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# A MULTIPLICITY BOUND FOR GRADED RINGS AND A CRITERION FOR THE COHEN-MACAULAY PROPERTY

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Abstract. Let $R$ be a polynomial ring over a field. We prove an upper bound for the multiplicity of $R/I$ when $I$ is a homogeneous ideal of the form $I = J+(F)$, where $J$ is a Cohen-Macaulay ideal and $F \notin J$. The bound is given in terms of two invariants of $R/J$ and the degree of $F$. We show that ideals achieving this upper bound have high depth, and provide a purely numerical criterion for the Cohen-Macaulay property. Applications to quasi-Gorenstein rings and almost complete intersections are given.

## 1. Introduction

Bounds on the multiplicity of a ring $S$ in terms of other invariants of $S$ (and analyses of the rings achieving these bounds) have attracted strong interest over the last 130 years, from the classical lower bound $\deg X \geq \operatorname{codim} X + 1$ for non-degenerate projective varieties to the still open Eisenbud-Green-Harris Conjecture [3] or the Huneke-Srinivasan Multiplicity Conjecture, proved a few years ago by Eisenbud and Schreyer [5].

In the present paper, we prove a new upper bound for the multiplicity of a wide class of graded rings and study the defining ideals achieving this bound. Let $R$ be a polynomial ring over a field $k$, $J$ a homogeneous Cohen-Macaulay ideal, $F \notin J$ a homogeneous element and $I = J+(F)$. Let $e(S)$ denote the multiplicity of a graded ring $S$. If $F$ is regular on $R/J$, it is well-known that $e(R/I) = e(R/J) \cdot \deg(F)$. If $F$ is a zero-divisor on $R/J$ (that is, $\operatorname{ht} I = \operatorname{ht} J$) one has the elementary inequality $e(R/I) \leq e(R/J) - 1$. In the present paper, we prove the sharper upper bound

$$e(R/I) \leq e(R/J) - \max\{1, s - \deg(F) + 1\}, \tag{1.1}$$

where $s = s(R/J)$ is the difference between the smallest graded shift appearing in the last step of a minimal graded free resolution of $R/J$ and $\operatorname{ht} J$. This bound, by its nature, is more restrictive when $\deg(F)$ is small or when $R/J$ is level, a particular instance of which is when $R/J$ is Gorenstein.

The inequality (1.1) generalizes a previous result of Engheta bounding the multiplicity of a homogeneous almost complete intersection in terms of the degrees of its minimal generators [6, Theorem 1]; see Corollary 2.3.

We name ideals $I$ achieving equality in (1.1) *ideals of maximal multiplicity*. Our second main result states that these ideals define factor rings with high depth.

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More precisely, $R/I$ is Cohen-Macaulay if the degree of $F$ is sufficiently small, namely, $\deg(F) \leq s$. In the remaining cases, that is, when $\deg(F) > s$, we show $\operatorname{depth}(R/I) = \dim(R/I) - 1$, provided $R/J$ is Gorenstein (Theorem 2.7). A consequence of this result is that almost complete intersections generated in a single degree and having maximal multiplicity are Cohen-Macaulay (Corollary 2.9), a result that we employ in [11]. Also, one should contrast this result with the abundance of examples of almost complete intersection ideals $I$ (even generated in a single degree) of any other multiplicity that are not Cohen-Macaulay (see Remark 2.10).

We then provide a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity and exhibit examples, which include rational normal curves and $\mathfrak{m}$-primary ideals (see Theorem 2.13, Examples 2.15 and 2.17, and Corollary 2.16). Among the applications, we use linkage to deduce a lower bound for the multiplicity of any graded quasi-Gorenstein ring $S$ in terms of its $a$-invariant and dimension (Proposition 3.2). If this lower bound is achieved, then $S$ is actually Gorenstein (see Theorem 3.3). We also prove that ideals of maximal multiplicity are unmixed if and only if they are Cohen-Macaulay (Corollary 3.5).

As a final application, we remark that part of the material from this paper is employed in the forthcoming paper [11], where, motivated by a question of Stillman [18], we prove a close-to-optimal upper bound on the projective dimension of any ideal generated by 4 quadratic polynomials. One of our original motivations in the present paper was, in fact, to find a more structural reason for the fact (proved in an earlier draft of [11]) that any almost complete intersection ideal (not necessarily unmixed) of multiplicity 6 generated by 4 quadrics is Cohen-Macaulay.

The structure of the paper is the following: in Section 2, we prove the three main results, namely the upper bound (1.1), the high depth properties of ideals of maximal multiplicity, and a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. We employ these results to obtain a sufficient condition for almost complete intersections of quadrics to be Cohen-Macaulay and provide examples. In Section 3, we prove a lower bound for the multiplicity of quasi-Gorenstein rings, a multiplicity-based sufficient condition for quasi-Gorenstein rings to be Gorenstein, and analogies between ideals of maximal multiplicity and ideals of multiplicity one.

## 2. The main results

Throughout this paper $R$ is a polynomial ring over a field $k$ and $\mathfrak{m}$ denotes its unique homogeneous maximal ideal. Since we may harmlessly replace $k$ by $k(X)$, by base change we may always assume $|k| = \infty$.

Recall that the *Hilbert function* of a finitely generated graded $R$-module $M$ is the numerical function given by $HF_M(i) = \dim_k M_i$ for all $i \in \mathbb{Z}$. If $A$ is a graded artinian factor ring of $R$, then $e(A) = \sum_{i=0}^{\infty} HF_A(i)$, and the *socle* of $A$ is the $k$-vector space $\operatorname{Soc}(A) = 0 :_A \mathfrak{m}_A$, where $\mathfrak{m}_A = \mathfrak{m}A$. A *socle element* is an element of $\operatorname{Soc}(A)$.

