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CORES FOR QUASICONVEX ACTIONS

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Abstract. We prove that any full relatively quasiconvex subgroup of a relatively hyperbolic group acting on a CAT(0) cube complex has a convex cocompact core. We give an application towards separability of quasiconvex subgroups of the fundamental group of a special cube complex.

1. Introduction

The aim of this paper is the following theorem. We refer to Section 4 for the definitions related to relative quasiconvexity.

Theorem 1.1. *Let $\widetilde{X}$ be a CAT(0) cube complex with a proper cocompact action by G. Suppose that G is hyperbolic relative to subgroups $\{P_1, \dots, P_r\}$. Let J be a full relatively quasiconvex subgroup. For each compact subspace $Q \subset \widetilde{X}$, there exists a J-cocompact convex subcomplex $\widetilde{Y}$ that contains Q.*

In the nonrelative case (i.e. when G is a hyperbolic group and J is a quasiconvex subgroup), the above theorem was proved independently by Haglund [6], who obtained the following:

Theorem 1.2. *Let G be a group acting on a finite-dimensional locally-finite δ-hyperbolic CAT(0) cube complex $\widetilde{X}$, and suppose that the action is quasiconvex. There exists a convex subcomplex of $\widetilde{X}$ on which G acts cocompactly.*

There are several situations where analogues of Theorem 1.1 hold (e.g. certain small-cancellation groups, certain groups with simplicial nonpositive curvature, Kleinian groups). However, outside some stronger combinatorial or geometric context, it is not known whether convex cocompact cores always exist for a quasiconvex subgroup H of a word-hyperbolic group G acting properly and cocompactly on a CAT(0) space.

The very simplest version of the above core theorems is the widely used 1-dimensional observation that the covering spaces of graphs corresponding to finitely generated subgroups have compact cores. The idea is implicit in Scott's work [15] which was generalized in [1], and appeared for certain 2-dimensional nonpositively curved square complexes in [17].

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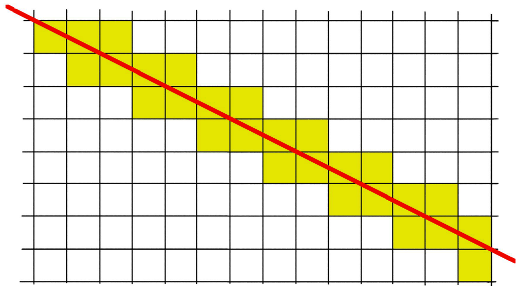

FIGURE 1

The fullness condition is necessary, as there are simple examples of infinite-index quasi-isometrically embedded subgroups J of $G = \pi_1 X$ where X is a compact nonpositively curved cube complex, such that no convex proper subcomplex contains $(J\widetilde{x})$. For instance, when X is the n-torus T^n, for any totally diagonal cyclic subgroup $J \subset \mathbb{Z}^n$ there is no proper J-invariant convex subcomplex. See Figure 1. Another example to bear in mind are subgroups like $\langle at, bt \rangle \subset \langle a, b, t \mid [a,t], [b,t] \rangle$.

We give applications towards separable subgroups of $G = \pi_1 X$ when X is compact and G is relatively hyperbolic. Other applications arise in the relatively hyperbolic case of the results in [16], and in the cubulation result in [10].

2. CAT(0) cube complex definitions

Definition 2.1 (Nonpositively curved cube complexes and local-isometries)**.** The standard 0-cube is a point. The *standard n-cube* is the subspace $[-1,1]^n \subset \mathbb{R}^n$. Its *codimension-$i$ faces* are the subspaces obtained by restricting i-coordinates to ± 1. We regard each codimension-i face as an $(n-i)$-cube. A *cube complex* is a CW complex where closed n-cells are identified with standard n-cubes, and where the attaching map of each n-cell is a combinatorial map whose restriction to each codimension-i face is an $(n-i)$-cell. So, roughly speaking, a cube complex is obtained from a collection of cubes by identifying some of their faces by isometries.

A *flag complex* is a simplicial complex with the property that any collection of $n+1$ pairwise adjacent vertices spans an n-simplex. A cube complex is *nonpositively curved* if the link of each vertex is a flag complex.

A combinatorial map $\phi : A \to B$ between cube complexes is a *local-isometry* if for each $a \in A^0$ mapping to $b \in B^0$ the corresponding map $\phi : \text{link}_A(a) \to \text{link}_B(b)$ is injective and adjacency preserving. As observed in [13], local isometries of nonpositively curved cube complexes are π_1-injective and lift to combinatorial isometries between their universal covers.

