A MULTIPLICITY BOUND FOR GRADED RINGS AND
A CRITERION FOR THE COHEN-MACAULAY PROPERTY

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Abstract. Let \( R \) be a polynomial ring over a field. We prove an upper
bound for the multiplicity of \( R/I \) when \( I \) is a homogeneous ideal of the form
\( I = J + (F) \), where \( J \) is a Cohen-Macaulay ideal and \( F \notin J \). The bound is given
in terms of two invariants of \( R/J \) and the degree of \( F \). We show that ideals
achieving this upper bound have high depth, and provide a purely numerical
criterion for the Cohen-Macaulay property. Applications to quasi-Gorenstein
rings and almost complete intersections are given.

1. Introduction

Bounds on the multiplicity of a ring \( S \) in terms of other invariants of \( S \) (and
analyses of the rings achieving these bounds) have attracted strong interest over
the last 130 years, from the classical lower bound \( \deg X \geq \text{codim } X + 1 \) for non-
degenerate projective varieties to the still open Eisenbud-Green-Harris Conjecture
[3] or the Huneke-Srinivasan Multiplicity Conjecture, proved a few years ago by
Eisenbud and Schreyer [5].

In the present paper, we prove a new upper bound for the multiplicity of a wide
class of graded rings and study the defining ideals achieving this bound. Let \( R \) be
a polynomial ring over a field \( k \), \( J \) a homogeneous Cohen-Macaulay ideal, \( F \notin J \) a
homogeneous element and \( I = J + (F) \). Let \( e(S) \) denote the multiplicity of a graded
ring \( S \). If \( F \) is regular on \( R/J \), it is well-known that \( e(R/I) = e(R/J) \cdot \deg(F) \). If
\( F \) is a zero-divisor on \( R/J \) (that is, \( \text{ht } I = \text{ht } J \)) one has the elementary inequality
\( e(R/I) \leq e(R/J) - 1 \). In the present paper, we prove the sharper upper bound
\[
(1.1) \quad e(R/I) \leq e(R/J) - \max\{1, s - \deg(F) + 1\},
\]
where \( s = s(R/J) \) is the difference between the smallest graded shift appearing in
the last step of a minimal graded free resolution of \( R/J \) and \( \text{ht } J \). This bound,
by its nature, is more restrictive when \( \deg(F) \) is small or when \( R/J \) is level, a
particular instance of which is when \( R/J \) is Gorenstein.

The inequality (1.1) generalizes a previous result of Engheta bounding the mul-
tiplicity of a homogeneous almost complete intersection in terms of the degrees of
its minimal generators [6, Theorem 1]; see Corollary 2.3.

We name ideals \( I \) achieving equality in (1.1) ideals of maximal multiplicity. Our
second main result states that these ideals define factor rings with high depth.

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More precisely, $R/I$ is Cohen-Macaulay if the degree of $F$ is sufficiently small, namely, $\deg(F) \leq s$. In the remaining cases, that is, when $\deg(F) > s$, we show $\text{depth}(R/I) = \dim(R/I) - 1$, provided $R/J$ is Gorenstein (Theorem 2.7). A consequence of this result is that almost complete intersections generated in a single degree and having maximal multiplicity are Cohen-Macaulay (Corollary 2.9), a result that we employ in [11]. Also, one should contrast this result with the abundance of examples of almost complete intersection ideals $I$ (even generated in a single degree) of any other multiplicity that are not Cohen-Macaulay (see Remark 2.10).

We then provide a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity and exhibit examples, which include rational normal curves and $m$-primary ideals (see Theorem 2.13, Examples 2.15 and 2.17, and Corollary 2.16). Among the applications, we use linkage to deduce a lower bound for the multiplicity of any graded quasi-Gorenstein ring $S$ in terms of its $a$-invariant and dimension (Proposition 3.2). If this lower bound is achieved, then $S$ is actually Gorenstein (see Theorem 3.3). We also prove that ideals of maximal multiplicity are unmixed if and only if they are Cohen-Macaulay (Corollary 3.5).

As a final application, we remark that part of the material from this paper is employed in the forthcoming paper [11], where, motivated by a question of Stillman [18], we prove a close-to-optimal upper bound on the projective dimension of any ideal generated by 4 quadratic polynomials. One of our original motivations in the present paper was, in fact, to find a more structural reason for the fact (proved in an earlier draft of [11]) that any almost complete intersection ideal (not necessarily unmixed) of multiplicity 6 generated by 4 quadrics is Cohen-Macaulay.

The structure of the paper is the following: in Section 2, we prove the three main results, namely the upper bound (1.1), the high depth properties of ideals of maximal multiplicity, and a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. We employ these results to obtain a sufficient condition for almost complete intersections of quadrics to be Cohen-Macaulay and provide examples. In Section 3, we prove a lower bound for the multiplicity of quasi-Gorenstein rings, a multiplicity-based sufficient condition for quasi-Gorenstein rings to be Gorenstein, and analogies between ideals of maximal multiplicity and ideals of multiplicity one.

2. THE MAIN RESULTS

Throughout this paper $R$ is a polynomial ring over a field $k$ and $m$ denotes its unique homogeneous maximal ideal. Since we may harmlessly replace $k$ by $k(X)$, by base change we may always assume $|k| = \infty$.