Let $J$ be a homogeneous Cohen-Macaulay $R$-ideal of height $g$. We consider the invariant

$$s(R/J) = \min\{i \mid \operatorname{Tor}_g^R(R/J, k)_i \neq 0\} - g.$$

We recall here two additional interpretations of this invariant. First, when $|k| = \infty$, $s(R/J)$ is the smallest degree of a non-zero homogeneous socle element of a general artinian reduction of $R/J$ (a *general artinian reduction* of $T = R/J$ is a ring $T/(L_1, \ldots, L_d)$ where $L_i$ are general linear forms and $d = \dim(T)$). Second, one has $s(R/J) = c(R/J) + \dim(R/J)$, where $c(R/J) = -\max\{i \mid [k \otimes_{R/J} \omega_{R/J}]_i \neq 0\}$ and $\omega_{R/J}$ denotes the graded canonical module of $R/J$.

We also recall that the *unmixed part* of an $R$-ideal $K$, denoted $K^{un}$, is the intersection of the primary components of $K$ of minimal height. The ideal $K$ is *unmixed* if $K = K^{un}$. The associativity formula yields $e(R/K) = e(R/K^{un})$ and the following easy remark:

*Remark* 2.1. If $K \subseteq L$ are unmixed ideals of the same height, then $e(R/K) \geq e(R/L)$, and $e(R/K) = e(R/L)$ if and only if $K = L$.

We now prove our first main result.

**Theorem 2.2.** *Let $J$ be a homogeneous Cohen-Macaulay $R$-ideal, and let $F \notin J$ be homogeneous. Set $I = J + (F)$ and assume* $\operatorname{ht} I = \operatorname{ht} J$. *Then*

(i) $e(R/I) \leq e(R/J) - \max\{1, s(R/J) - \deg(F) + 1\}$;
(ii) *if equality is achieved in* (i) *and* $\deg(F) \leq s(R/J)$, *then $R/I$ is Cohen-Macaulay.*

*Proof.* We prove both statements at once. Set $\delta = \deg(F)$ and $s = s(R/J)$. First, assume $1 > s - \delta + 1$. We need to show that $e(R/I) \leq e(R/J) - 1$. This follows by the equality $e(R/I) = e(R/I^{un})$, the inclusions $J \subsetneq I \subseteq I^{un}$ and Remark 2.1. Hence, we may assume $1 \leq s - \delta + 1$, that is, $\delta \leq s$. We need to show that $e(R/I) \leq e(R/J) - (s - \delta + 1)$ and, if equality is achieved, then $R/I$ is Cohen-Macaulay.

Let $A$ be a general artinian reduction of $R/J$, and denote by $^-$ images in $A$. We claim that both statements hold if $\overline{F} \neq 0$ in $A$. Indeed, in this case, we set $n = \max\{t \in \mathbb{N}_0 \mid \overline{F}\mathfrak{m}_A^t \neq 0\} < \infty$ and observe that $HF_{A/(\overline{F})}(i) \leq HF_A(i) - 1$ for every $\delta \leq i \leq \delta + n$. This proves $e(A/(\overline{F})) \leq e(A) - (n+1)$. Let $G$ be a homogeneous element of degree $n$ such that $\overline{FG} \neq 0$ in $A$. By the definition of $n$, we have $0 \neq \overline{FG} \in \operatorname{Soc}(A)$, whence $\delta + n = \deg(\overline{FG}) \geq s$. This proves $s - \delta \leq n$ and gives $e(A/(\overline{F})) \leq e(A) - (n+1) \leq e(A) - (s - \delta + 1)$.

A well-known result of Serre (e.g. [1, Theorem 4.7.10]) yields $e(A) = e(R/J)$ and $e(R/I) \leq e(A/(\overline{F}))$; hence we obtain the inequalities

$$e(R/I) \leq e(A/(\overline{F})) \leq e(A) - (s - \delta + 1) = e(R/J) - (s - \delta + 1).$$

If, moreover, $e(R/I) = e(R/J) - (s - \delta + 1)$, then we have the equality $e(R/I) = e(A/(\overline{F}))$ and, by [1, Theorem 4.7.10], $R/I$ is Cohen-Macaulay.

Therefore, to finish the proof it suffices to prove $\overline{F} \neq 0$ in $A$. Set $\widetilde{R} = R/J$ and let $^\sim$ denote images in $\widetilde{R} = R/J$. Since $A$ is a general artinian reduction of $\widetilde{R}$, to show $\overline{F} \neq 0$ in $A$, we need to prove that $\widetilde{F}$ is not in the intersection of all the (general) minimal reductions of $\widetilde{\mathfrak{m}}$, that is, $\widetilde{F} \notin \operatorname{core}(\widetilde{\mathfrak{m}})$ in $\widetilde{R} = R/J$, where $\operatorname{core}(\mathfrak{m})$ denotes the core of $\mathfrak{m}$. Since $\widetilde{R} = R/J$ is Cohen-Macaulay, it follows by work of Fouli, Polini and Ulrich [8, Corollary 4.3(b)] that $\operatorname{core}(\widetilde{\mathfrak{m}}) \subseteq \widetilde{\mathfrak{m}}^{s+1}$. The inequality $s \geq \delta$ now yields $\widetilde{F} \notin \operatorname{core}(\widetilde{\mathfrak{m}})$ for degree reasons. $\square$

As a first application we recover an upper bound (first proved by Engheta [6]) for the multiplicity of $R/I$, where $I$ is any homogeneous almost complete intersection.

