EXISTENCE AND ANALYTICITY OF LEI-LIN SOLUTION TO THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we prove the recent work of Lei-Lin in a slightly different setting, which enables us to prove analyticity of the solution.

1. Introduction

In this paper, we study the incompressible Navier-Stokes equations in $\mathbb{R}^3$:

$$v_t + v \cdot \nabla v - \mu \Delta v + \nabla p = 0, \quad \nabla \cdot v = 0,$$

where $v$ is the velocity field, $p$ is the pressure, and $\mu > 0$ is the viscosity coefficient.

The aim of this paper is to prove the recent work of Lei-Lin [15] in a slightly different fashion. To this end, we reformulate the problem as follows. Since $\nabla \cdot v = 0$, we can rewrite (1.1) by projecting it onto the divergence-free space. Namely, we apply the orthogonal projection $P = \text{Id} - \nabla(-\Delta)^{-1} \text{div}$ in $L^2$ over divergence-free vector fields to (1.1). Then, we obtain

$$v_t + P \nabla \cdot (v \otimes v) - \mu \Delta v = 0.$$ 

Formally, we can express a solution $v$ of (1.2) in the integral form:

$$v(t) = e^{\mu t \Delta} v_0 - \int_0^t \left[ e^{\mu (t-s) \Delta} P \nabla \cdot (v \otimes v)(s) \right] ds.$$ 

Any solution satisfying this integral form is called a mild solution, and we can find it by using a fixed point argument for the function $v \mapsto F(v)$, where

$$F(v)(t) = e^{\mu t \Delta} v_0 - \int_0^t \left[ e^{\mu (t-s) \Delta} P \nabla \cdot (v \otimes v)(s) \right] ds.$$ 

The function spaces for solving this integral equation correspond to a scale invariance property of the equation. Assume that $(v, p)$ solves (1.1). Then, the same is true for rescaled functions:

$$v_\lambda(t, x) = \lambda v(\lambda^2 t, \lambda x), \quad p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x).$$

Under these scalings, $L^3, H^{1\frac{1}{2}}, W^{3-1,p}_\gamma, B^{3-1}_p$, and $BMO^{-1}$ are critical spaces for initial data $(t = 0)$ to name a few, and one can find various well-posedness results for small data in these critical spaces in [3–6,8,13,14,20].
We now state the result in [15], where Lei and Liu introduced the following function spaces:

\[ X^{-1} = \left\{ f \in \mathcal{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{-1} |\hat{f}(\xi)| \, d\xi < \infty \right\}, \]

\[ X^1 = \left\{ f \in \mathcal{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi| |\hat{f}(\xi)| \, d\xi < \infty \right\}, \]

where \( X^{-1} \) is also a critical space for initial data.

**Theorem 1.1** ([15]). For any initial data in \( X^{-1} \) with \( \|v_0\|_{X^{-1}} < \mu \), there exists a unique global in time solution \( v \in C(\mathbb{R}^+; X^{-1}) \cap L^1(\mathbb{R}^+; X^1) \) such that

\[ \sup_{0 \leq t < \infty} \left[ \|v(t)\|_{X^{-1}} + (\mu - \|v_0\|_{X^{-1}}) \int_0^t \|Dv(s)\|_{L^\infty} \, ds \right] \leq \|v_0\|_{X^{-1}}. \]

In this paper, we prove a similar result to [15] in the following time dependent spaces:

\[ \mathcal{X}^{-1} = \left\{ f \in \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^3) : \int_{\mathbb{R}^3} \left[ \sup_{0 \leq t < \infty} |\xi|^{-1} |\hat{f}(t, \xi)| \right] \, d\xi < \infty \right\}, \]

\[ \mathcal{X}^1 = \left\{ f \in \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^3) : \int_{\mathbb{R}^3} \left[ \int_0^\infty |\xi| |\hat{f}(t, \xi)| \, dt \right] \, d\xi < \infty \right\}. \]

We note that we define these spaces by switching the order of time and Fourier variables to prove the existence result (Theorem [12]) in the integral form [13]. Although we cannot take the size of initial data precisely in terms of the size of the viscosity coefficient, our result allows us to establish analyticity of the solution (Theorem [13]). (For various analyticity results, see [12, 7, 10, 12, 18, 19] and the references therein.) For simplicity, we take \( \mu = 1 \) for the rest of the paper.

