PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 7, July 2015, Pages 3155–3163
S 0002-9939(2015)12107-7
Article electronically published on March 18, 2015

HOMOCLINIC ORBITS FOR A CLASS OF DISCRETE PERIODIC HAMILTONIAN SYSTEMS

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(Communicated by Yingfei Yi)

Abstract. In this paper we establish new criteria for the existence of nontrivial homoclinic orbits to a class of discrete Hamiltonian systems. Our results do not need to suppose that the system satisfies the well-known global Ambrosetti-Rabinowitz superquadratic assumption.

1. Introduction

Consider the second-order self-adjoint discrete Hamiltonian system

$$\triangle[p(n)\triangle u(n-1)] - L(n)u(n) + \nabla W(n, u(n)) = 0, \tag{1.1}$$

where $n \in \mathbb{Z}$, $u \in \mathbb{R}^{\mathcal{N}}$, $\triangle u(n) = u(n+1) - u(n)$ is the forward difference, $p, L : \mathbb{Z} \to \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ and $W : \mathbb{Z} \times \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$ satisfy

(P) $p(n)$ is an $\mathcal{N} \times \mathcal{N}$ real symmetric positive definite matrix for all $n \in \mathbb{Z}$, and $p(n+N) = p(n)$, $\forall n \in \mathbb{Z}$;

(L) $L(n)$ is an $\mathcal{N} \times \mathcal{N}$ real symmetric positive definite matrix for all $n \in \mathbb{Z}$, $L(n+N) = L(n)$, $\forall n \in \mathbb{Z}$, and there exists a constant $\beta > 0$ such that

$$(L(n)x, x) \geq \beta |x|^2, \quad \forall\ (n, x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}};$$

(W1) $W(n, x)$ is continuously differentiable in x for every $n \in \mathbb{Z}$, $W(n, 0) = 0$, $W(n, x) \geq 0$ and $W(n+N, x) = W(n, x)$ for all $(n, x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}$, where N is a given positive integer;

(W2) $\lim_{|x|\to 0} \frac{\nabla W(n,x)}{|x|} = 0$ uniformly for $n \in \mathbb{Z}$.

In general, system (1.1) may be regarded as a discrete analogue of the following second-order Hamiltonian system

$$[p(t)u'(t)]' - L(t)u(t) + \nabla W(t, u(t)) = 0 \tag{1.2}$$

which has been extensively studied in the literature on the existence and multiplicity of its homoclinic orbits (see, for example [6], [10], [11], [16] and references therein). Moreover, system (1.1) also comes from actual applications; see the monographs [1], [2].

Received by the editors November 18, 2012 and, in revised form, December 4, 2012.
2010 *Mathematics Subject Classification.* Primary 58E05; Secondary 70H05.
This project was supported by the Doctoral Program Foundation of the Ministry of Education of China (20104410110001).

As usual, we say that a solution $u(n)$ of system (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \not\equiv 0$, then $u(n)$ is called a nontrivial homoclinic solution.

It is well known that homoclinic orbits play a very important role in the study of chaos in dynamical systems. It has been proved that the system must be chaotic provided it has the transversely intersected homoclinic orbits. Therefore, it possesses important theoretical significance and practical value to investigate the existence of homoclinic orbits of system (1.1) emanating from zero.

In recent years, the existence of homoclinic solutions of system (1.1) or their special forms have been investigated by many authors; see, for example [13], [15] where $W(n,x)$ is subquadratic as $|x| \to \infty$ and [4], [5], [7], [8], [9], [11], [14], [17], [18] where $W(n,x)$ is superquadratic. Moreover, in the superquadratic case, almost all the existing results always need the next well-known global Ambrosetti-Rabinowitz superquadratic condition:

(AR) there exists $\mu > 2$ such that

$$0 < \mu W(n,x) \le (\nabla W(n,x), x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}} \setminus \{0\},$$

where and in the sequel, $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^{\mathcal{N}}$ and $|\cdot|$ is the induced norm.