**Corollary 2.3** (Engheta [6, Theorem 1]). *Let $I = (f_1, \ldots, f_{g+1})$ be a homogeneous almost complete intersection of height $g$ in $R$, where $f_1, \ldots, f_g$ form a homogeneous regular sequence. Set $d_i = \deg(f_i)$ for every $i = 1, \ldots, g+1$. Then,*

$$e(R/I) \leq \prod_{i=1}^{g} d_i - \max\left\{1, \sum_{i=1}^{g}(d_i - 1) - (d_{g+1} - 1)\right\}.$$

*In particular, if $I$ is generated by forms of the same degree $d$ and $g > 1$, then*

$$e(R/I) \leq d^g - (d-1)(g-1).$$

*Proof.* Since $J = (f_1, \ldots, f_g)$ is a complete intersection, we have $s(R/J) = \sum_{i=1}^{g}(d_i - 1)$. Now, Theorem 2.2 applied to $I = J + (f_{g+1})$ proves the statement. □

Although the bound of Corollary 2.3 can be sharp (see Remark 2.10), G. Caviglia remarked that if this is the case, then either $g = 2$ or $g = 3$ and $d_1 = d_2 = d_3 = 2$. Instead, the bound of Theorem 2.2 is more general and, indeed, is achieved in a wider variety of situations (see, for instance, Examples 2.15 and 2.17 and Corollary 2.16).

Before stating our second main result, we recall a few definitions. An ideal $I$ is called *almost Cohen-Macaulay* if $\operatorname{depth}(R/I) \geq \dim(R/I) - 1$. Also, a homogeneous Cohen-Macaulay ideal $J$ of height $g$ is said to be *level* if there exists only one positive integer $i$ such that $[\operatorname{Tor}_g^R(R/J, k)]_i \neq 0$. An ideal $J$ is *Gorenstein* if $R/J$ is Gorenstein, that is, $\operatorname{Tor}_g^R(R/J, k) \cong k$. Clearly, every homogeneous Gorenstein ideal is level.

We will also need a few definitions and results from liaison theory. Two (homogeneous) ideals $J$ and $K$ in $R$ are *linked*, denoted $J \sim K$, if there exists a (homogeneous) Gorenstein ideal $G$ with $K = G :_R J$ and $J = G :_R K$. Sometimes we say $J$ and $K$ are linked *by* $G$. Linkage (by complete intersections) has been studied since the nineteenth century, although its first modern treatment appeared in the ground-breaking paper by Peskine and Szpiro [19]. Properties of liaison by Gorenstein ideals were then studied in [23]. We refer the interested reader to [16], [12] and their references.

**Proposition 2.4** (Peskine-Szpiro, Schenzel [19], [23]). *Let $J$ be an unmixed ideal of $R$ of height $g$. If $G \subseteq J$ is a height $g$ Gorenstein ideal, and $K = G : J$, then $J \sim K$.*

We will use the following simple fact: If $G$ is a Gorenstein ideal contained in an ideal $J$ and $\operatorname{ht} G = \operatorname{ht} J$, then $G : J = G : J^{un}$; that is, $G : J$ is linked to $J^{un}$.

**Proposition 2.5** (Peskine-Szpiro, Golod, Schenzel [19], [9], [23]). *If $J \sim K$, then*

(a) *$S/J$ is Cohen-Macaulay if and only if $S/K$ is Cohen-Macaulay;*
(b) *$e(S/J) + e(S/K) = e(S/G)$, where $G$ is the ideal defining the link $J \sim K$.*

The following linkage result is well-known (its proof follows, for instance, along the same lines of the proof of [7, Theorem 3]).

**Lemma 2.6.** *Let $I = J + (F)$, where $J$ is Gorenstein ideal with* $\operatorname{ht}(J) = \operatorname{ht}(I)$, *and $F \notin J$. If $L$ is any ideal linked to $I^{un}$, then* $\operatorname{pd}(R/I) \leq \operatorname{pd}(R/L) + 1$.

In the setting of Theorem 2.2, we say that $I$ has *maximal multiplicity* if there exist a Cohen-Macaulay ideal $J$ and an element $F \notin J$ with $I = J + (F)$, $\operatorname{ht}(I) = \operatorname{ht}(J)$

and $e(R/I) = e(R/J) - \max\{1, s(R/J) - \deg(F) + 1\}$. If $J$ and $F$ are as above, we say they form a *maximal decomposition* of $I$.

We can now state our second main result, proving the high depth of $R/I$.

**Theorem 2.7.** *Let $I$ be a homogeneous $R$-ideal of maximal multiplicity, and let $I = J + (F)$ be a maximal decomposition of $I$.*

(a) *If $\deg(F) \leq s(R/J)$, then $R/I$ is Cohen-Macaulay.*
(b) *If $R/J$ is level, then $R/I$ is Cohen-Macaulay if and only if $\deg(F) \leq s(R/J)$.*

Note that although $J$ is Cohen-Macaulay, the ideal $I$ may not even be unmixed.