**Theorem 1.2.** There exists a positive constant \( \epsilon_0 > 0 \) such that for any initial data in \( X^{-1} \) with \( \|v_0\|_{X^{-1}} < \epsilon_0 \) there exists a unique global in time solution \( v \in \mathcal{X}^{-1} \cap \mathcal{X}^1 \) such that

\[ \|v\|_{\mathcal{X}^{-1}} + \|v\|_{\mathcal{X}^1} \lesssim \|v_0\|_{X^{-1}} \]

**Theorem 1.3.** There exists a positive constant \( \epsilon_0 > 0 \) such that for any initial data in \( X^{-1} \) with \( \|v_0\|_{X^{-1}} < \epsilon_0 \), the solution in Theorem [1.2] is analytic in the sense that

\[ \left\| e^{\sqrt{t}|D|} v \right\|_{X^{-1}} + \left\| e^{\sqrt{t}|D|} v \right\|_{X^1} \lesssim \|v_0\|_{X^{-1}}, \]

where \( e^{\sqrt{t}|D|} \) is a Fourier multiplier whose symbol is given by \( e^{\sqrt{t}|\xi|} \).

2. Proof of Theorem [1.2]

Here we provide only a priori estimate (1.7). The iteration can be performed by a regularization argument and we skip the iteration step.

**Step 1.** We first take the Fourier transform to (1.3):

\[ \hat{v}(t, \xi) = e^{-t|\xi|^2} \hat{v}_0(\xi) \]

\[ -\int_0^t \left[ e^{-(t-s)|\xi|^2} P_{\xi} \left( \int_{\mathbb{R}^3} \hat{v}(s, \xi - \eta) \hat{v}(s, \eta) \, d\eta \right) \right] \, ds. \]
Since the Fourier multiplier is a bounded Fourier multiplier, we ignore this term in the estimations. We first estimate \( v \) in \( \mathcal{X}^{-1} \). By multiplying (2.1) by \(|\xi|^{-1}\), we have
\[
|||\xi|^{-1}\widehat{v}(t, \xi)|| \leq e^{-t||\xi||^2}|||\xi|^{-1}\widehat{v}_0(\xi)||
\]
(2.2)
\[
+ \int_0^t \left[ e^{-t-s}||\xi||^2 \left( \int_{\mathbb{R}^3} |\widehat{v}(s, \xi - \eta)||\widehat{v}(s, \eta)| \, d\eta \right) \right] \, ds.
\]
Since \( 1 \leq \frac{|\xi - \eta|}{|\eta|} + \frac{|\eta|}{|\xi - \eta|} \), we can rewrite the nonlinear term as follows:
\[
\int_0^t \left[ \left( \int_{\mathbb{R}^3} \left( \frac{|\xi - \eta|}{|\eta|} + \frac{|\eta|}{|\xi - \eta|} \right) |\widehat{v}(t, \xi - \eta)||\widehat{v}(t, \eta)| \, d\eta \right) \right] \, ds
\]
(2.3)
\[
\leq \int_0^t \left[ \left( \int_{\mathbb{R}^3} \left( \frac{|\xi - \eta|}{|\eta|} + \frac{|\eta|}{|\xi - \eta|} \right) |\widehat{v}(s, \xi - \eta)||\widehat{v}(s, \eta)| \, d\eta \right) \right] \, ds.
\]
We take the \( L^\infty \) norm in time to (2.2). Then,
\[
\sup_{0 \leq t < \infty} |||\xi|^{-1}\widehat{v}(t, \xi)||
\]
(2.4)
\[
\leq |||\xi|^{-1}\widehat{v}_0(\xi)||
\]
\[
+ \int_0^\infty \left[ \left( \int_{\mathbb{R}^3} \left( \frac{|\xi - \eta|}{|\eta|} + \frac{|\eta|}{|\xi - \eta|} \right) |\widehat{v}(t, \xi - \eta)||\widehat{v}(t, \eta)| \, d\eta \right) \right] \, dt
\]
\[
\lesssim |||\xi|^{-1}\widehat{v}_0(\xi)|| + \left[ \int_0^\infty |\xi|\widehat{v}(t, \xi) \right] *_\xi \left( \sup_{0 \leq t < \infty} |||\xi|^{-1}\widehat{v}(t, \xi)|| \right).
\]
We take the \( L^1 \) norm in \( \xi \) to (2.4). By Young’s inequality,
\[
\|v\|_{\mathcal{X}^{-1}} \lesssim \|v_0\|_{\mathcal{X}^{-1}} + \|v\|_{\mathcal{X}^{-1}}
\]
(2.5)
\[
\|v\|_{\mathcal{X}^{-1}}<\infty.
\]
Step 2. We next estimate \( v \) in \( \mathcal{X}^{-1} \). We multiply (2.1) by \(|\xi|\). Then,
\[
|||\xi|v(t, \xi)|| \leq |||\xi|^{2}e^{-t||\xi||^2}|||\xi|^{-1}\widehat{v}_0(\xi)||
\]
(2.7)
\[
+ \int_0^t \left[ |||\xi|^{2}e^{-t-s}||\xi||^2 \left( \int_{\mathbb{R}^3} |\widehat{v}(s, \xi - \eta)||\widehat{v}(s, \eta)| \, d\eta \right) \right] \, ds.
\]
As before, we rewrite the nonlinear term as (2.3) and we take the \( L^1 \) norm in time to (2.7). Using
\[
\int_0^\infty |||\xi|^{2}e^{-t}||\xi||^2 \, dt < C
\]
and Young’s inequality, we have
\[
\int_0^\infty |||\xi|\widehat{v}(t, \xi)|| \, dt
\]
(2.8)
\[
\leq |||\xi|^{-1}\widehat{v}_0(\xi)||
\]
\[
+ \int_0^\infty \left[ \left( \int_{\mathbb{R}^3} \left( \frac{|\xi - \eta|}{|\eta|} + \frac{|\eta|}{|\xi - \eta|} \right) |\widehat{v}(t, \xi - \eta)||\widehat{v}(t, \eta)| \, d\eta \right) \right] \, dt
\]
\[
\lesssim |||\xi|^{-1}\widehat{v}_0(\xi)|| + \left[ \int_0^\infty |\xi|\widehat{v}(t, \xi) \right] *_\xi \left( \sup_{0 \leq t < \infty} |||\xi|^{-1}\widehat{v}(t, \xi)|| \right).
By taking the $L^1$ norm in $\xi$ to (2.8) and using Young’s inequality, we have
\begin{equation}
\|v\|_{X^1} \lesssim \|v_0\|_{X^{-1}} + \|v\|_{X^1} \|v\|_{X^{-1}}.
\end{equation}