Recall that the Hilbert function of a finitely generated graded $R$-module $M$ is the numerical function given by $HF_M(i) = \dim_k M_i$ for all $i \in \mathbb{Z}$. If $A$ is a graded artinian factor ring of $R$, then $e(A) = \sum_{i=0}^{\infty} HF_A(i)$, and the socle of $A$ is the $k$-vector space $\text{Soc}(A) = 0 :_A \mathfrak{m}_A$, where $\mathfrak{m}_A = \mathfrak{m}A$. A socle element is an element of $\text{Soc}(A)$.

Let $J$ be a homogeneous Cohen-Macaulay $R$-ideal of height $g$. We consider the invariant

$$s(R/J) = \min\{i \mid \text{Tor}_g^R(R/J, k)_i \neq 0\} - g.$$ 

We recall here two additional interpretations of this invariant. First, when $|k| = \infty$, $s(R/J)$ is the smallest degree of a non-zero homogeneous socle element of a general
an artinian reduction of $R/J$ (a general artinian reduction of $T = R/J$ is a ring $T/(L_1, \ldots, L_d)$ where $L_i$ are general linear forms and $d = \dim(T)$). Second, one has $s(R/J) = c(R/J) + \dim(R/J)$, where $c(R/J) = \min\{ k \in \mathbb{N} | [k \otimes_{R/J} \omega_{R/J}], \neq 0 \}$ and $\omega_{R/J}$ denotes the graded canonical module of $R/J$.

We also recall that the unmixed part of an $R$-ideal $K$, denoted $K^{un}$, is the intersection of the primary components of $K$ of minimal height. The ideal $K$ is unmixed if $K = K^{un}$. The associativity formula yields $e(R/K) = e(R/K^{un})$ and the following easy remark:

**Remark 2.1.** If $K \subseteq L$ are unmixed ideals of the same height, then $e(R/K) \geq e(R/L)$, and $e(R/K) = e(R/L)$ if and only if $K = L$.

We now prove our first main result.

**Theorem 2.2.** Let $J$ be a homogeneous Cohen-Macaulay $R$-ideal, and let $F \notin J$ be homogeneous. Set $I = J + (F)$ and assume $\text{ht} I = \text{ht} J$. Then

1. $e(R/I) \leq e(R/J) - \max\{ 1, s(R/J) - \deg(F) + 1 \}$;
2. if equality is achieved in (i) and $\deg(F) \leq s(R/J)$, then $R/I$ is Cohen-Macaulay.

**Proof.** We prove both statements at once. Set $\delta = \deg(F)$ and $s = \deg(R/J)$. First, assume $1 > s - \delta + 1$. We need to show that $e(R/I) \leq e(R/J) - 1$. This follows by the equality $e(R/I) = e(R/I^{un})$, the inclusions $J \subsetneq I \subseteq I^{un}$ and Remark 2.1. Hence, we may assume $1 \leq s - \delta + 1$, that is, $\delta \leq s$. We need to show that $e(R/I) \leq e(R/J) - (s - \delta + 1)$ and, if equality is achieved, then $R/I$ is Cohen-Macaulay.

Let $A$ be a general artinian reduction of $R/J$, and denote by $\bar{a}$ images in $A$. We claim that both statements hold if $\bar{F} \neq 0$ in $A$. Indeed, in this case, we set $n = \max\{ t \in \mathbb{N}_0 | |\bar{F}^n|_A \neq 0 \} < \infty$ and observe that $HF_{A/(\bar{F})}(i) \leq HF_A(i) - 1$ for every $\delta \leq i \leq \delta + n$. This proves $e(A/(\bar{F})) \leq e(A) - (n + 1)$. Let $G$ be a homogeneous element of degree $n$ such that $\bar{F}G \neq 0$ in $A$. By the definition of $n$, we have $0 \neq \bar{F}G \in \text{Soc}(A)$, whence $\delta + n = \deg(\bar{F}G) \geq s$. This proves $s - \delta \leq n$ and gives $e(A/(\bar{F})) \leq e(A) - (n + 1) \leq e(A) - (s - \delta + 1)$.

A well-known result of Serre (e.g. [1] Theorem 4.7.10] yields $e(A) = e(R/J)$ and $e(R/I) \leq e(A/(\bar{F}))$; hence we obtain the inequalities

$$e(R/I) \leq e(A/(\bar{F})) \leq e(A) - (s - \delta + 1) = e(R/J) - (s - \delta + 1).$$

If, moreover, $e(R/I) = e(R/J) - (s - \delta + 1)$, then we have the equality $e(R/I) = e(A/(\bar{F}))$ and, by [1] Theorem 4.7.10], $R/I$ is Cohen-Macaulay.

Therefore, to finish the proof it suffices to prove $\bar{F} \neq 0$ in $A$. Set $\bar{R} = R/J$ and let $\bar{a}$ denote images in $\bar{R} = R/J$. Since $A$ is a general artinian reduction of $\bar{R}$, to show $\bar{F} \neq 0$ in $A$, we need to prove that $\bar{F}$ is not in the intersection of all the (general) minimal reductions of $\bar{m}$, that is, $\bar{F} \notin \text{core}(\bar{m})$ in $\bar{R} = R/J$, where $\text{core}(\bar{m})$ denotes the core of $\bar{m}$. Since $\bar{R} = R/J$ is Cohen-Macaulay, it follows by work of Fouli, Polini and Ulrich [3] Corollary 4.3(b)] that $\text{core}(\bar{m}) \subseteq \bar{m}^{t+1}$. The inequality $s \geq \delta$ now yields $\bar{F} \notin \text{core}(\bar{m})$ for degree reasons.