In this paper, we introduce the following weaker assumptions than (AR):

(W3) $\lim_{|x|\to\infty} \frac{|W(n,x)|}{|x|^2} = \infty$ for all $n \in \mathbb{Z}$;

(W4) $\widetilde{W}(n,x) := \frac{1}{2}(\nabla W(n,x), x) - W(n,x) > 0$, $\forall\ (n,x) \in \mathbb{Z} \times (\mathbb{R}^{\mathcal{N}} \setminus \{0\})$, and there exist $c_1 > 0$ and $R_0 > 0$ such that

$$W(n,x) \le c_1 |x|^2 \widetilde{W}(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, \ \ |x| \ge R_0.$$

Now, we are ready to state the main result of this paper.

Theorem 1.1. *Assume that p, L and W satisfy* (P), (L), (W1), (W2), (W3) *and* (W4). *Then system* (1.1) *possesses a nontrivial homoclinic solution.*

Remark 1.2. If $W(n,x)$ satisfies (AR), then there exists a constant $C_0 > 0$ such that

$$W(n,x) \ge C_0 |x|^{\mu}, \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, \ \ |x| \ge 1;$$

moreover $\widetilde{W}(n,x) > 0$ for all $(n,x) \in \mathbb{Z} \times (\mathbb{R}^{\mathcal{N}} \setminus \{0\})$, and

$$W(n,x) \le \frac{2}{\mu - 2} |x|^2 \widetilde{W}(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, \ \ |x| \ge 1.$$

These show that (W3) and (W4) are satisfied.

Remark 1.3. There are many functions $W(n,x)$ that satisfy (W3) and (W4) but not (AR). For example:

$$W(n,x) = a(n)|x|^2 \ln(1 + |x|), \tag{1.3}$$

$$W(n,x) = b(n)\left(|x|^4 + 2|x|^3 \sin^2 |x|\right), \tag{1.4}$$

where $a(n)$ and $b(n)$ are N-periodic positive functions.

2. Preliminaries

Throughout this section, we always assume that p and L satisfy (P) and (L). Let

$$S = \left\{ \{u(n)\}_{n\in\mathbb{Z}} \ : \ u(n) \in \mathbb{R}^{\mathcal{N}}, \ n \in \mathbb{Z} \right\},$$

$$E = \left\{ u \in S \ : \ \sum_{n\in\mathbb{Z}} \left[(p(n+1)\triangle u(n), \triangle u(n)) + (L(n)u(n), u(n)) \right] < +\infty \right\},$$

and for $u, v \in E$, let

$$\langle u, v\rangle = \sum_{n\in\mathbb{Z}} \left[(p(n+1)\triangle u(n), \triangle v(n)) + (L(n)u(n), v(n)) \right].$$

Then E is a Hilbert space with the above inner product, and the corresponding norm is

$$\|u\| = \left\{ \sum_{n\in\mathbb{Z}} \left[(p(n+1)\triangle u(n), \triangle u(n)) + (L(n)u(n), u(n)) \right] \right\}^{1/2}, \qquad u \in E.$$

As usual, for $1 \le q < +\infty$, set

$$l^q(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}) = \left\{ u \in S \ : \ \sum_{n\in\mathbb{Z}} |u(n)|^q < +\infty \right\},$$

and

$$l^\infty(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}) = \left\{ u \in S \ : \ \sup_{n\in\mathbb{Z}} |u(n)| < +\infty \right\},$$

and their norms are defined by

$$\|u\|_q = \left(\sum_{n\in\mathbb{Z}} |u(n)|^q \right)^{1/q}, \quad \forall\, u \in l^q(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}); \quad \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)|, \quad \forall\, u \in l^\infty(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}),$$

respectively. Evidently, E is continuously embedded into $l^q(\mathbb{Z}, \mathbb{R}^{\mathcal{N}})$ for $2 \le q \le \infty$; i.e., there exists $\gamma_q > 0$ such that

$$\|u\|_q \le \gamma_q \|u\|, \qquad \forall\, u \in E. \tag{2.1}$$

Lemma 2.1 ([9]). *For $u \in E$, one has*

$$\|u\|_\infty \le \frac{1}{\sqrt[4]{(\beta + 4\alpha)\beta}} \|u\|, \tag{2.2}$$

where $\alpha = \inf\{(p(n)x, x) : n \in \mathbb{Z}, \ x \in \mathbb{R}^{\mathcal{N}}, \ |x| = 1\}$.