Definition 2.2 (Hyperplanes, halfspaces, and hulls)**.** A *midcube* is the subspace of an n-cube $[-1,1]^n$ obtained by restricting exactly one of its coordinates to 0. A *hyperplane* H is a nonempty connected subspace of a CAT(0) cube complex $\widetilde{X}$ with the property that its intersection with each cube is either $\varnothing$ or consists of a midcube. The *open carrier* $N^o(H)$ of a hyperplane H is the union of all open cubes intersecting H. A *halfspace* is a component of $\widetilde{X} - N^o(H)$. Note that each hyperplane is convex relative to the CAT(0) metric geometry, and each halfspace is convex with respect to both combinatorial and metric geometry. As shown in [14], every midcube of $\widetilde{X}$ lies in a unique hyperplane, and each hyperplane separates $\widetilde{X}$ into precisely two components.

Let $D \subset \widetilde{X}$. The *hull* of D is the intersection of all halfspaces containing D. If no halfspace contains D, then define $\mathrm{Hull}(D) = \widetilde{X}$. Note that $\mathrm{Hull}(D)$ is a convex CAT(0) subcomplex of $\widetilde{X}$.

3. Proof of Theorem 1.2

Before going into the relative case, we first present a proof of Theorem 1.2. We do this for the sake of completeness and because the relative version is built on this proof.

We need the following elementary lemma:

Lemma 3.1. *Consider $\mathbb{R}^n$ with the standard basis $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$. Let $\theta_n = \arcsin(1/\sqrt{n})$. If L is a ray emanating from the origin, then there is a codimension-1 subspace H spanned by $d-1$ vectors in $\mathcal{E}$, such that $\sphericalangle(L, H) \geq \theta_n$.*

Proof. We show that the angle with one of the hyperplanes is $\geq \theta_n$. Consider the unit vector $\vec{v}$ in the direction of L, and let $(v_1, \ldots, v_n)$ be the coordinates of $\vec{v}$ relative to the standard basis. Since $\Sigma v_i^2 = 1$, there exists i such that $|v_i| \geq 1/\sqrt{n}$. Let ζ denote the acute angle between $\vec{v}$ and $\pm\vec{e}_i$. Since $\zeta \leq \arccos(1/\sqrt{n})$ the angle between $\vec{v}$ and the plane spanned by $\mathcal{E} - \{\vec{e}_i\}$ is at least $\arcsin(1/\sqrt{n})$. $\square$

We will employ the following immediate consequence of Lemma 3.1 (see Figure 2):

Remark 3.2. Let p be a vertex of an n-cube σ with $n \geq 1$. Let γ be a ray in σ emanating from p. Then there is a midcube H of σ such that γ intersects H at a point b such that $\sphericalangle(\gamma, H) \geq \theta_n$ and $\mathsf{d}(p, b) \leq \sqrt{n}$.

The group J acts *quasiconvexly* on $\widetilde{X}$ if for each $x \in \widetilde{X}$, there exists R such that each geodesic with endpoints on Jx lies within $\mathcal{N}_R(Jx)$. Note that when $\widetilde{X}$ is δ-hyperbolic, for any R there exists $\mu = \mu(R)$ such that any geodesic with endpoints in $\mathcal{N}_R(Jx)$ actually lies within $\mathcal{N}_\mu(Jx)$.

Suppose that J acts quasiconvexly on $\widetilde{X}$, and let $\mathcal{N}_R(Jx)$ be a neighborhood of an orbit Jx, such that geodesics between points of Jx lie in $\mathcal{N}_R(Jx)$. We will show the following:

Proposition 3.3. *Let J act quasiconvexly on the δ-hyperbolic, finite-dimensional CAT(0) cube complex $\widetilde{X}$. For each $x \in \widetilde{X}$ and $R > 0$ there exists S such that $\mathrm{Hull}(\mathcal{N}_R(Jx)) \subset \mathcal{N}_S(Jx)$.*

In the event that $\widetilde{X}$ is locally-finite, J acts cocompactly on $\mathcal{N}_S(Jx)$. It thus follows from Proposition 3.3 that J acts cocompactly on the CAT(0) cube complex $\mathrm{Hull}(\mathcal{N}_R(Jx))$ and Theorem 1.2 follows.

Proof. Let $d = \dim(\widetilde{X})$. Let δ be the hyperbolicity constant for $\widetilde{X}$. Let $\theta = \theta_d$ be as in Lemma 3.1. Without loss of generality, we assume R is large enough that $\mathcal{N}_R(Jx)$ is connected. By the quasiconvexity of Jx, there exists μ so that any geodesic joining points within $\mathcal{N}_{R+1}(Jx)$ lies entirely within $\mathcal{N}_\mu(Jx)$. Let $S = 2\sqrt{d} + \mu + \delta\csc(\theta/2) + \delta$.