*Proof.* Set $g = \operatorname{ht} J = \operatorname{ht} I$. Assertion (a) was proved in Theorem 2.2 (ii). To prove assertion (b) we need to show that if $R/I$ is Cohen-Macaulay, then $\deg(F) \leq s(R/J)$. Let $L_1, \ldots, L_d$ be general linear forms, where $d = \dim(R/J) = \dim(R/I)$, and let $^-$ denote images modulo $L_1, \ldots, L_d$. Since $R/I$ is Cohen-Macaulay, by Theorem [1, Theorem 4.7.10], we have $e(R/I) = e(\overline{R}/\overline{I})$. Moreover, since $R/J$ is level, we have $\operatorname{Soc}(\overline{R}/\overline{J}) = \overline{\mathfrak{m}_{R/J}}^s$, where $s = s(R/J)$. Now, assume by contradiction that $\deg(F) > s(R/J)$. Then $\overline{F} \in \overline{\mathfrak{m}_{R/J}}^{s+1} = 0$ in $\overline{R}/\overline{J}$, that is, $\overline{R}/\overline{I} = \overline{R}/\overline{J}$. This implies that

$$e(R/I) = e(\overline{R}/\overline{I}) = e(\overline{R}/\overline{J}) = e(R/J).$$

Since both $I$ and $J$ are unmixed, Remark 2.1 implies $J = I$, which is a contradiction.

We now prove assertion (c). The assumptions imply $e(R/I) = e(R/J) - 1$. Let $L = J : I$ and note that, by Lemma 2.6, $\operatorname{pd}(R/I) \leq \operatorname{pd}(R/L) + 1$. Moreover, $R/L$ is unmixed with $e(R/L) = e(R/J) - e(R/I) = 1$ (by Proposition 2.5); therefore, by a well-known result of Samuel (see Proposition 3.4), $R/L$ is Cohen-Macaulay. Then $\operatorname{pd}(R/I) \leq g + 1$. Hence, by the Auslander-Buchsbaum formula, we have $\operatorname{depth}(R/I) \geq \dim(R/I) - 1$. Finally, by assertion (b), $R/I$ is not Cohen-Macaulay, because $\deg(F) > s(R/J)$. This yields $\operatorname{depth}(R/I) = \dim(R/I) - 1$. □

The next simple example shows that the bound given in Theorem 2.2 can be (trivially) sharp, and there are ideals of maximal multiplicity that are not Cohen-Macaulay.

**Example 2.8.** Let $R = k[x, y, z]$, $J = (x^2, xy, y^2)$ and $F = xz$. Then $I = J + (F)$ is not unmixed; $R/I$ has maximal multiplicity and is almost Cohen-Macaulay.

We now apply Theorem 2.7 to almost complete intersection ideals generated by quadrics.

**Corollary 2.9.** *Let $I$ be an almost complete intersection of height $g \geq 1$ generated by homogeneous elements of the same degree 2. If $e(R/I) = 2^g - (g - 1)$, then $R/I$ has maximal multiplicity and is Cohen-Macaulay.*

*Proof.* It follows by Corollary 2.3 together with Theorem 2.7 (b). □

If $I$ is an almost complete intersection generated by 4 quadrics (that is, $g = 3$ and $d = 2$), then Corollary 2.3 gives $e(R/I) \leq 6$, and Corollary 2.9 yields that $R/I$ is Cohen-Macaulay if $e(R/I) = 6$. This fact is employed in [11].

The following remark shows that, without further assumptions, no other values of $e(R/I)$ guarantee the Cohen-Macaulayness of $R/I$.

*Remark* 2.10. Let $I$ be a height three ideal generated by 4 quadrics.

(a) If $e(R/I) = 6$, then $R/I$ is Cohen-Macaulay.
(b) For every $1 \leq e \leq 6$, there are examples of $I$ with $e(R/I) = e$, and if $e \neq 6$, $R/I$ is not Cohen-Macaulay.

Assertion (a) follows by Corollary 2.9. Assertion (b) can be seen, for instance, as follows. Take $R = k[a, b, c, x, y, z]$,

(1) if $I = (ax, by, cz, x^2 + y^2 + z^2)$, then $e(R/I) = 1$ and $\mathrm{pd}(R/I) = 4$;
(2) if $I = (ax, by, xy+xz+yz, x^2+y^2+z^2)$, then $e(R/I) = 2$ and $\mathrm{pd}(R/I) = 4$;
(3) if $I = (ax + by + cz, x^2, y^2, z^2)$, then $e(R/I) = 3$ and $\mathrm{pd}(R/I) = 6$;
(4) if $I = (ax, x^2, y^2, z^2)$, then $e(R/I) = 4$ and $\mathrm{pd}(R/I) = 4$;
(5) if $I = (ax + by + cz, bx + cy + az, cx + ay + bz, bx + cy - bz - cz)$, then $e(R/I) = 5$ and $\mathrm{pd}(R/I) = 4$;
(6) if $I = (x^2, y^2, z^2, xy)$, then $e(R/I) = 6$ and, by part (a), $\mathrm{pd}(R/I) = \mathrm{ht}\, I = 3$.

Next, we want to provide a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. The first step consists of describing the structure of the colon ideal $J : F$.