Lemma 2.2 ([3]). *Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \eta_0 < \eta \le \inf_{\|u\|=\rho} I(u),$$

for some $\eta_0 < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \ \gamma(1) = e\}$ is the set of continuous paths joining 0 to e; then there exists $\{u_k\}_{k\in\mathbb{N}} \subset E$ such that

$$I(u_k) \to c \quad \text{and} \quad (1 + \|u_k\|)\|I'(u_k)\|_{E^*} \to 0 \quad \text{as } k \to \infty.$$

Now we define a functional Φ on E by

(2.3)
$$\Phi(u)=\frac{1}{2}\sum_{n\in\mathbb{Z}}[(p(n+1)\triangle u(n),\triangle u(n))+(L(n)u(n),u(n))]-\sum_{n\in\mathbb{Z}}W(n,u(n)).$$

For any $u\in E$, there exists an $n_1\in\mathbb{N}$ such that $|u(n)|\le 1$ for $|n|\ge n_1$. Hence, under assumptions (P), (L), (W1) and (W2), the functional Φ is of class $C^1(E,\mathbb{R})$. Moreover,

$$\Phi(u)=\frac{1}{2}\|u\|^2-\sum_{n\in\mathbb{Z}}W(n,u(n)),\quad \forall\, u\in E, \tag{2.4}$$

$$\langle\Phi'(u),v\rangle=\langle u,v\rangle-\sum_{n\in\mathbb{Z}}(\nabla W(n,u(n)),v(n)),\quad \forall\, u,v\in E. \tag{2.5}$$

Furthermore, the critical points of Φ in E are solutions of system (1.1) with $u(\pm\infty)=0$; see [9], [13].

Lemma 2.3. *Under assumptions* (P), (L), (W1), (W2) *and* (W3), *there exist a constant $c>0$ and a sequence $\{u_k\}\subset E$ satisfying*

$$\Phi(u_k)\to c,\quad \|\Phi'(u_k)\|(1+\|u_k\|)\to 0. \tag{2.6}$$

Proof. From (W2), there exists $\delta_0>0$ such that

$$|\nabla W(t,x)|\le\frac{1}{2\gamma_2^2}|x|,\quad \forall\,(n,x)\in\mathbb{Z}\times\mathbb{R}^{\mathcal{N}},\ \ |x|\le\delta_0.$$

Since $W(n,0)=0$, it follows that

$$W(n,x)\le\frac{1}{4\gamma_2^2}|x|^2,\quad \forall\,(n,x)\in\mathbb{Z}\times\mathbb{R}^{\mathcal{N}},\ \ |x|\le\delta_0. \tag{2.7}$$

If $\|u\|=\sqrt[4]{(\beta+4\alpha)\beta}\,\delta_0:=\rho$, then by Lemma 2.1, $|u(n)|\le\delta_0$ for all $n\in\mathbb{Z}$. Hence, from (2.4) and (2.7), one has

$$\begin{aligned}\Phi(u)&=\frac{1}{2}\|u\|^2-\sum_{n\in\mathbb{Z}}W(n,u(n))\\&\ge\frac{1}{2}\|u\|^2-\frac{1}{4\gamma_2^2}\sum_{n\in\mathbb{Z}}|u(n)|^2\\&=\frac{1}{2}\|u\|^2-\frac{1}{4\gamma_2^2}\|u\|_2^2\\&\ge\frac{1}{4}\|u\|^2=\frac{1}{4}\rho^2,\quad \forall\,u\in E,\ \ \|u\|=\rho.\end{aligned}$$

Choose $u_0\in E$ as follows:

$$u_0(0)=(1,0,\cdots,0)^\top\in\mathbb{R}^{\mathcal{N}},\quad u_0(n)=0,\ \ \forall\,n\ne 0.$$

Then it follows from (W1), (W3) and (2.4) that

$$\begin{aligned}\Phi(su_0) &= \frac{s^2}{2}\|u_0\|^2 - \sum_{n\in\mathbb{Z}} W(n, su_0(n)) \\ &= \frac{s^2}{2}\|u_0\|^2 - W(0, su_0(0)) \\ &= s^2\left[\frac{1}{2}\|u_0\|^2 - \frac{W(0, su_0(0))}{|su_0(0)|^2}\right] \\ &\leq 0, \quad \text{for large } s > 0.\end{aligned}$$

Choose $s_1 > 1$ such that $s_1\|u_0\| > \rho$ and $\Phi(s_1u_0) \leq 0$. Let $e = s_1u_0$; then $e \in E$, $\|e\| > \rho$ and $\Phi(e) \leq 0$. By Lemma 2.2, there exist a constant $c \geq \rho^2/4$ and a sequence $\{u_k\} \subset E$ such that (2.6) holds. ☐

Lemma 2.4. *Under assumptions* (P), (L), (W1), (W2), (W3) *and* (W4), *any sequence* $\{u_k\} \subset E$ *satisfying*

$$\Phi(u_k) \to c > 0, \qquad \langle \Phi'(u_k), u_k\rangle \to 0 \tag{2.8}$$

is bounded in E.