As each $a \in \mathrm{Hull}(\mathcal{N}_R(Jx))$ lies within distance $\sqrt{d}$ of some 0-cube $p \in \mathrm{Hull}(\mathcal{N}_R(Jx))$, it suffices to show that $\mathsf{d}(p, Jx) \leq S - \sqrt{d}$ for each

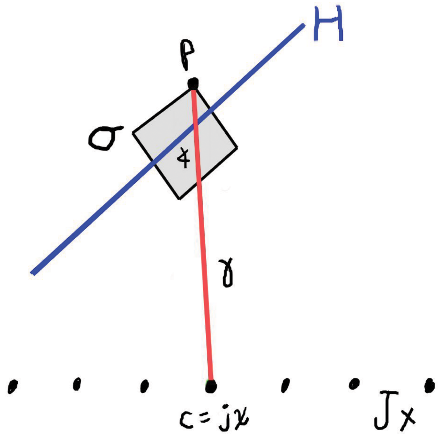

FIGURE 2. H cuts through a cube σ that γ passes through.

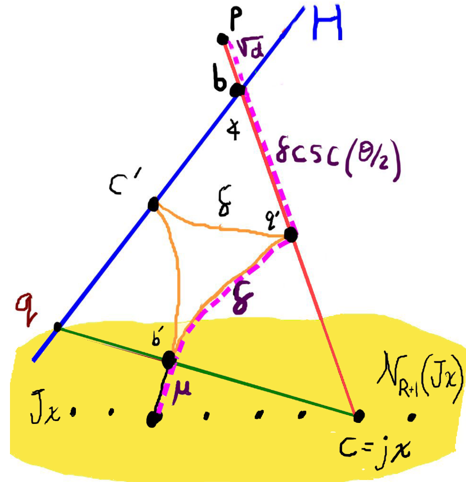

FIGURE 3. If $\mathsf{d}(p, Jx) > S$, then p and Jx lie in opposite halfspaces of H.

0-cube $p \in \text{Hull}(\mathcal{N}_R(Jx))$. Suppose this is not the case. We refer the reader to Figure 3. Let p be a 0-cube of $\widetilde{X}$ such that $\mathsf{d}(p, Jx) \geq S - \sqrt{d}$ and let γ denote a geodesic from p to $c = jx$ where $|\gamma| = \mathsf{d}(p, Jx)$. Let σ be a cube containing an open neighborhood about p in γ. Following Remark 3.2, there exists a hyperplane H passing through γ and with $\sphericalangle(H, \gamma) \geq \theta$. Let $b = H \cap \gamma$ and note that $\mathsf{d}(p, b) \leq \sqrt{d}$.

Suppose $\mathcal{N}_R(Jx)$ does not lie entirely in a halfspace of H. Then there is a point $q \in H \cap \mathcal{N}_{R+1}(Jx)$. Consider the geodesic from c to q, and consider the δ-thin triangle $\Delta(bcq)$. Let b', c', q' denote points in cq, bq, bc that are pairwise equidistant from the vertices and within δ of each other. Observe that $\mathsf{d}(b, q') \leq \delta \csc(\theta/2)$ since the isosceles triangle $\Delta(bq'c')$ has legs of length $\leq \delta \csc(\theta/2)$ since its angle is $\geq \theta$ and its base has length $\leq \delta$. We have

$$\mathsf{d}(p, Jx) \;\leq\; \mathsf{d}(p, b) + \mathsf{d}(b, q') + \mathsf{d}(q', b') + \mathsf{d}(b', Jx) \;\leq\; \sqrt{d} + \delta \csc(\theta/2) + \delta + \mu$$

and this contradicts that $\mathsf{d}(p, Jx) > S - \sqrt{d} = \sqrt{d} + \mu + \delta \csc(\theta/2) + \delta$. $\square$

4. BACKGROUND ON RELATIVE HYPERBOLICITY AND QUASICONVEXITY

Following Bowditch [2], a group G is hyperbolic relative to subgroups $\{P_1, \dots, P_r\}$ if it acts cocompactly and with finite edge stabilizers on a graph K_G such that the following holds: Each edge stabilizer is finite; each infinite vertex stabilizer is a conjugate of some P_i; the graph K_G is hyperbolic; and K_G is *fine* in the sense that each edge of K_G lies in finitely many length n cycles for each n. In this setting, a subgroup J of G is *relatively quasiconvex* if there is a connected J-cocompact subgraph $K_J \subset K_G$ that is quasi-isometrically embedded. We refer the reader to [8] for a survey of the various equivalent approaches to relative hyperbolicity and accompanying notions of quasiconvexity, and to [12] for an explanation that the relative quasiconvexity notion above is equivalent to those surveyed in [8].

Among the various characteristic features of relatively hyperbolic groups is the following thin triangle property from [3, Sec 8.1.3].