**Lemma 2.11.** *Let $I$ be a Cohen-Macaulay homogeneous ideal of height $g$ having maximal multiplicity. If $J$ and $F$ form a maximal decomposition of $I$, then $J : F = (x_1, \ldots, x_{g-1}, q)$, for some linearly independent linear forms $x_1, \ldots, x_{g-1}$ and an element $q \notin (x_1, \ldots, x_{g-1})$.*

*Proof.* From the short exact sequence

$$(\star) \qquad 0 \longrightarrow R/J : F[-\deg(F)] \longrightarrow R/J \longrightarrow R/I \longrightarrow 0$$

one obtains $e(R/J) - e(R/I) = e((R/J : F)[-\deg(F)]) = e(R/J : F)$. First, assume $I$ is $\mathfrak{m}$-primary. If $\deg(F) \geq s(R/J)$, then, by assumption of maximal multiplicity, $e(R/I) = e(R/J) - 1$, whence $e(R/J : F) = 1$. Then $J : F = \mathfrak{m}$, and the statement follows. We may then assume $\deg(F) < s(R/J)$. Since $J$ and $F$ form a maximal decomposition of $I$, the proof of Theorem 2.2 gives $HF_{R/I}(i) = HF_{R/J}(i) - 1$ for all $\deg(F) \leq i \leq s(R/J)$. This yields that $HF_{R/J:F}(i) = 1$ if $0 \leq i \leq s(R/J) - \deg(F)$; hence, there exist linearly independent linear forms $x_1, \ldots, x_{g-1}, x_g$ such that $J : F = (x_1, \ldots, x_{g-1}, x_g^c)$, and the statement follows.

Now assume $\dim(R/I) = d > 0$. Let $L_1, \ldots, L_d$ be general linear forms in $R$, set $L = (L_1, \ldots, L_d)$ and let $^-$ denote images in $\overline{R} = R/L$. Since $I$ is Cohen-Macaulay, $L_1, \ldots, L_d$ form a regular sequence on $R/I$; then, applying the functor $- \otimes_R R/L$ to the short exact sequence $(\star)$, one obtains the short exact sequence

$$0 \longrightarrow \overline{R}/\overline{J : F} \longrightarrow \overline{R}/\overline{J} \longrightarrow \overline{R}/\overline{I} \longrightarrow 0.$$

Clearly, $\overline{R} = R/L$ is still a polynomial ring, $\overline{I}$ is a homogeneous $\overline{\mathfrak{m}}$-primary ideal, $\overline{R}/\overline{I}$ has maximal multiplicity, and $\overline{J}$ and $\overline{F}$ form a maximal decomposition of $\overline{I}$. Then, by the above, one has $\overline{J : F} = \overline{J} : \overline{F} = (\overline{z_1}, \ldots, \overline{z_{g-1}}, \overline{z_g}^c)$ for some linearly independent linear forms $z_1, \ldots, z_g$ of $\overline{R}$. The statement now follows by lifting this equality back to $R$. $\square$

For the rest of the paper, a complete intersection $C$ of height $g$ containing $g - 1$ linearly independent linear forms is called an *almost linear complete intersection*.

**Proposition 2.12.** *Let $I$ be a Cohen-Macaulay homogeneous ideal of height $g$ having maximal multiplicity. Then there exists an almost linear complete intersection $C'$ such that $(I+C')/C'$ is cyclic.*

*Proof.* If $I$ is contained in an almost linear complete intersection, then $(I+C')/C'=0$ and the statement follows trivially. We may then assume $I$ is not contained in any almost linear complete intersection. Let $J$ and $F$ form a maximal decomposition of $I$. By Lemma 2.11, the ideal $C'=J:F$ is an almost linear complete intersection. Note that $J\subseteq I\cap C'$. Since $I$ is not contained in $C'$, then $(I+C')/C'$ is non-zero. Now, the natural mapping $I/J\to I/I\cap C'\cong (I+C')/C'\to 0$ together with the assumption that $I/J$ is cyclic yields that also $(I+C')/C'$ is cyclic. □

Next, we prove a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. The assumption of $(I+C')/C'$ being cyclic is necessary, by Proposition 2.12.

**Theorem 2.13.** *Let $I$ be a homogeneous Cohen-Macaulay ideal of height $g$. If there exists an almost linear complete intersection $C'$ satisfying the following two conditions:*

(i) *$(I+C')/C'$ is non-zero and cyclic, generated by an element of degree $\delta\geq 1$, and*
(ii) *$e(R/C')\leq \max\{1, s(R/I)-\delta+1\}$,*

*then $I$ has maximal multiplicity.*

*Proof.* Let $F$ be a homogeneous element of $I$ of degree $\delta\geq 1$ whose image generates the cyclic module $(I+C')/C'$. Set $J=C'\cap I$ and note that $I=J+(F)$. Set $C=J:F=C':F$. Since $C'\subseteq C$ are unmixed of the same height, the ideal $C$ is again an almost linear complete intersection and $e(R/C)\leq e(R/C')$. Write $C=(x_1,\ldots,x_{g-1},q)$ and note that $e(R/C)=\deg(q)\leq e(R/C')\leq \max\{1,s(R/I)-\delta+1\}$. We need to show that $J$ is Cohen-Macaulay and $e(R/I)=e(R/J)-\max\{1,s(R/J)-\delta+1\}$.