**Step 3.** Combining (2.5) and (2.9), we finally have
\begin{equation}
\|v\|_{X^{-1}} + \|v\|_{X^1} \lesssim \|v_0\|_{X^{-1}} + \left(\|v\|_{X^{-1}} + \|v\|_{X^1}\right)^2,
\end{equation}
which implies the existence of a global-in-time solution in $X^{-1} \cap X^1$ for small data in $X^{-1}$.

### 3. Proof of Theorem 1.3

The proof of Theorem 1.3 is identical to the proof in [16], where Lemarié-Rieusset proved analyticity of the solution constructed by Le Jan–Sznitman [17]:
\[ \sup_{0 < t < \infty} \sup_{\xi \in \mathbb{R}^3} e^{\sqrt{t} |\xi|^2} |\hat{\nu}(t, \xi)| < \infty. \]

Here we provide details of the proof of Theorem 1.3 for the reader’s convenience. Again, we provide only a priori estimate (1.8) and we skip the iteration step. Let
\[ \hat{V}(t, \xi) := e^{\sqrt{t} |\xi|^2} \hat{v}(t, \xi). \]

Then, $\hat{V}(t, \xi)$ satisfies that
\begin{equation}
\hat{V}(t, \xi) = e^{\sqrt{t} |\xi|-t|\xi|^2} \hat{v}_0(\xi) - \int_0^t \left[ e^{\sqrt{t} |\xi|-(t-s)|\xi|^2} \xi \cdot \left( \int_{\mathbb{R}^3} \hat{v}(s, \xi - \eta) \hat{v}(s, \eta) d\eta \right) \right] ds.
\end{equation}

Since $e^{\sqrt{t} |\xi| - \frac{1}{2} t |\xi|^2}$ is uniformly bounded in time, the linear term behaves like the linear term of (2.1). Thus, we only focus on the nonlinear term:
\begin{align*}
&\int_0^t \left[ e^{\sqrt{t} |\xi|-(t-s)|\xi|^2} \xi \cdot \left( \int_{\mathbb{R}^3} \hat{v}(s, \xi - \eta) \hat{v}(s, \eta) d\eta \right) \right] ds \\
&= \int_0^t \left[ e^{(\sqrt{t} |\xi| - \sqrt{t} |\xi| - \frac{1}{2} (t-s)|\xi|^2}) e^{-\frac{1}{2} (t-s)|\xi|^2} \xi \cdot \left( \int_{\mathbb{R}^3} \hat{v}(s, \xi - \eta) \hat{v}(s, \eta) d\eta \right) \right] ds \\
&= \int_0^t \left[ e^{(\sqrt{t} |\xi| - \sqrt{t} |\xi| - \frac{1}{2} (t-s)|\xi|^2}) e^{-\frac{1}{2} (t-s)|\xi|^2} \xi \cdot \left( \int_{\mathbb{R}^3} \hat{v}(s, \xi - \eta) \hat{v}(s, \eta) d\eta \right) \right] ds \\
&\leq e^{-\frac{1}{2} |\xi|^2} \hat{v}_0(\xi) + \int_0^t \left[ e^{-\frac{1}{2} (t-s)|\xi|^2} \left( \int_{\mathbb{R}^3} |\hat{V}(s, \xi - \eta)| \left| \hat{V}(s, \eta) \right| d\eta \right) \right] ds.
\end{align*}

Therefore, we can obtain (1.8) by following the proof of Theorem 1.2 line by line. This completes the proof of Theorem 1.3.
References


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