Theorem 4.1. *Let $(G, \mathbb{P})$ be relatively hyperbolic, and let Γ be the Cayley graph of G with respect to some finite generating set. For each ϵ there is a constant δ such that if Δ is an ϵ–quasigeodesic triangle with sides c_0, c_1 and c_2, then there is either:*

1. *a point p such that $\mathcal{N}_{\delta/2}(p)$ intersects each side of Δ or*
2. *a peripheral coset gP such that $\mathcal{N}_\delta(gP)$ intersects each side of Δ.*

In the second case, each side c_i of Δ has a subpath c_i' that lies in $\mathcal{N}_\delta(gP)$ such that (coefficients mod 3) the terminal endpoint of c_i' and the initial endpoint of c_{i+1}' are mutually within a distance δ.

The following tighter form of Theorem 4.1 is available for an action on a CAT(0) space:

Proposition 4.2 (Relatively thin triangles)**.** *Suppose G is hyperbolic relative to $\{P_1, \ldots, P_r\}$. Let G act properly and cocompactly on a CAT(0) space $\widetilde{X}$. There exists δ with the following property: Let $\Delta(abc)$ be a geodesic triangle in $\widetilde{X}$. Either $ab \subset \mathcal{N}_\delta(bc \cup ca)$ or there is a translate of an orbit $F = gP_ix$ where $g \in G$ and $1 \leq i \leq r$ such that ab lies in $\mathcal{N}_\delta(F \cup bc \cup ca)$.*

Proof. We now use that there is a G-equivariant quasi-isometry between $\widetilde{X}$ and Γ. The geodesic triangle Δ in $\widetilde{X}$ corresponds to an ϵ'-quasigeodesic triangle Δ' in Γ. Theorem 4.1 holds for Δ' with some constant δ'. In case (1), there is a point p that lies within $\delta'/2$ of each side of Δ'. It follows that there is a point in X that lies uniformly close to each side of Δ. It then follows from the CAT(0) inequality that each side lies in a uniform neighborhood of the other two sides. In case (2), each side of Δ' contains a subpath that lie within $\delta'/2$ of a coset gP, the endpoints of these subpaths are pairwise within $2\delta'$ of each other. The corresponding pairs of points in Δ are uniformly close, and thus the three tails of Δ are uniformly thin by the CAT(0) inequality. Furthermore, the corresponding inner subpaths lie uniformly close to a corresponding orbit Px. We are thus able to choose the desired δ. ☐

Let G be hyperbolic relative to $\{P_1, \ldots, P_r\}$. A subgroup J is *full* if $J \cap P_i^g$ is either finite or of finite-index in P_i^g for each P_i and each $g \in G$.

The following statements hold because, for the relatively hyperbolic group G, quasigeodesics in its Cayley graph Γ uniformly fellow travel relative to cosets of peripheral subgroups, and cosets of peripheral subgroups are uniformly coarsely isolated from each other. This fellow-traveling property is already implicit in Farb's original exposition in [5], and has been revisited in [3].

Full relatively quasiconvex subgroups behave like quasiconvex subgroups of hyperbolic groups in the following sense:

Lemma 4.3. *Let J be a full relatively quasiconvex subgroup of a relatively hyperbolic group G. There is a number μ such that every geodesic in Γ with endpoints on J lies in $\mathcal{N}_\mu(J)$.*

Proof. This follows from [8, Cor 8.16] since the peripheral subgroups $J \cap gPg^{-1}$ of H are quasiconvex in the corresponding peripheral subgroups gPg^{-1} of G. It can also be deduced from [3, Prop. 8.28]. ☐

The following holds because (by the pigeon-hole principle) a long coarse overlap would imply an infinite coarse overlap.

Lemma 4.4. *Let J be a full relatively quasiconvex subgroup of a relatively hyperbolic group G. There is a number $B = B(J, \mu, \delta)$ such that diameter $\big(\mathcal{N}_\delta(gP_i) \cap \mathcal{N}_\mu(J)\big) \leq B$ unless $[P_i^g : P_i^g \cap J] < \infty$.*

Remark 4.5 (Quasiadjustment). For the case when the group G acts properly and cocompactly on a CAT(0) space $\widetilde{X}$, Lemma 4.3 and Lemma 4.4 hold with analogous statements: Firstly, a geodesic in $\widetilde{X}$ with endpoints in Jx actually lies in $\mathcal{N}_\mu(Jx)$. Secondly, there is a number B such that diameter $\big(\mathcal{N}_\delta(gP_ix) \cap \mathcal{N}_\mu(Jx)\big) \leq B$ unless $[P_i^g : J \cap P_i^g] < \infty$.

We conclude that, as there are finitely many J-conjugacy classes of infinite parabolic intersections $J \cap P_i^g$, there is a uniform upper bound on any finite-index $[P_i^g : J \cap P_i^g]$, and hence there exists κ such that whenever diameter $\big(\mathcal{N}_\delta(gP_ix) \cap \mathcal{N}_\mu(Jx)\big) > B$ we have $gP_ix \subset \mathcal{N}_\kappa(Jx)$.