Proof. To prove the boundedness of $\{u_k\}$, arguing by contradiction, we suppose that $\|u_k\| \to \infty$. Let $v_k = u_k/\|u_k\|$. Then $\|v_k\| = 1$ and $\|v_k\|_q \leq \gamma_q\|v_k\| = \gamma_q$ for $2 \leq q \leq \infty$. Observe that for k large

$$c + 1 \geq \Phi(u_k) - \frac{1}{2}\langle \Phi'(u_k), u_k\rangle = \sum_{n\in\mathbb{Z}} \widetilde{W}(n, u_k(n)). \tag{2.9}$$

It follows from (2.4) and (2.8) that

$$\frac{1}{2} \leq \limsup_{k\to\infty} \sum_{n\in\mathbb{Z}} \frac{|W(n, u_k(n))|}{\|u_k\|^2}. \tag{2.10}$$

For $0 \leq a < b$, let

$$\Omega_k(a, b) = \{n \in \mathbb{Z} : a \leq |u_k(n)| < b\}. \tag{2.11}$$

Passing to a subsequence, we may assume that $v_k \rightharpoonup v$ in E; then $v_k(n) \to v(n)$ for all $n \in \mathbb{Z}$.

Let $\delta := \limsup_{k\to\infty} \|v_k\|_\infty$. By (W1) and (W2), there exist $a_0 \in (0, R_0)$ and $\Theta > 0$ such that

$$|W(n, x)| \leq \frac{1}{3\gamma_2}|x|^2, \quad \forall\ (n, x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, \ \ |x| \leq a_0,$$

and

$$|W(n, x)| \leq \Theta|x|^2, \quad \forall\ (n, x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}, \ \ |x| \leq R_0; \tag{2.12}$$

consequently,

$$\sum_{n\in\Omega_k(0,a_0)} \frac{|W(n, u_k(n))|}{|u_k(n)|^2}|v_k(n)|^2 \leq \frac{1}{3\gamma_2} \sum_{n\in\Omega_k(0,a_0)} |v_k(n)|^2 \leq \frac{1}{3}, \qquad k \in \mathbb{N}. \tag{2.13}$$

Let

$$\vartheta = \inf\left\{\frac{\widetilde{W}(n, x)}{|x|^2} : n \in \mathbb{Z},\ x \in \mathbb{R}^{\mathcal{N}},\ a_0 \leq |x| \leq R_0\right\}.$$

Since $W(n,x)$ depends periodically on n and $\widetilde{W}(n,x) > 0$ if $x \neq 0$, one has $\vartheta > 0$ and

$$\widetilde{W}(n, u_k(n)) \geq \vartheta |u_k(n)|^2, \quad \forall\, n \in \Omega_k(a_0, R_0). \tag{2.14}$$

From (W4), (2.9) and (2.14), we have

$$c + 1 \geq \sum_{n\in\mathbb{Z}} \widetilde{W}(n, u_k(n)) \geq \vartheta \sum_{n\in\Omega_k(a_0,R_0)} |u_k(n)|^2. \tag{2.15}$$

It follows from (2.12) that

$$\begin{aligned}
\sum_{n\in\Omega_k(a_0,R_0)} \frac{|W(n,u_k(n))|}{|u_k(n)|^2}|v_k(n)|^2 &\leq \Theta \sum_{n\in\Omega_k(a_0,R_0)} |v_k(n)|^2 \\
&= \frac{\Theta}{\|u_k\|^2} \sum_{n\in\Omega_k(a_0,R_0)} |u_k(n)|^2 \\
&\leq \frac{(c+1)\Theta}{\vartheta\|u_k\|^2} \to 0, \quad k \to \infty. \qquad (2.16)
\end{aligned}$$