Since $I$ and $C$ are Cohen-Macaulay ideals of height $g$, the short exact sequence

$$(\star)\qquad 0\longrightarrow (R/C)[-\delta]\xrightarrow{\cdot F} R/J\longrightarrow R/I\longrightarrow 0$$

implies that $R/J$ is Cohen-Macaulay too. Now, if $1\geq s(R/I)-\delta+1$, then $e(R/C)\leq 1$, and since $C$ is a proper ideal, then $e(R/C)=1$, yielding that $e(R/I)=e(R/J)-e(R/J:F)=e(R/J)-1$. This proves that $I$ has maximal multiplicity. We may then assume $s(R/I)-\delta+1>1$. The Horseshoe Lemma applied to $(\star)$ gives $s(R/J)\geq \min\{s(R/I), s(R/C[-\delta])\}$. Since $C$ is an almost linear complete intersection, we have

$$s(R/C[-\delta])=\delta+s(R/C)=\delta+(\deg(q)+g-1)-g=\delta+\deg(q)-1,$$

whence we obtain

$$s(R/J)\geq\min\{s(R/I),s(R/C[-\delta])\}=\min\{s(R/I),\delta+\deg(q)+-1\}=\delta+\deg(q)-1,$$

where the last equality holds because the inequalities

$$\deg(q)=e(R/C)\leq e(R/C')\leq \max\{1,s(R/I)-\delta+1\}=s(R/I)-\delta+1$$

imply that $\delta+\deg(q)-1 \leq s(R/I)$. Then, we have obtained $\delta+\deg(q)-1 \leq s(R/J)$, that is, $\deg(q) \leq s(R/J) - \delta + 1$. We now apply Theorem 2.2 and obtain

$$\begin{array}{lll} e(R/J)-e(R/C) & = e(R/I) & \leq e(R/J) - \max\{1, s(R/J) - \delta + 1\} \\ & \leq e(R/J) - \max\{1, \deg(q)\} & = e(R/J) - \deg(q) \\ & = e(R/J) - e(R/C) & \end{array}$$

proving that $e(R/I) = e(R/J) - \max\{1, s(R/J) - \delta + 1\}$. □

Note that the proof of Theorem 2.13 is constructive, in the sense that if $(I + C')/C'$ is non-zero, cyclic and $e(R/C') \leq \max\{1, s(R/I) - \delta + 1\}$, then one can explicitly construct a maximal decomposition $I = J + (F)$ of the ideal $I$.

We isolate the special case where $C'$ is a *linear prime*, that is, where $C'$ is a prime ideal generated by linear forms.

**Corollary 2.14.** *Let $I$ be a homogeneous Cohen-Macaulay ideal of height $g$. If there exists a linear prime $C'$ of height $g$ such that $(I + C')/C'$ is cyclic and non-zero, then $I$ has maximal multiplicity.*

*Proof.* By assumption $e(R/C') = 1$, so the multiplicity condition of Theorem 2.13 is trivially satisfied. □

We now exhibit two classes of ideals having maximal multiplicity.

**Example 2.15.** Let $I$ be any homogeneous $\mathfrak{m}$-primary ideal. Then $I$ has maximal multiplicity.

*Proof.* Let $\{F_1, \dots, F_r\}$ be a minimal generating set of $I$, let $I' = (F_1, \dots, F_{r-1})$, $F = F_r$, and let $J$ be the $\mathfrak{m}$-primary ideal $J = I' + F \cdot \mathfrak{m}$. Then $I$ has maximal multiplicity because $I = J + (F)$ and $e(R/I) = e(R/J) - 1$. □

Also ideals generated by the 2 by 2 minors of catalecticant matrices have maximal multiplicity. For instance, ideals defining rational normal curves have maximal multiplicity.

**Corollary 2.16.** *Fix integers $d \geq 1$, $r \geq 2$ and $N \geq 3$. Then the ideal $I = I_2(A)$ generated by the 2-minors of the matrix*

$$A = \begin{pmatrix} x_1^d & x_2^d & \dots & x_{N-r+2}^d \\ x_2^d & x_3^d & \dots & x_{N-r+1}^d \\ \vdots & \vdots & \vdots & \vdots \\ x_r^d & x_{r+1}^d & \dots & x_{N+1}^d \end{pmatrix}$$

*has maximal multiplicity.*

*In particular, the defining ideals of the rational normal curves of $\mathbb{P}^N$ for any $N \geq 3$ have maximal multiplicity.*

*Proof.* It is known that $\mathrm{ht}(I) = N - 1$. Set $C' = (x_2, \dots, x_N)$, and note that $(I + C')/C'$ is cyclic and non-zero, generated by the image of $F = x_1^d x_{N+1}^d - x_r^d x_{N-r+2}^d$ in $R/C'$. Then, by Corollary 2.14, $I$ has maximal multiplicity, and a maximal decomposition of $I$ is given by $I = J + (F)$, where $J$ is the ideal generated by all the 2 by 2 minors of $A$ except for $F$. Rational normal curves consist of the special case where $r = 2$ and $d = 1$. □

We remark that, in contrast with the ideals $I$ satisfying equality in Corollary 2.3, these ideals can be generated in arbitrarily high degrees and can have arbitrarily large heights.

We conclude this section with a class of almost complete intersections $I$ of height 3, generated in a single degree $d \geq 4$ and having maximal multiplicity (compare with the discussion after Corollary 2.3). Note that, for these ideals, there is no linear prime $C'$ of height 3 such that $(I + C')/C'$ is cyclic; hence one cannot use Corollary 2.14 to prove that $I$ has maximal multiplicity.

**Example 2.17.** Fix $t \geq 1$, and let $f_i$, $g_i$, $h_i$, where $i = 1, 2$, be irreducible polynomials of the same degree $t + 1$ in disjoint sets of variables such that $f_1$ is contained in a linear prime of height 2. Set $F = h_1h_2$ and $J = (f_1f_2, g_1g_2, f_1g_1h_1)$. Then $I = J + (F)$ is a Cohen-Macaulay almost complete intersection that has maximal multiplicity.