As a first application we recover an upper bound (first proved by Engheta [3]) for the multiplicity of $R/I$, where $I$ is any homogeneous almost complete intersection.


Corollary 2.3 (Engheta [6] Theorem 1]). Let \( I = (f_1, \ldots, f_{g+1}) \) be a homogeneous almost complete intersection of height \( g \) in \( R \), where \( f_1, \ldots, f_g \) form a homogeneous regular sequence. Set \( d_i = \deg(f_i) \) for every \( i = 1, \ldots, g+1 \). Then,
\[
e(R/I) \leq \prod_{i=1}^{g} d_i - \max \left\{ 1, \sum_{i=1}^{g} (d_i - 1) - (d_{g+1} - 1) \right\}.
\]
In particular, if \( I \) is generated by forms of the same degree \( d \) and \( g > 1 \), then
\[
e(R/I) \leq d^g - (d - 1)(g - 1).
\]

Proof. Since \( J = (f_1, \ldots, f_g) \) is a complete intersection, we have \( s(R/J) = \sum_{i=1}^{g} (d_i - 1) \). Now, Theorem 2.2 applied to \( I = J + (f_{g+1}) \) proves the statement. \( \square \)

Although the bound of Corollary 2.3 can be sharp (see Remark 2.10), G. Caviglia remarked that if this is the case, then either \( g = 2 \) or \( g = 3 \) and \( d_1 = d_2 = d_3 = 2 \). Instead, the bound of Theorem 2.2 is more general and, indeed, is achieved in a wider variety of situations (see, for instance, Examples 2.15 and 2.17 and Corollary 2.10).

Before stating our second main result, we recall a few definitions. An ideal \( I \) is called almost Cohen-Macaulay if \( \dim(R/I) \geq \dim(R/I) - 1 \). Also, a homogeneous Cohen-Macaulay ideal of height \( g \) is said to be level if there exists only one positive integer \( i \) such that \( \operatorname{Tor}_g^R(R/J, k) \neq 0 \). An ideal \( J \) is Gorenstein if \( R/J \) is a Gorenstein ideal with \( \operatorname{dim}(R/J) = 1 \). Sometimes, we say \( J \) is linked by \( G \) (by complete intersections) has been studied since the nineteenth century, although its first modern treatment appeared in the ground-breaking paper by Peskine and Szpiro [19]. Properties of liaison by Gorenstein ideals were then studied in [23]. We refer the interested reader to [16], [12] and their references.

Proposition 2.4 (Peskine-Szpiro, Schenzel [19, 23]). Let \( J \) be an unmixed ideal of \( R \) of height \( g \). If \( G \subseteq J \) is a height \( g \) Gorenstein ideal, and \( K = G : J \), then \( J \sim K \).

We will also need a few definitions and results from liaison theory. Two (homogeneous) ideals \( J \) and \( K \) in \( R \) are linked, denoted \( J \sim K \), if there exists a (homogeneous) Gorenstein ideal \( G \) with \( K = G :_R J \) and \( J = G :_R K \). Sometimes we say \( J \) and \( K \) are linked by \( G \). Linkage (by complete intersections) has been studied since the nineteenth century, although its first modern treatment appeared in the ground-breaking paper by Peskine and Szpiro [19]. Properties of liaison by Gorenstein ideals were then studied in [23]. We refer the interested reader to [16], [12] and their references.

Proposition 2.5 (Peskine-Szpiro, Golod, Schenzel [19, 9, 23]). If \( J \sim K \), then
(a) \( S/J \) is Cohen-Macaulay if and only if \( S/K \) is Cohen-Macaulay.
(b) \( e(S/J) + e(S/K) = e(S/G) \), where \( G \) is the ideal defining the link \( J \sim K \).

The following linkage result is well-known (its proof follows, for instance, along the same lines of the proof of [7, Theorem 3]).

Lemma 2.6. Let \( I = J + (F) \), where \( J \) is Gorenstein ideal with \( \operatorname{ht}(J) = \operatorname{ht}(I) \), and \( F \notin J \). If \( L \) is any ideal linked to \( I^{un} \), then \( \operatorname{pd}(R/I) \leq \operatorname{pd}(R/L) + 1 \).

In the setting of Theorem 2.2, we say that \( I \) has maximal multiplicity if there exist a Cohen-Macaulay ideal \( J \) and an element \( F \notin J \) with \( I = J + (F) \), \( \operatorname{ht}(I) = \operatorname{ht}(J) \).
and $e(R/I) = e(R/J) - \max \{1, s(R/J) - \deg(F) + 1\}$. If $J$ and $F$ are as above, we say they form a maximal decomposition of $I$.

We can now state our second main result, proving the high depth of $R/I$.

**Theorem 2.7.** Let $I$ be a homogeneous $R$-ideal of maximal multiplicity, and let $I = J + (F)$ be a maximal decomposition of $I$.

(a) If $\deg(F) \leq s(R/J)$, then $R/I$ is Cohen-Macaulay.

(b) If $R/J$ is level, then $R/I$ is Cohen-Macaulay if and only if $\deg(F) \leq s(R/J)$.

Note that although $J$ is Cohen-Macaulay, the ideal $I$ may not even be unmixed.

