frac{1}{2}}$ and choose an integer N such that

$$(5.7) \quad 2N\rho_0 < \text{diam}(M) \leq 2(N+1)\rho_0.$$

We can assume that $N \geq 1$. Otherwise, we have

$$diam(M) \leq 2\rho_0 = 2\left(\frac{A(M)}{W(M)}\right)^{\frac{1}{2}} \leq \frac{1}{2\pi}(A(M)W(M))^{\frac{1}{2}}.$$

Here we have used $\frac{W(M)}{4\pi} \geq 1$ which is deduced from inequality (2.4).

For $N \geq 1$, we can choose $\{y_0, \dots, y_N\}$ in M , such that $d(y_i, y_j) \geq 2\rho_0$ for $i \neq j$ which implies the balls $M_{\rho_0}(y_i) \doteq \{x \in M : d(x, y_i) < \rho_0\}$ are disjoint. Using inequality (5.6) for each $M_{\rho_0}(y_i)$ and summing i from 0 to N , we derive

$$(N + 1)\pi \leq (\rho_0^{-2} + \frac{1}{2})A(M) + \frac{1}{4}W(M).$$

Combining with (5.7), we get

$$\begin{aligned} diam(M) &\leq \frac{2\rho_0}{\pi}[(\rho_0^{-2} + \frac{1}{2})A(M) + \frac{W(M)}{4}] \\ &= \frac{2}{\pi}[(\frac{W(M)}{A(M)})^{\frac{1}{2}} + \frac{1}{2}(\frac{A(M)}{W(M)})^{\frac{1}{2}}]A(M) + \frac{1}{2\pi}(A(M)W(M))^{\frac{1}{2}} \\ &\leq \frac{2}{\pi}[(\frac{W(M)}{A(M)})^{\frac{1}{2}} + \frac{1}{2}(\frac{W(M)}{A(M)})^{\frac{1}{2}}]A(M) + \frac{1}{2\pi}(A(M)W(M))^{\frac{1}{2}} \\ &= \frac{7}{2\pi}A^{\frac{1}{2}}(M)W^{\frac{1}{2}}(M). \end{aligned}$$

In the third line, we have used the inequality $\frac{A(M)}{W(M)} \leq 1$ which is also deduced from inequality (2.4). □

Now we are in position to give a lower bound for the area of a dumbbell in terms of its length d , which must be satisfied for its neck not to pinch off under the flow. Then we construct a sphere in the shape of a dumbbell, violating the lower bound, and we get a contradiction.

Proof of Theorem 1.3. Suppose M_0 is a topological sphere embedded in \mathbb{H}^3 which encloses two round spheres B_0 , both of radius 1, with centres separated by a distance $d > 0$. Then the diameter of M_0 satisfies

$$diam(M_0) \geq d + 2.$$

By the comparison principle of the flow (cf. Proposition 10.4 in [16]), we have the estimate of the diameter

$$(5.8) \quad diam(M_t) \geq d + 2r(t),$$

where $r(t)$ satisfies the following equation given by containment principle

$$e^t \cosh r(t) = \cosh 1,$$

for $t < T_0 \doteq \ln \cosh 1$.

Combining (5.1) with (5.8), we get

$$d + 2r(t) \leq \frac{7}{2\pi}(A(M_t)W(M_t))^{\frac{1}{2}} = \frac{7}{2\pi}[A(-\frac{1}{2}\frac{d}{dt}A)]^{\frac{1}{2}} = \frac{7}{4\pi}[-\frac{d}{dt}A^2(M_t)]^{\frac{1}{2}},$$

which implies

$$\frac{16\pi^2}{49}(d + 2r(t))^2 \leq -\frac{d}{dt}A^2(M_t).$$

If there is non-singularity of M_t before T_0 , we have

$$\frac{16\pi^2}{49} \int_0^{T_0} (d + 2r)^2 dt \leq A^2(M_0) - A^2(M_{T_0}) \leq A^2(M_0),$$

which yields

$$A^2(M_0) \geq \frac{16\pi^2}{49} \int_0^{T_0} (d^2 + 4r^2 + 4rd) dt,$$

in particular that

$$(5.9) \quad A^2(M_0) \geq \frac{16\pi^2 T_0 d^2}{49}.$$

If (5.9) fails to hold, then a singularity must develop in the flow before time

$$t_0 = \frac{49A^2(M_0)}{16\pi^2 d^2}.$$

But we can choose M_0 to be the shape of dumbbell with a thin cylinder such that $A(M_0)$ satisfies

$$A(M_0) \leq 2Vol(B_0) + \epsilon d.$$

It is easy to see that inequality (5.9) is violated for some sufficiently large d , whenever $\epsilon < 4\pi T_0/7$. This completes the proof of Theorem 1.3. \square

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