5. Cores in the relatively hyperbolic case

We now give a proof of Theorem 1.1. The proof proceeds in the exact same way as the proof of Theorem 1.2 utilizing Proposition 3.3: We choose R such that $Q \subset \mathcal{N}_R(x)$ where x is the basepoint. We then show that $\mathrm{Hull}(\mathcal{N}_R(Jx)) \subset \mathcal{N}_S(Jx)$ where $S = 2\sqrt{d} + \mu + \delta\csc(\theta/2) + \delta + (B + 2\delta)\csc(\theta/2) + \kappa$. Here d, δ, and θ play the same role as they did in Proposition 3.3. Namely, $d = \dim(\widetilde{X})$, and δ is a hyperbolicity constant for $\widetilde{X}$, and θ is the constant provided by Lemma 3.1. The constants B, μ, and κ are as in Remark 4.5: The constant μ has the property that any geodesic in $\widetilde{X}$ with endpoints in Jx lies in $\mathcal{N}_\mu(Jx)$; The constant B is such that diameter $\big(\mathcal{N}_\delta(gP_ix) \cap \mathcal{N}_\mu(Jx)\big) \leq B$ unless $[P_i^g : J \cap P_i^g] < \infty$. And the constant κ has the property that whenever diameter $\big(\mathcal{N}_\delta(gP_ix) \cap \mathcal{N}_\mu(Jx)\big) > B$ we have $gP_ix \subset \mathcal{N}_\kappa(Jx)$.

The initial part of the proof remains the same: Suppose $\mathsf{d}(p, Jx) > S - \sqrt{d}$, let γ be a geodesic from p to $c = jx$ with $|\gamma| = \mathsf{d}(p, Jx)$, let σ be the open cube that γ initially passes through, and let H be a hyperplane cutting through σ with point of intersection $b = H \cap \gamma$ and angle of intersection $\sphericalangle(\gamma, H) \geq \theta$.

The second part of the proof uses relatively thin triangles in place of the ordinary thin triangle argument given in Proposition 3.3. It is for this reason that the constant S has been increased as above.

Consider the geodesic triangle $\Delta(bqc)$. By Proposition 4.2 either the three sides are δ-close to each other, or there exists $F = gP_ix$ such that each side is δ-close to the union of the other sides with gP_ix. In the former case, we proceed exactly as in the proof of Proposition 3.3. The latter case splits into two subcases according to whether or not P_i^g has a finite-index subgroup in J. We refer the reader to Figure 4.

If $[P_i^g : P_i^g \cap J] < \infty$, then we let q' and c' denote corresponding points on bq and bc farthest from b such that $\mathsf{d}(q', c') \leq \delta$. Observe that $\mathsf{d}(b, q') \leq \delta\csc(\theta/2)$. Observe that $\mathsf{d}(q', F) \leq \delta$ and so $\mathsf{d}(q', Jx) \leq \delta + \kappa$. Thus $\mathsf{d}(p, Jx) \leq \mathsf{d}(p, b) + \mathsf{d}(b, q') + \mathsf{d}(q', Jx) \leq \sqrt{d} + \delta\csc(\theta/2) + \delta + \kappa < S - \sqrt{d}$ which is impossible.

If $[P_i^g : P_i^g \cap J] = \infty$, then we let q'' and c'' denote the points in qc that are closest to q and c and have the property that $\mathsf{d}(q'', F) \leq \delta$ and $\mathsf{d}(c'', F) \leq \delta$. Note