**Proof.** Set $g = \text{ht } J = \text{ht } I$. Assertion (a) was proved in Theorem 2.2 (ii). To prove assertion (b) we need to show that if $R/I$ is Cohen-Macaulay, then $\deg(F) \leq s(R/J)$. Let $L_1, \ldots, L_d$ be general linear forms, where $d = \dim(R/J) = \dim(R/I)$, and let $\overline{-}$ denote images modulo $L_1, \ldots, L_d$. Since $R/I$ is Cohen-Macaulay, by Theorem 4.7.10, we have $e(R/I) = e(\overline{R/I})$. Moreover, since $R/J$ is level, we have $\text{Soc}(\overline{R/J}) = \overline{m_{R/J} s}$, where $s = s(R/J)$. Now, assume by contradiction that $\deg(F) > s(R/J)$. Then $\overline{F} \in \overline{m_{R/J}}^{s+1} = 0$ in $\overline{R/J}$, that is, $\overline{R/I} = \overline{R/J}$. This implies that

$$e(R/I) = e(\overline{R/I}) = e(\overline{R/J}) = e(R/J).$$

Since both $I$ and $J$ are unmixed, Remark 2.1 implies $J = I$, which is a contradiction.

We now prove assertion (c). The assumptions imply $e(R/I) = e(R/J) - 1$. Let $L = J : I$ and note that, by Lemma 2.6, $p \text{d}(R/I) \leq p \text{d}(R/L) + 1$. Moreover, $R/L$ is unmixed with $e(R/L) = e(R/J) - e(R/I) = 1$ (by Proposition 2.5); therefore, by a well-known result of Samuel (see Proposition 3.4), $R/L$ is Cohen-Macaulay. Then $p \text{d}(R/I) \leq g + 1$. Hence, by the Auslander-Buchsbaum formula, we have $\text{depth}(R/I) \geq \dim(R/I) - 1$. Finally, by assertion (b), $R/I$ is not Cohen-Macaulay, because $\deg(F) > s(R/J)$. This yields $\text{depth}(R/I) = \dim(R/I) - 1$. \hfill $\square$

The next simple example shows that the bound given in Theorem 2.2 can be (trivially) sharp, and there are ideals of maximal multiplicity that are not Cohen-Macaulay.

**Example 2.8.** Let $R = k[x, y, z]$, $J = (x^2, xy, y^2)$ and $F = xz$. Then $I = J + (F)$ is not unmixed; $R/I$ has maximal multiplicity and is almost Cohen-Macaulay.

We now apply Theorem 2.7 to almost complete intersection ideals generated by quadrics.

**Corollary 2.9.** Let $I$ be an almost complete intersection of height $g \geq 1$ generated by homogeneous elements of the same degree $2$. If $e(R/I) = 2^g - (g - 1)$, then $R/I$ has maximal multiplicity and is Cohen-Macaulay.

**Proof.** It follows by Corollary 2.3 together with Theorem 2.7 (b). \hfill $\square$

If $I$ is an almost complete intersection generated by 4 quadrics (that is, $g = 3$ and $d = 2$), then Corollary 2.3 gives $e(R/I) \leq 6$, and Corollary 2.9 yields that $R/I$ is Cohen-Macaulay if $e(R/I) = 6$. This fact is employed in 11.

The following remark shows that, without further assumptions, no other values of $e(R/I)$ guarantee the Cohen-Macaulayness of $R/I$.
Remark 2.10. Let $I$ be a height three ideal generated by 4 quadrics.

(a) If $e(R/I) = 6$, then $R/I$ is Cohen-Macaulay.
(b) For every $1 \leq e \leq 6$, there are examples of $I$ with $e(R/I) = e$, and if $e \neq 6$, $R/I$ is not Cohen-Macaulay.

Assertion (a) follows by Corollary 2.9. Assertion (b) can be seen, for instance, as follows. Take $R = k[a, b, c, x, y, z]$.

1. if $I = (ax, by, cz, x^2 + y^2 + z^2)$, then $e(R/I) = 1$ and $pd(R/I) = 4$;
2. if $I = (ax, by, xy + xz + yz, x^2 + y^2 + z^2)$, then $e(R/I) = 2$ and $pd(R/I) = 4$;
3. if $I = (ax + by + cz, x^2, y^2, z^2)$, then $e(R/I) = 3$ and $pd(R/I) = 6$;
4. if $I = (ax, x^2, y^2, z^2)$, then $e(R/I) = 4$ and $pd(R/I) = 4$;
5. if $I = (ax + by + cz, bx + cy + az, cx + ay + bz, bx + cy - bz - cz)$, then $e(R/I) = 5$ and $pd(R/I) = 4$;
6. if $I = (x^2, y^2, z^2, xy)$, then $e(R/I) = 6$ and, by part (a), $pd(R/I) = ht I = 3$.

Next, we want to provide a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. The first step consists of describing the structure of the colon ideal $J : F$.

Lemma 2.11. Let $I$ be a Cohen-Macaulay homogeneous ideal of height $g$ having maximal multiplicity. If $J$ and $F$ form a maximal decomposition of $I$, then $J : F = (x_1, \ldots, x_{g-1}, q)$, for some linearly independent linear forms $x_1, \ldots, x_{g-1}$ and an element $q \notin (x_1, \ldots, x_{g-1})$.