*Proof.* We first show that $I$ is Cohen-Macaulay. Observe that the ideals $(f_l, g_i, h_j)$ with $1 \leq l \leq 2$, $1 \leq l = i \leq 2$ and $1 \leq j \leq 2$ are all prime. Since

$$I = \left( \bigcap_{1 \leq i \leq 2, 1 \leq j \leq 2} (f_1, g_i, h_j) \right) \cap (f_2, g_1, h_1) \cap (f_2, g_1, h_2) \cap (f_2, g_2, h_1),$$

then $I$ is unmixed. Also, since $C_0 = (f_1f_2, g_1g_2, h_1h_2)$ is a complete intersection of height 3 contained in $I$, then $I \sim C_0 : I$ by Proposition 2.4. Since $C_0 : I = (f_2, g_2, h_2)$ is a complete intersection, then by Proposition 2.5, the ideal $I$ is Cohen-Macaulay.

Let $(x_1, x_2)$ be a linear prime of height 2 containing $f_1$, and set $C' = (x_1, x_2, g_1)$. Then $(I + C')/C'$ is cyclic, generated by the image of $F = h_1h_2$ in $R/C'$. We have $s(R/I) = 5t + 2$; hence $s(R/I) - \deg(F) + 1 = (5t + 2) - (2t + 2) + 1 = 3t + 1 > 3$ for every $t \geq 1$. Then

$$e(R/C') = 2 < 3t + 1 = \max\{1, s(R/I) - \deg(F) + 1\}.$$

Then $I$ has maximal multiplicity by Theorem 2.13. $\square$

For instance, let $R = k[x_1, \ldots, x_8, y_1, \ldots, y_8, z_1, \ldots, z_8]$, $f_1 = x_1^t x_2 - x_3^t x_4$, $g_1 = y_1^t y_2 - y_3^t y_4$, $h_1 = z_1^t z_2 - z_3^t z_4$, $f_2 = x_5^t x_6 - x_7^t x_8$, $g_2 = y_5^t y_6 - y_7^t y_8$, $h_2 = z_5^t z_6 - z_7^t z_8$, and $F = h_1h_2$. Then $I = (f_1f_2, g_1g_2, f_1g_1h_1, F)$ is a Cohen-Macaulay almost complete intersection that has maximal multiplicity and is generated in degree $2t + 2 \geq 4$ for any $t \geq 1$.

## 3. Application to quasi-Gorenstein rings

In this section we prove a lower bound for the multiplicity of quasi-Gorenstein rings and a sufficient condition for a quasi-Gorenstein ring to be Gorenstein.

We first recall the definition of graded quasi-Gorenstein rings (also known as 1-Gorenstein rings). Recall that the canonical module of a $d$-dimensional graded ring $S$ with homogeneous maximal ideal $\mathfrak{m}_S$ is defined as $\omega_S = \operatorname{Hom}_k(H^d_{\mathfrak{m}_S}(S), E)$, where $E = E_S(S/\mathfrak{m}_S)$ is the injective envelope of the residue field of $S$.

A graded ring $S$ is *quasi-Gorenstein* if $\omega_S \cong S(a)$ for some integer $a$. An ideal $Q$ is said to be *quasi-Gorenstein* if $R/Q$ is quasi-Gorenstein. The number $a = a(R/Q)$ is the *a-invariant* of $R/Q$. Note that quasi-Gorenstein ideals are unmixed (indeed, their factor rings satisfy Serre's property $(S_2)$).

Quasi-Gorenstein rings arise naturally in several contexts, for instance, (extended) Rees algebras (cf. [24, Theorem 2.8], [14, Theorem 3.2], and [10, Theorem 6.1]) or coordinate rings of cones over abelian surfaces (see, for instance, [21]). From the definition, it follows that an ideal $Q$ of $R$ is Gorenstein if and only if $Q$ is quasi-Gorenstein and Cohen-Macaulay. We will employ the following result, essentially proved by Schenzel [22, Proposition 1].

**Proposition 3.1.** *Let $Q$ be an $R$-ideal. The following are equivalent:*

(i) *$Q$ is quasi-Gorenstein;*
(ii) *there exist a Gorenstein ideal $G \subseteq Q$ and $h \in R$ so that $Q \sim G + hR$ by $G$.*

We now give a lower bound for the multiplicity of graded quasi-Gorenstein rings.

**Proposition 3.2.** *If $Q$ is a homogeneous quasi-Gorenstein ideal, then*

$$e(R/Q) \geq \max\{1, a(R/Q) + \dim(R/Q) + 1\}.$$

*Proof.* We may assume $Q$ is proper. By Proposition 3.1, there exists a Gorenstein ideal $G \subseteq Q$ with $\operatorname{ht} G = \operatorname{ht} Q$ and $Q \sim I$, where $I = G + hR$ for some element $h$. Note that $h \notin G$ (otherwise $I = G$ and then, by Proposition 2.4, $Q = G : G = R$, contradicting $Q$ is proper) and $\operatorname{ht} I = \operatorname{ht} G$. Then, by Proposition 2.5 (b) and Theorem 2.2, we have

$$e(R/Q) = e(R/G) - e(R/I) \geq \max\{1, s(R/G) - \deg(h) + 1\}.$$

Now, from the standard exact sequence from linkage

$$0 \to G \to I \to \omega_{R/Q}(-a(R/G)) \to 0$$

we have $\omega_{R/Q} \cong I/G[a(R/G)]$. Since $I/G$ is generated by the image of $h$, we obtain $a(R/Q) = a(R/G) - \deg(h)$. This fact, together with the equalities $s(R/G) = \dim(R/G) + a(R/G)$ and $\dim(R/G) = \dim(R/Q)$, yields

$$\begin{aligned} e(R/Q) &\geq \max\{1, s(R/G) - \deg(h) + 1\} \\ &= \max\{1, \dim(R/G) + a(R/G) - \deg(h) + 1\} \\ &= \max\{1, \dim(R/Q) + a(R/Q) + 1\}. \end{aligned}$$

□

We now prove a multiplicity-based sufficient condition for $R/Q$ to be Gorenstein.