If $\delta = 0$, then from (W4) and (2.9), one has

$$\begin{aligned}
&\sum_{n\in\Omega_k(R_0,\infty)} \frac{|W(n,u_k(n))|}{|u_k(n)|^2}|v_k(n)|^2 \\
\leq\ & \|v_k\|_\infty \sum_{n\in\Omega_k(R_0,\infty)} \frac{|W(n,u_k(n))|}{|u_k(n)|^2} \\
\leq\ & c_1\|v_k\|_\infty \sum_{n\in\Omega_k(R_0,\infty)} \widetilde{W}(n,u_k(n)) \\
\leq\ & c_1(c+1)\|v_k\|_\infty \to 0, \quad k \to \infty. \qquad (2.17)
\end{aligned}$$

From (2.13), (2.16) and (2.17), we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{|W(n,u_k(n))|}{\|u_k\|^2} &= \sum_{n\in\Omega_k(0,a_0)} \frac{|W(n,u_k(n))|}{|u_k(n)|^2}|v_k(n)|^2 \\
&\quad + \sum_{n\in\Omega_k(a_0,R_0)} \frac{|W(n,u_k(n))|}{|u_k(n)|^2}|v_k(n)|^2 \\
&\quad + \sum_{n\in\Omega_k(R_0,\infty)} \frac{|W(n,u_k(n))|}{|u_k(n)|^2}|v_k(n)|^2 \\
&\leq \frac{1}{3} + o(1), \quad k \to \infty,
\end{aligned}$$

which contradicts (2.10). Thus $\delta > 0$.

Going to a subsequence if necessary, we may assume the existence of $n_k \in \mathbb{Z}$ such that

$$|v_k(n_k)| = \|v_k\|_\infty > \frac{\delta}{2}.$$

Choose integers i_k and m_k with $0 \leq m_k \leq N-1$ such that $n_k = i_k N + m_k$. Let $w_k(n) = v_k(n + i_k N)$; then

$$|w_k(m_k)| > \frac{\delta}{2}, \quad \forall\, k \in N. \tag{2.18}$$

Now we define $\tilde{u}_k(n) = u_k(n+i_kN)$. Since $p(n)$, $L(n)$ and $W(n,x)$ are N-periodic on n, then $\tilde{u}_k(n)/\|u_k\| = w_k(n)$ and $\|\tilde{u}_k\| = \|u_k\|$. Passing to a subsequence, we have $w_k \rightharpoonup w$ in E; then $w_k(n) \to w(n)$ for all $n \in \mathbb{Z}$. Thus (2.18) implies that $w(n) \neq 0$ for some $n \in \{0, 1, \dots, N-1\}$.

It is obvious that $w(n) \neq 0$ implies $\lim_{k\to\infty} |\tilde{u}_k(n)| = \infty$. Hence, it follows from (2.4) and (W3) that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \frac{c+o(1)}{\|u_k\|^2} = \lim_{k\to\infty} \frac{\Phi(u_k)}{\|u_k\|^2} \\
&= \lim_{k\to\infty} \left[\frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{W(n,u_k(n))}{|u_k(n)|^2}|v_k(n)|^2\right] \\
&= \lim_{k\to\infty} \left[\frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{W(n+i_kN,u_k(n+i_kN))}{|u_k(n+i_kN)|^2}|v_k(n+i_kN)|^2\right] \\
&= \lim_{k\to\infty} \left[\frac{1}{2} - \sum_{n\in\mathbb{Z}} \frac{W(n,\tilde{u}_k(n))}{|\tilde{u}_k(n)|^2}|w_k(n)|^2\right] \\
&\leq \frac{1}{2} - \liminf_{k\to\infty} \sum_{n\in\mathbb{Z}} \frac{W(n,\tilde{u}_k(n))}{|\tilde{u}_k(n)|^2}|w_k(n)|^2 \\
&\leq \frac{1}{2} - \liminf_{k\to\infty} \sum_{n=0}^{N-1} \frac{W(n,\tilde{u}_k(n))}{|\tilde{u}_k(n)|^2}|w_k(n)|^2 \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Thus $\{u_k\}$ is bounded in E. □