Proof. From the short exact sequence

$0 \to R/J : F[-\deg(F)] \to R/J \to R/I \to 0$

one obtains $e(R/J) - e(R/I) = e((R/J : F)[-\deg(F)]) = e(R/J : F)$. First, assume $I$ is $\mathfrak{m}$-primary. If $\deg(F) \geq s(R/J)$, then, by assumption of maximal multiplicity, $e(R/I) = e(R/J) - 1$, whence $e(R/J : F) = 1$. Then $J : F = \mathfrak{m}$, and the statement follows. We may then assume $\deg(F) < s(R/J)$. Since $J$ and $F$ form a maximal decomposition of $I$, the proof of Theorem 2.2 gives $HF_{R/J}(i) = HF_{R/I}(i) - 1$ for all $\deg(F) \leq i \leq s(R/J)$. This yields that $HF_{R/J,F}(i) = 1$ if $0 \leq i \leq s(R/J) - \deg(F)$; hence, there exist linearly independent linear forms $x_1, \ldots, x_{g-1}, x_g$ such that $J : F = (x_1, \ldots, x_{g-1}, x_g)$, and the statement follows.

Now assume $\dim(R/I) = d > 0$. Let $L_1, \ldots, L_d$ be general linear forms in $R$, set $L = (L_1, \ldots, L_d)$ and let $\overline{\cdot}$ denote images in $\overline{R} = R/L$. Since $I$ is Cohen-Macaulay, $L_1, \ldots, L_d$ form a regular sequence on $R/I$; then, applying the functor $- \otimes_R R/L$ to the short exact sequence $(\ast)$, one obtains the short exact sequence

$0 \to \overline{R}/\overline{J} : \overline{F} \to \overline{R}/\overline{J} \to \overline{R}/\overline{I} \to 0$.

Clearly, $\overline{R} = R/L$ is still a polynomial ring, $\overline{I}$ is a homogeneous $\overline{\mathfrak{m}}$-primary ideal, $\overline{R}/\overline{I}$ has maximal multiplicity, and $\overline{J}$ and $\overline{F}$ form a maximal decomposition of $\overline{I}$. Then, by the above, one has $J : F = \overline{J} : \overline{F} = (\overline{z_1}, \ldots, \overline{z_{g-1}}, \overline{z_g})$ for some linearly independent linear forms $z_1, \ldots, z_g$ of $\overline{R}$. The statement now follows by lifting this equality back to $R$. \hfill $\square$

For the rest of the paper, a complete intersection $C$ of height $g$ containing $g - 1$ linearly independent linear forms is called an almost linear complete intersection.
Proposition 2.12. Let $I$ be a Cohen-Macaulay homogeneous ideal of height $g$ having maximal multiplicity. Then there exists an almost linear complete intersection $C'$ such that $(I + C')/C'$ is cyclic.

Proof. If $I$ is contained in an almost linear complete intersection, then $(I + C')/C' = 0$ and the statement follows trivially. We may then assume $I$ is not contained in any almost linear complete intersection. Let $J$ and $F$ form a maximal decomposition of $I$. By Lemma 2.11 the ideal $C' = J : F$ is an almost linear complete intersection. Note that $J \subseteq I \cap C'$. Since $I$ is not contained in $C'$, then $(I + C')/C'$ is non-zero. Now, the natural mapping $I/J \to I/I \cap C' \cong (I + C')/C' \to 0$ together with the assumption that $I/J$ is cyclic yields that also $(I + C')/C'$ is cyclic. □

Next, we prove a sufficient condition for Cohen-Macaulay ideals to have maximal multiplicity. The assumption of $(I + C')/C'$ being cyclic is necessary, by Proposition 2.12.

Theorem 2.13. Let $I$ be a homogeneous Cohen-Macaulay ideal of height $g$. If there exists an almost linear complete intersection $C'$ satisfying the following two conditions:

(i) $(I + C')/C'$ is non-zero and cyclic, generated by an element of degree $\delta \geq 1$, and

(ii) $e(R/C') \leq \max\{1, s(R/I) - \delta + 1\}$,

then $I$ has maximal multiplicity.

Proof. Let $F$ be a homogeneous element of $I$ of degree $\delta \geq 1$ whose image generates the cyclic module $(I + C')/C'$. Set $J = C' \cap I$ and note that $I = J + (F)$. Set $C = J : F = C' : F$. Since $C' \subseteq C$ are unmixed of the same height, the ideal $C$ is again an almost linear complete intersection and $e(R/C) \leq e(R/C')$. Write $C = (x_1, \ldots, x_{g-1}, q)$ and note that $e(R/C) = \deg(q) \leq e(R/C') \leq \max\{1, s(R/I) - \delta + 1\}$. We need to show that $J$ is Cohen-Macaulay and $e(R/I) = e(R/J) = \max\{1, s(R/J) - \delta + 1\}$.

Since $I$ and $C$ are Cohen-Macaulay ideals of height $g$, the short exact sequence

\[(*) \quad 0 \to (R/C)[-\delta] \xrightarrow{\cdot F} R/J \to R/I \to 0\]

implies that $R/J$ is Cohen-Macaulay too. Now, if $1 \leq s(R/I) - \delta + 1$, then $e(R/C) \leq 1$, and since $C$ is a proper ideal, then $e(R/C) = 1$, yielding that $e(R/I) = e(R/J) - e(R/J : F) = e(R/J) - 1$. This proves that $I$ has maximal multiplicity. We may then assume $s(R/I) - \delta + 1 > 1$. The Horseshoe Lemma applied to $(*)$ gives $s(R/J) \geq \min\{s(R/I), s(R/C[-\delta])\}$. Since $C$ is an almost linear complete intersection, we have