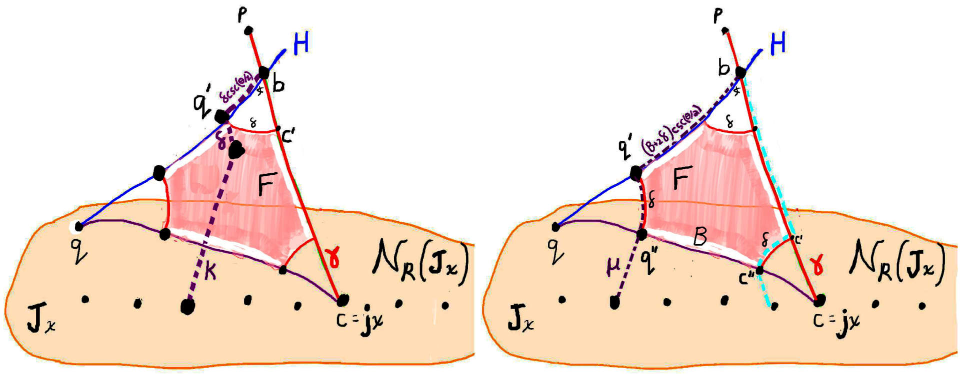

FIGURE 4

that there are points q' and c' on bq and bc with $\mathsf{d}(q',q'') \leq \delta$ and $\mathsf{d}(c',c'') \leq \delta$. Observe that $\mathsf{d}(q'',c'') \leq B$ by Lemma 4.4, since diameter $\big(\mathcal{N}_\delta(Jx) \cap \mathcal{N}_\mu(F)\big) \leq B$. Thus $\mathsf{d}(q',c') \leq \mathsf{d}(q',q'') + \mathsf{d}(q'',c'') + \mathsf{d}(c'',c) \leq 2\delta + B$. Consideration of $\Delta(bq'c')$ shows that at least one of $\mathsf{d}(b,q')$ and $\mathsf{d}(b,c')$ is bounded above by $(2\delta+B)\csc(\theta/2)$. Suppose $\mathsf{d}(b,c') \leq (2\delta+B)\csc(\theta/2)$ and the other possibility is analogous. Then $\mathsf{d}(p,Jx) \leq \mathsf{d}(p,b) + \mathsf{d}(b,c') + \mathsf{d}(c',Jx) \leq \sqrt{d} + (2\delta+B)\csc(\theta/2) + \mu + \delta < S - \sqrt{d}$.

6. Application to separability

The goal of this section is to give applications towards separability of relatively quasiconvex subgroups of a group $G \cong \pi_1 X$ where X is a compact special cube complex.

The following was proven in [7]:

Proposition 6.1. *Let X be a special cube complex. Let $Y \to X$ be a local isometry of nonpositively curved cube complexes where Y is compact. Then there is a finite cover $\widehat{X} \to X$ such that $Y \to X$ lifts to an embedding $Y \hookrightarrow \widehat{X}$ and there is a retraction $\widehat{X} \to Y$.*

Since $\pi_1 X$ is residually finite, we see that the virtual retract $\pi_1 Y$ is separable or closed in the profinite topology of $\pi_1 X$.

The following was proven by Martinez-Pedroza in [11, Thm 1.1]:

Proposition 6.2. *For a relatively quasiconvex subgroup J of G and a maximal parabolic subgroup P of G, there is a constant $C = C(J,P) \geq 0$ with the following property: Let M be a subgroup of P with*

(1) *$J \cap P$ is a subgroup of M and*
(2) *$\mathsf{d}_G(1,g) \geq C$ for any $g \in M - J$.*

Then the natural homomorphism $J *_{J\cap M} M \to G$ is injective, and its image is a relatively quasiconvex subgroup. Moreover, every parabolic subgroup of $\langle J \cup M\rangle$ is conjugate within $\langle J \cup M\rangle$ to a subgroup of J or a subgroup of M.

The following is a natural consequence of Proposition 6.2.

Corollary 6.3. *Let G be hyperbolic relative to $\{P_1,\ldots,P_r\}$. Let J be a relatively quasiconvex subgroup of the group G. Suppose that $J \cap P_i^g$ is separable in P_i^g*

whenever it is infinite. Then there is a sequence $\{J_n\}$ *of fully quasiconvex subgroups such that* $J = \cap_n J_n$.

Proof. There are finitely many representatives of distinct J-conjugates of infinite parabolic intersections $K_s = J \cap P_{i_s}^{g_s}$. For each K_s, the separability hypothesis allows us to choose a finite-index subgroup $M_{sn}^{g_s}$ of $P_{i_s}^{g_s}$ that contains K_s and such that $\mathsf{d}_G(1, g) \geq C(K_s, P_{i_s}^{g_s})$ whenever $g \in M_{sn}^{g_s} - K_s$. We then let J_n denote the group that splits as a tree of groups whose central vertex is J and whose edge groups are K_s and whose other vertices are leaves with vertex group $M_{sn}^{g_s}$.

Now suppose that M_{sn} is a descending sequence of subgroups for each s, so $M_{s1}^{g_s} \supsetneq M_{s2}^{g_s} \supsetneq M_{s3}^{g_s} \supsetneq \cdots$, and suppose $\bigcap M_{si}^{g_s} = K_s$. The natural maps $J_n \hookrightarrow G$ factor through $J_1 \supsetneq J_2 \supsetneq J_3 \cdots$ and then $\bigcap J_i = J$ by the normal form theorem for graphs of groups. $\square$

Corollary 6.4. *Let* X *be a compact special cube complex. Suppose* $G = \pi_1 X$ *is hyperbolic relative to subgroups* $\{P_1, \ldots, P_r\}$. *Let* J *be a relatively quasiconvex subgroup of* G. *Suppose that* $J \cap P_i^g$ *is separable in* P_i^g *for each* P_i *and each* $g \in G$. *Then* J *is separable in* G.