3. Proof of the theorem

Proof of Theorem 1.1. Lemma 2.3 implies the existence of a sequence $\{u_k\} \subset E$ satisfying (2.6), and so (2.8). By Lemma 2.4, $\{u_k\}$ is bounded in E. Thus there exists a constant $C_1 > 0$ such that

$$\sqrt[4]{(\beta+4\alpha)\beta}\,\|u_k\|_\infty \leq \|u_k\| \leq C_1, \quad \forall\, k \in \mathbb{N}. \tag{3.1}$$

Hence, by (W1) and (W2), there exists a constant $C_2 > 0$ such that

$$|\widetilde{W}(n,x)| \leq \frac{c}{2C_1^2\gamma_2^2}|x|^2 + C_2|x|^3, \quad \forall\, (n,x) \in \mathbb{Z}\times\mathbb{R}^{\mathcal{N}}, \ |x| \leq \frac{1}{\sqrt[4]{(\beta+4\alpha)\beta}}C_1. \tag{3.2}$$

If $\delta := \limsup_{k\to\infty} \|u_k\|_\infty = 0$, then for $q > 2$,

$$\sum_{n\in\mathbb{Z}} |u_k(n)|^q \leq \|u_k\|_\infty^{q-2} \sum_{n\in\mathbb{Z}} |u_k(n)|^2 \leq \gamma_2^2 \|u_k\|_\infty^{q-2} \to 0, \quad k \to \infty. \tag{3.3}$$

From (2.4), (2.5), (2.8), (3.2) and (3.3), one has

$$\begin{aligned} c &= \Phi(u_k) - \frac{1}{2}\langle \Phi'(u_k), u_k\rangle + o(1) \\ &= \sum_{n\in\mathbb{Z}} \widetilde{W}(n, u_k(n) + o(1) \\ &\le \frac{c}{2C_1^2\gamma_2^2}\sum_{n\in\mathbb{Z}} |u_k(n)|^2 + C_2 \sum_{n\in\mathbb{Z}} |u_k(n)|^3 + o(1) \\ &\le \frac{c}{2} + o(1), \qquad k\to\infty. \end{aligned}$$

This contradiction shows that $\delta > 0$.

Going to a subsequence if necessary, we may assume the existence of $n_k \in \mathbb{Z}$ such that

$$|u_k(n_k)| = \|u_k\|_\infty > \frac{\delta}{2}.$$

Choose integers i_k and m_k with $0 \le m_k \le N-1$ such that $n_k = i_k N + m_k$. Let $v_k(n) = u_k(n + i_k N)$; then

$$|v_k(m_k)| > \frac{\delta}{2}, \qquad \forall\, k \in \mathbb{N}. \tag{3.4}$$

Since $p(n)$, $L(n)$ and $W(n,x)$ are N-periodic on n, we have $\|v_k\| = \|u_k\|$ and

$$\Phi(v_k) \to c, \qquad \|\Phi'(v_k)\|(1 + \|v_k\|) \to 0. \tag{3.5}$$

Passing to a subsequence, we have $v_k \rightharpoonup v$ in E, $v_k(n) \to v(n)$ for all $n \in \mathbb{Z}$. Thus, (3.4) implies that $v \ne 0$. Let

$$E_0 = \{u \in E : \{n \in \mathbb{Z} : |u(n)| > 0\} \text{ is finite set}\}.$$

Then for every $w \in E_0$, there exists an $n_0 \in \mathbb{N}$ such that $w(n) = 0$ for all $|n| > n_0$. Hence, it follows from (2.5) and (3.5) that

$$\begin{aligned} \langle \Phi'(v), w\rangle &= \sum_{n\in\mathbb{Z}} [(p(n+1)\triangle v(n), \triangle w(n)) + (L(n)v(n), w(n)) \\ &\qquad\qquad - (\nabla W(n, v(n)), w(n))] \\ &= \sum_{|n|\le n_0} [(p(n+1)\triangle v(n), \triangle w(n)) + (L(n)v(n), w(n)) \\ &\qquad\qquad - (\nabla W(n, v(n)), w(n))] \\ &= \lim_{k\to\infty} \sum_{|n|\le n_0} [(p(n+1)\triangle v_k(n), \triangle w(n)) + (L(n)v_k(n), w(n)) \\ &\qquad\qquad - (\nabla W(n, v_k(n)), w(n))] \\ &= \lim_{k\to\infty} \langle \Phi'(v_k), w\rangle = 0. \end{aligned}$$

Since E_0 is dense in E, $\Phi'(v) = 0$. Thus the proof is finished. □

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