\[
s(R/C[-\delta]) = \delta + s(R/C) = \delta + (\deg(q) + g - 1) - g = \delta + \deg(q) - 1,
\]

whence we obtain

\[
s(R/J) \geq \min\{s(R/I), s(R/C[-\delta])\} = \min\{s(R/I), \delta + \deg(q) + 1\} = \delta + \deg(q) - 1,
\]

where the last inequality holds because the inequalities

\[
\deg(q) = e(R/C) \leq e(R/C') \leq \max\{1, s(R/I) - \delta + 1\} = s(R/I) - \delta + 1
\]
imply that $\delta + \deg(q) - 1 \leq s(R/I)$. Then, we have obtained $\delta + \deg(q) - 1 \leq s(R/J)$, that is, $\deg(q) \leq s(R/J) - \delta + 1$. We now apply Theorem 2.2 and obtain
\[
\begin{align*}
\delta(R/J) - \delta(R/C) &= \delta(R/I) \\
&\leq \delta(R/J) - \max\{1, \deg(q)\} = \delta(R/J) - \delta(\deg(q)) \\
&= \delta(R/J) - \delta(R/C)
\end{align*}
\]
proving that $\delta(R/I) = \delta(R/J) - \max\{1, s(R/J) - \delta + 1\}$. \hfill $\square$

Note that the proof of Theorem 2.13 is constructive, in the sense that if $(I + C')/C'$ is non-zero, cyclic and $\delta(R/C') \leq \max\{1, s(R/I) - \delta + 1\}$, then one can explicitly construct a maximal decomposition $I = J + (F)$ of the ideal $I$.

We isolate the special case where $C'$ is a linear prime, that is, where $C'$ is a prime ideal generated by linear forms.

**Corollary 2.14.** Let $I$ be a homogeneous Cohen-Macaulay ideal of height $g$. If there exists a linear prime $C'$ of height $g$ such that $(I + C')/C'$ is cyclic and non-zero, then $I$ has maximal multiplicity.

**Proof.** By assumption $\delta(R/C') = 1$, so the multiplicity condition of Theorem 2.13 is trivially satisfied. \hfill $\square$

We now exhibit two classes of ideals having maximal multiplicity.

**Example 2.15.** Let $I$ be any homogeneous $m$-primary ideal. Then $I$ has maximal multiplicity.

**Proof.** Let $\{F_1, \ldots, F_r\}$ be a minimal generating set of $I$, let $I' = (F_1, \ldots, F_{r-1})$, $F = F_r$, and let $J$ be the $m$-primary ideal $J = I' + F \cdot m$. Then $I$ has maximal multiplicity because $I = J + (F)$ and $\delta(R/I) = \delta(R/J) - 1$. \hfill $\square$

Also ideals generated by the 2 by 2 minors of catalecticant matrices have maximal multiplicity. For instance, ideals defining rational normal curves have maximal multiplicity.

**Corollary 2.16.** Fix integers $d \geq 1$, $r \geq 2$ and $N \geq 3$. Then the ideal $I = I_2(A)$ generated by the 2-minors of the matrix
\[
A = \begin{pmatrix}
x_1^d & x_2^d & \cdots & x_{N-r+1}^d \\
x_2^d & x_3^d & \cdots & x_{N-r+2}^d \\
\vdots & \vdots & \ddots & \vdots \\
x_r^d & x_{r+1}^d & \cdots & x_{N+1}^d
\end{pmatrix}
\]
has maximal multiplicity.

In particular, the defining ideals of the rational normal curves of $\mathbb{P}^N$ for any $N \geq 3$ have maximal multiplicity.

**Proof.** It is known that $\text{ht}(I) = N - 1$. Set $C' = (x_2, \ldots, x_N)$, and note that $(I + C')/C'$ is cyclic and non-zero, generated by the image of $F = x_1^d x_{N+1} - x_2^d x_{N-r+2}$ in $R/C'$. Then, by Corollary 2.14, $I$ has maximal multiplicity, and a maximal decomposition of $I$ is given by $I = J + (F)$, where $J$ is the ideal generated by all the 2 by 2 minors of $A$ except for $F$. Rational normal curves consist of the special case where $r = 2$ and $d = 1$. \hfill $\square$
We remark that, in contrast with the ideals $I$ satisfying equality in Corollary 2.3, these ideals can be generated in arbitrarily high degrees and can have arbitrarily large heights.

We conclude this section with a class of almost complete intersections $I$ of height 3, generated in a single degree $d \geq 4$ and having maximal multiplicity (compare with the discussion after Corollary 2.3). Note that, for these ideals, there is no linear prime $C'$ of height 3 such that $(I + C')/C'$ is cyclic; hence one cannot use Corollary 2.14 to prove that $I$ has maximal multiplicity.

**Example 2.17.** Fix $t \geq 1$, and let $f_1, g_i, h_i$, where $i = 1, 2$, be irreducible polynomials of the same degree $t + 1$ in disjoint sets of variables such that $f_1$ is contained in a linear prime of height 2. Set $F = h_1h_2$ and $J = (f_1f_2, g_1g_2, f_1g_1h_1)$. Then $I = J + (F)$ is a Cohen-Macaulay almost complete intersection that has maximal multiplicity.