Proof. By Corollary 6.3, the subgroup J is the intersection of a collection $\{J_n\}$ of full quasiconvex subgroups. By Theorem 1.1, each J_n acts freely and cocompactly on a convex subcomplex $\widetilde{Y} \subset \widetilde{X}$ containing the basepoint of $\widetilde{X}$. Thus $J_n = \pi_1 Y_n$ where $Y_n = J_n \backslash \widetilde{Y}$. By Proposition 6.1, the subgroup J_n is separable in G. Consequently J is separable since it is the intersection of separable subgroups. $\square$

7. When G is hyperbolic relative to free-abelian subgroups

In the motivating case when $\pi_1 X$ is hyperbolic relative to virtually free-abelian subgroups, the picture is simplified and several additional conclusions can be drawn.

7.1. **Cosparse actions.**

Definition 7.1 (Cosparse actions)**.** An m-dimensional *quasiflat* $F \subset \widetilde{X}$ is a convex combinatorial subcomplex that is quasi-isometric to $\mathbb{E}^m$. We say G acts *cosparsely* on $\widetilde{X}$ if there is a compact space K and finitely many quasiflats $F_1, \ldots, F_r$ such that:

(1) $\widetilde{X} = GK \cup_i GF_i$.
(2) Each hyperplane in $\widetilde{X}$ crosses GK.
(3) $hF_i \cap kF_j \subset GK$ unless $i = j$ and $k^{-1}h \in \text{Stabilizer}(F_i)$.
(4) Quasiflats are D-isolated in the sense that $hF_i \cap kF_j$ has diameter $< D$ unless $hF_i = kF_j$.

A *cosparse core* for the J-action on $\widetilde{X}$ is a convex subcomplex $\widetilde{Y} \subset \widetilde{X}$ such that J stabilizes and acts cosparsely on $\widetilde{Y}$.

The following holds by a variation on the proof of Theorem 1.1:

Theorem 7.2. *Let* G *be hyperbolic relative to a collection of virtually free-abelian subgroups. Suppose that* G *acts properly and cosparsely on a CAT(0) cube complex* $\widetilde{X}$. *Let* J *be a relatively quasiconvex subgroup of* G. *Let* Q *be a compact subspace of* $\widetilde{X}$. *Then* J *acts cosparsely on* $\text{Hull}(JQ)$.

Moreover, J acts cosparsely on the convex CAT(0) subcomplex

$$\widetilde{Y}_\infty = \mathrm{Hull}(Jx \cup gP_ix)$$

where P_i^g varies over the maximal parabolic subgroups with $P_i^g \cap J$ infinite.

Sketch. Let $E_1, \ldots, E_s$ represent the finitely many distinct J-orbits of quasiflats having infinite coarse intersection with JQ. Each of these corresponds to an infinite parabolic subgroup of J, and there are finitely many J conjugacy classes of these, since J is relatively quasiconvex.

The key point is that $\mathrm{Hull}(JQ \cup \bigcup_i JE_i) \subset \mathcal{N}_d(JQ \cup \bigcup_i JE_i)$ for some d, and hence they are coarsely equal. This is proven following the method in the proof of Theorem 1.1. We describe the adjustments below.

The $\{gF_i\}$ play the role of the $\{gP_ix\}$ in the proof of Theorem 1.1. The subspace $JQ \cup \bigcup_i JE_i$ is coarsely isolated from other quasiflats in $\widetilde{X}$. This substitutes for the fullness property of J.

As in the proof of Theorem 1.1, the argument examines a geodesic triangle Δ in terms of two cases: In the first case Δ is δ-thin. In the second case it is δ-thin relative to $F = gP_ix$. In our setting Δ is δ-thin relative to a quasiflat F. In the proof of Theorem 1.1, the second case breaks into two subcases according to whether $P_i^g \cap J$ is finite or infinite and hence of finite-index in P_i^g by fullness. In our setting, these two subcases correspond to whether the coarse intersection of F with JQ is finite or infinite.

Having verified that $\mathrm{Hull}(JQ \cup \bigcup_i JE_i)$ equals a thickening of $(JQ \cup \bigcup_i JE_i)$, the desired $\mathrm{Hull}(JQ)$ is obtained from $\mathrm{Hull}(JQ \cup \bigcup_i JE_i)$ by removing from any jE_i those halfspaces that are disjoint from JQ. This truncation of $\mathrm{Hull}(JQ \cup \bigcup_i JE_i)$ is the desired J-cosparse core. □

7.2. **Virtual retracts.** The following was observed independently by Chesebro, DeBlois, and Wilton in [4]:

Theorem 7.3. *Suppose $G = \pi_1 X$ is hyperbolic relative to free-abelian subgroups, and X is special and compact. Then every relatively quasiconvex subgroup J of G is a retract of a finite-index subgroup of G.*

Proof. For each K_s in the proof of Corollary 6.3, we choose a finite-index subgroup M_{s*} so that K_s is a retract of M_{s*}. We let J_* be the tree of groups centered at J, and we note that J is a retract of J_*. Finally, we note that J_* is a retract of the finite-index subgroup G' of G provided by Proposition 6.1. □

7.3. **Cocompact convex subspaces.** The following was proven in [9] (where there is a more general relatively hyperbolic version as well). The idea is that each quasiflat can be convexly truncated sufficiently far away from the cocompact part GK.