**Theorem 3.3.** *Let $Q$ be a homogeneous quasi-Gorenstein ideal. If*

$$e(R/Q) = \max\{1, a(R/Q) + \dim(R/Q) + 1\},$$

*then $R/Q$ is Gorenstein.*

*Proof.* We may assume $Q$ is proper. Let $G \subseteq Q$ be a Gorenstein ideal with $\operatorname{ht} G = \operatorname{ht} Q$, and set $I = G : Q$. From the proof of Proposition 3.2, the given equality implies that $I$ has maximal multiplicity.

Since $\deg(h) = a(R/G) - a(R/Q)$ and $-a(R/Q) \leq \dim(R/Q)$, then we have $\deg(h) \leq s(R/G)$. By Theorem 2.7 (b), this yields that $R/I$ is Cohen-Macaulay. Thus, by Proposition 2.5 (a), $R/Q$ is Cohen-Macaulay and, then, Gorenstein. □

Next, we would like to point out an analogy between the two extremal values of $e(R/I)$. Assume $I = J + (F)$, where $J$ is Gorenstein and $F \notin J$ is a zero divisor on $R/J$; then

$$1 \leq e(R/I) \leq e(R/J) - \max\{1, s(R/J) - \deg(F) + 1\}.$$

If $I$ has maximal multiplicity, by Theorem 2.7 either $R/I$ is Cohen-Macaulay or it is almost Cohen-Macaulay. When $e(R/I) = 1$ a similar statement holds.

**Proposition 3.4.** *Let $J$ be a homogeneous Gorenstein ideal, and let $F \notin J$ be a homogeneous element such that* $\operatorname{ht} I = \operatorname{ht} J$*, where* $I = J+(F)$*. Assume* $e(R/I) = 1$.

(a) [*Samuel* [20], [17]] *$I$ is unmixed if and only if $I$ is Cohen-Macaulay.*
(b) *$I$ is not unmixed if and only if* $\operatorname{depth}(R/I) = \dim(R/I) - 1$ (*that is, $R/I$ is almost Cohen-Macaulay*).

*Proof.* We prove only assertion (b). If $\operatorname{depth}(R/I) = \dim(R/I) - 1$, then $R/I$ is not Cohen-Macaulay; hence, by assertion (a), $I$ is not unmixed. Next, assume $I$ is not unmixed. Set $L = J : I$ and note that $L \sim I^{un}$, by Proposition 2.4. Since $e(R/I^{un}) = e(R/I) = 1$, the ideal $I^{un}$ is Cohen-Macaulay, by assertion (a). By Proposition 2.5 (a), $R/L$ is also Cohen-Macaulay. Then, Lemma 2.6 gives $\operatorname{pd}(R/I) \leq \operatorname{grade} I + 1$. An application of the Auslander-Buchsbaum formula now proves $\operatorname{depth}(R/I) \geq \dim(R/I)-1$. Since $I$ is not unmixed, we have $\operatorname{depth}(R/I) \neq \dim(R/I)$, yielding $\operatorname{depth}(R/I) = \dim(R/I) - 1$. □

It is then natural to ask whether the analogue of Proposition 3.4 holds: assume $I$ has maximal multiplicity; is it true that $I$ is unmixed if and only if $I$ is Cohen-Macaulay? Corollary 3.5 provides a positive answer.

**Corollary 3.5.** *Let $J$ be a homogeneous Gorenstein ideal, and let $F \notin J$ be a homogeneous element such that* $\operatorname{ht} I = \operatorname{ht} J$*, where* $I = J + (F)$*. Assume $I$ has maximal multiplicity.*

(a) *$I$ is unmixed if and only if $I$ is Cohen-Macaulay if and only if* $\deg(F) \leq s(R/J)$.
(b) *$I$ is not unmixed if and only if* $\operatorname{depth}(R/I) = \dim(R/I) - 1$ *if and only if* $\deg(F) > s(R/J)$.

*Proof.* (a) By Theorem 2.7 (c) we only need to prove that if $I$ is unmixed, then $I$ is Cohen-Macaulay. If $I$ is unmixed, the ideal $Q = J : I$ is quasi-Gorenstein by Proposition 3.1. As in the proof of Proposition 3.2, the maximal multiplicity of $I$ yields $e(R/Q) = \max\{1, a(R/Q) + \dim(R/Q) + 1\}$. Now, Theorem 3.3 implies that $Q$ is Gorenstein and then, by Proposition 2.5 (a), $I$ is Cohen-Macaulay.

Assertion (b) follows from assertion (a) and Theorem 2.7 (a) and (c). □

We conclude with a question. If $I$ is an ideal of maximal multiplicity, in general there could be several different maximal decompositions $I = J+(F)$. Moreover, if $F$ and $F'$ are minimal generators of $I$, there could be a maximal decomposition of $I$ of the form $I = J+(F)$, but no maximal decomposition for $I$ of the form $I = J'+(F')$. These observations raise the following question: *Is there an implicit characterization of ideals $I$ of maximal multiplicity?* Theorem 2.13 gives a sufficient condition when $I$ is Cohen-Macaulay, and Proposition 2.12 gives a necessary condition, but we don't know a necessary and sufficient condition.

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