**Proof.** We first show that $I$ is Cohen-Macaulay. Observe that the ideals $(f_1, g_i, h_j)$ with $1 \leq i \leq 2, 1 \leq j \leq 2$ are all prime. Since

$$I = \left( \bigcap_{1 \leq i \leq 2, 1 \leq j \leq 2} (f_1, g_i, h_j) \right) \cap (f_2, g_1, h_1) \cap (f_2, g_1, h_2) \cap (f_2, g_2, h_1),$$

then $I$ is unmixed. Also, since $C_0 = (f_1f_2, g_1g_2, h_1h_2)$ is a complete intersection of height 3 contained in $I$, then $I \sim C_0 : I$ by Proposition 2.14. Since $C_0 : I = (f_2, g_2, h_2)$ is a complete intersection, then by Proposition 2.15, the ideal $I$ is Cohen-Macaulay.

Let $(x_1, x_2)$ be a linear prime of height 2 containing $f_1$, and set $C' = (x_1, x_2, g_1)$. Then $(I + C')/C'$ is cyclic, generated by the image of $F = h_1h_2$ in $R/C'$. We have $s(R/I) = 5t + 2$; hence $s(R/I) - \deg(F) + 1 = (5t + 2) - (2t + 2) + 1 = 3t + 1 > 3$ for every $t \geq 1$. Then

$$e(R/C') = 2 < 3t + 1 = \max\{1, s(R/I) - \deg(F) + 1\}.$$

Then $I$ has maximal multiplicity by Theorem 2.13. \qed

For instance, let $R = \mathbb{k}[x_1, \ldots, x_8, y_1, \ldots, y_8, z_1, \ldots, z_8]$, $f_1 = x_1^4x_2 - x_3^4x_4, g_1 = y_1^4y_2 - y_3^4y_4, h_1 = z_1^4z_2 - z_3^4z_4, f_2 = x_5^4x_6 - x_7^4x_8, g_2 = y_5^4y_6 - y_7^4y_8, h_2 = z_5^4z_6 - z_7^4z_8$, and $F = h_1h_2$. Then $I = (f_1f_2, g_1g_2, f_1g_1h_1, F)$ is a Cohen-Macaulay almost complete intersection that has maximal multiplicity and is generated in degree $2t + 2 \geq 4$ for any $t \geq 1$.

3. **Application to quasi-Gorenstein rings**

In this section we prove a lower bound for the multiplicity of quasi-Gorenstein rings and a sufficient condition for a quasi-Gorenstein ring to be Gorenstein.

We first recall the definition of graded quasi-Gorenstein rings (also known as 1-Gorenstein rings). Recall that the canonical module of a $d$-dimensional graded ring $S$ with homogeneous maximal ideal $\mathfrak{m}_S$ is defined as $\omega_S = \text{Hom}_k(H^d_{\mathfrak{m}_S}(S), E)$, where $E = E_S(S/m_S)$ is the injective envelope of the residue field of $S$.

A graded ring $S$ is quasi-Gorenstein if $\omega_S \cong S(a)$ for some integer $a$. An ideal $Q$ is said to be quasi-Gorenstein if $R/Q$ is quasi-Gorenstein. The number $a = a(R/Q)$ is the $a$-invariant of $R/Q$. Note that quasi-Gorenstein ideals are unmixed (indeed, their factor rings satisfy Serre’s property $(S_2)$).
Quasi-Gorenstein rings arise naturally in several contexts, for instance, (extended) Rees algebras (cf. [24 Theorem 2.8], [13 Theorem 3.2], and [10 Theorem 6.1]) or coordinate rings of cones over abelian surfaces (see, for instance, [21]). From the definition, it follows that an ideal \( Q \) of \( R \) is Gorenstein if and only if \( Q \) is quasi-Gorenstein and Cohen-Macaulay. We will employ the following result, essentially proved by Schenzel [22 Proposition 1].

**Proposition 3.1.** Let \( Q \) be an \( R \)-ideal. The following are equivalent:

(i) \( Q \) is quasi-Gorenstein;

(ii) there exist a Gorenstein ideal \( G \subseteq Q \) and \( h \in R \) so that \( Q \sim G + hR \) by \( G \).

We now give a lower bound for the multiplicity of graded quasi-Gorenstein rings.

**Proposition 3.2.** If \( Q \) is a homogeneous quasi-Gorenstein ideal, then

\[
e(R/Q) \geq \max\{1, a(R/Q) + \dim(R/Q) + 1\}.
\]

**Proof.** We may assume \( Q \) is proper. By Proposition 3.1 there exists a Gorenstein ideal \( G \subseteq Q \) with \( \text{ht} G = \text{ht} Q \) and \( Q \sim I \), where \( I = G + hR \) for some element \( h \). Note that \( h \notin G \) (otherwise \( I = G \) and then, by Proposition 2.3 \( Q = G : G = R \), contradicting \( Q \) is proper) and \( \text{ht} I = \text{ht} G \). Then, by Proposition 2.5 (b) and Theorem 2.2, we have

\[
e(R/Q) = e(R/G) - e(R/I) \geq \max\{1, s(R/G) - \deg(h) + 1\}.
\]

Now, from the standard exact sequence from linkage

\[
0 \to G \to I \to \omega_{R/Q}(-a(R/G)) \to 0
\]

we have \( \omega_{R/Q} \cong I/G[a(R/G)] \). Since \( I/G \) is generated by the image of \( h \), we obtain

\[
a(R/Q) = a(R/G) - \deg(h).
\]

This fact, together with the equalities \( s(R/G) = \dim(R/G) + a(R/G) \) and \( \dim(R/G) = \dim(R/Q) \), yields

\[
e(R/Q) \geq \max\{1, s(R/G) - \deg(h) + 1\} = \max\{1, \dim(R/G) + a(R/G) - \deg(h) + 1\} = \max\{1, \dim(R/Q) + a(R/Q) + 1\}.
\]

We now prove a multiplicity-based sufficient condition for \( R/Q \) to be Gorenstein.