Proposition 7.4 (CAT(0) truncation)**.** *Suppose G is hyperbolic relative to virtually free-abelian groups, and G acts properly and cosparsely on a CAT(0) cube complex $\widetilde{X}$. Then G acts properly and cocompactly on a convex subspace $\widetilde{Y} \subset \widetilde{X}$.*

We emphasize that the subspace $\widetilde{Y}$ of Proposition 7.4 might not be a subcomplex, and its convexity is only relative to the CAT(0) metric and not the natural cubical L^1 metric.

Combining Theorem 7.2 with Proposition 7.4 we obtain the following:

Corollary 7.5. *Let G be hyperbolic relative to virtually abelian groups. Suppose that G acts properly and cocompactly on the CAT(0) cube complex $\widetilde{X}$. Let J be a relatively quasiconvex subgroup of G. Then J acts properly and cocompactly on a convex CAT(0) subspace.*

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References

[1] I. Agol, D. D. Long, and A. W. Reid, *The Bianchi groups are separable on geometrically finite subgroups*, Ann. of Math. (2) **153** (2001), no. 3, 599–621, DOI 10.2307/2661363. MR1836283 (2002e:20099)

[2] B. H. Bowditch, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. **22** (2012), no. 3, 1250016, 66, DOI 10.1142/S0218196712500166. MR2922380

[3] Cornelia Druţu and Mark Sapir, *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), no. 5, 959–1058, DOI 10.1016/j.top.2005.03.003. With an appendix by Denis Osin and Sapir. MR2153979 (2006d:20078)

[4] Eric Chesebro, Jason DeBlois, and Henry Wilton, *Some virtually special hyperbolic 3-manifold groups*, (2009), 1–51.

[5] B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. **8** (1998), no. 5, 810–840, DOI 10.1007/s000390050075. MR1650094 (99j:20043)

[6] Frédéric Haglund, *Finite index subgroups of graph products*, Geom. Dedicata **135** (2008), 167–209, DOI 10.1007/s10711-008-9270-0. MR2413337 (2009d:20098)

[7] Frédéric Haglund and Daniel T. Wise, *Special cube complexes*, Geom. Funct. Anal. **17** (2008), no. 5, 1551–1620, DOI 10.1007/s00039-007-0629-4. MR2377497 (2009a:20061)

[8] G. Christopher Hruska, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. **10** (2010), no. 3, 1807–1856, DOI 10.2140/agt.2010.10.1807. MR2684983 (2011k:20086)

[9] G. C. Hruska and Daniel T. Wise, *Finiteness properties of cubulated groups*, Compos. Math. **150** (2014), no. 3, 453–506, DOI 10.1112/S0010437X13007112. MR3187627

[10] Tim Hsu and Daniel T. Wise, *Cubulating malnormal amalgams*, Inventiones Mathematicae, **199** (2015), no. 2, 293-331.

[11] Eduardo Martínez-Pedroza, *Combination of quasiconvex subgroups of relatively hyperbolic groups*, Groups Geom. Dyn. **3** (2009), no. 2, 317–342, DOI 10.4171/GGD/59. MR2486802 (2010c:20054)

[12] Eduardo Martínez-Pedroza and Daniel T. Wise, *Relative quasiconvexity using fine hyperbolic graphs*, Algebr. Geom. Topol. **11** (2011), no. 1, 477–501, DOI 10.2140/agt.2011.11.477. MR2783235 (2012c:20117)

[13] Lee Mosher, *Geometry of cubulated 3-manifolds*, Topology **34** (1995), no. 4, 789–814, DOI 10.1016/0040-9383(94)00050-6. MR1362788 (97i:57017)

[14] Michah Sageev, *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. (3) **71** (1995), no. 3, 585–617, DOI 10.1112/plms/s3-71.3.585. MR1347406 (97a:20062)

[15] Peter Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) **17** (1978), no. 3, 555–565. MR0494062 (58 #12996)

[16] Daniel T. Wise, *The structure of groups with a quasiconvex hierarchy*, Available at `http://www.math.mcgill.ca/wise/papers`, pp. 1–189. Submitted.

[17] Daniel T. Wise, *Subgroup separability of the figure 8 knot group*, Topology **45** (2006), no. 3, 421–463, DOI 10.1016/j.top.2005.06.004. MR2218750 (2007a:57010)

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