**Theorem 3.3.** Let \( Q \) be a homogeneous quasi-Gorenstein ideal. If

\[
e(R/Q) = \max\{1, a(R/Q) + \dim(R/Q) + 1\},
\]

then \( R/Q \) is Gorenstein.

**Proof.** We may assume \( Q \) is proper. Let \( G \subseteq Q \) be a Gorenstein ideal with \( \text{ht} G = \text{ht} Q \), and set \( I = G : Q \). From the proof of Proposition 3.2 the given equality implies that \( I \) has maximal multiplicity.

Since \( \deg(h) = a(R/G) - a(R/Q) \) and \( -a(R/Q) \leq \dim(R/Q) \), then we have \( \deg(h) \leq s(R/G) \). By Theorem 2.7 (b), this yields that \( R/I \) is Cohen-Macaulay. Thus, by Proposition 2.5 (a), \( R/Q \) is Cohen-Macaulay and, then, Gorenstein.

Next, we would like to point out an analogy between the two extremal values of \( e(R/I) \). Assume \( I = J + (F) \), where \( J \) is Gorenstein and \( F \notin J \) is a zero divisor on \( R/J \); then

\[
1 \leq e(R/I) \leq e(R/J) - \max\{1, s(R/J) - \deg(F) + 1\}.
\]
If $I$ has maximal multiplicity, by Theorem 2.7 either $R/I$ is Cohen-Macaulay or it is almost Cohen-Macaulay. When $e(R/I) = 1$ a similar statement holds.

**Proposition 3.4.** Let $J$ be a homogeneous Gorenstein ideal, and let $F \notin J$ be a homogeneous element such that $\text{ht } I = \text{ht } J$, where $I = J + (F)$. Assume $e(R/I) = 1$.

(a) [Samuel [20], [17]] $I$ is unmixed if and only if $I$ is Cohen-Macaulay.

(b) $I$ is not unmixed if and only if $\text{depth}(R/I) = \text{dim}(R/I) - 1$ (that is, $R/I$ is almost Cohen-Macaulay).

**Proof.** We prove only assertion (b). If $\text{depth}(R/I) = \text{dim}(R/I) - 1$, then $R/I$ is not Cohen-Macaulay; hence, by assertion (a), $I$ is not unmixed. Next, assume $I$ is not unmixed. Set $L = J : I$ and note that $L \sim I^u$, by Proposition 2.3. Since $e(R/I^u) = e(R/I) = 1$, the ideal $I^u$ is Cohen-Macaulay, by assertion (a). By Proposition 2.5 (a), $R/L$ is also Cohen-Macaulay. Then, Lemma 2.6 gives $\text{pd}(R/I) \leq \text{grade } I + 1$. An application of the Auslander-Buchsbaum formula now proves $\text{depth}(R/I) \geq \text{dim}(R/I) - 1$. Since $I$ is not unmixed, we have $\text{depth}(R/I) \neq \text{dim}(R/I)$, yielding $\text{depth}(R/I) = \text{dim}(R/I) - 1$. □

It is then natural to ask whether the analogue of Proposition 3.4 holds: assume $I$ has maximal multiplicity; is it true that $I$ is unmixed if and only if $I$ is Cohen-Macaulay? Corollary 3.5 provides a positive answer.

**Corollary 3.5.** Let $J$ be a homogeneous Gorenstein ideal, and let $F \notin J$ be a homogeneous element such that $\text{ht } I = \text{ht } J$, where $I = J + (F)$. Assume $I$ has maximal multiplicity.

(a) $I$ is unmixed if and only if $I$ is Cohen-Macaulay if and only if $\text{deg}(F) \leq s(R/J)$.

(b) $I$ is not unmixed if and only if $\text{depth}(R/I) = \text{dim}(R/I) - 1$ if and only if $\text{deg}(F) > s(R/J)$.

**Proof.** (a) By Theorem 2.7 (c) we only need to prove that if $I$ is unmixed, then $I$ is Cohen-Macaulay. If $I$ is unmixed, the ideal $Q = J : I$ is quasi-Gorenstein by Proposition 3.4. As in the proof of Proposition 3.2, the maximal multiplicity of $I$ yields $e(R/Q) = \max \{1, a(R/Q) + \text{dim}(R/Q) + 1\}$. Now, Theorem 3.3 implies that $Q$ is Gorenstein and then, by Proposition 2.5 (a), $I$ is Cohen-Macaulay.

Assertion (b) follows from assertion (a) and Theorem 2.7 (a) and (c). □

We conclude with a question. If $I$ is an ideal of maximal multiplicity, in general there could be several different maximal decompositions $I = J + (F)$. Moreover, if $F$ and $F'$ are minimal generators of $I$, there could be a maximal decomposition of $I$ of the form $I = J + (F)$, but no maximal decomposition for $I$ of the form $I = J' + (F')$. These observations raise the following question: Is there an implicit characterization of ideals $I$ of maximal multiplicity? Theorem 2.13 gives a sufficient condition when $I$ is Cohen-Macaulay, and Proposition 2.12 gives a necessary condition, but we don’t know a necessary and sufficient condition.

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